Transactions of NAS of Azerbaijan, Issue Mathematics, **37** (4), 12–23 (2017).
Series of Physical-Technical and Mathematical Sciences

$L_{p,\nu}$-boundedness of the vector-valued B-square functions

Simten Bayrakci · Seyda Keles · Fatai A. Isayev

Received: 10.06.2017 / Revised: 12.10.2017 / Accepted: 19.11.2017

Abstract. *The classical square functions play important role in Harmonic analysis and have a very direct connections L_2-estimates and Littlewood-Paley theory. In this paper we consider the generalized shift operator associated with the Laplace-Bessel differential operator*

$$\Delta_B = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \Big(\frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n} \frac{\partial}{\partial x_n} \Big), \;\; \nu > 0$$

and the relevant square functions. We introduce $B-$ square functions and then prove boundedness of newly defined $B-$ square functions from $L_{p,\nu}(\mathbb{R}^n_+, H_1)$ to $L_{p,\nu}(\mathbb{R}^n_+)$, for all $1 < p < \infty$ and H_1 separable Hilbert space.

Keywords. Laplace-Bessel differential operator, generalized translation.

Mathematics Subject Classification (2010): 42B25, 44A35

1 Introduction

The classical square functions are defined as follows. Let $S \equiv S(\mathbb{R}^n)$ be the Schwartzian test function space, $\Phi \in S$ and $\int_{\mathbb{R}^n} \Phi(x)dx = 0$. Denote

$$\Phi_t(x) = t^{-n} \Phi\left(\frac{x}{t}\right), \qquad t > 0.$$

The (non- linear) operator

$$S_\Phi(f)(x) = \left(\int_0^\infty |f * \Phi_t(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

S. Bayrakci
Department of Mathematics, Akdeniz University, Antalya, Turkey
E-mail: simten@akdeniz.edu.tr

S. Keles
Department of Mathematics, Akdeniz University, Antalya, Turkey
E-mail: seydaaltinkol@gmail.com

F. Isayev
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan Baku AZ 1141, Azerbaijan
E-mail: isayevfatai@yahoo.com

is called a square function generated by Φ. Here $*$ denotes the usual convolution. Together with the maximal functions and singular integrals, the square function plays important role in Harmonic Analysis and have a very direct connection with L_2 estimates and Littlewood-Paley theory (see [26]; p. 26-27, [27]). There are a lot of diverse variants of square functions and their various applications; see [4],[11],[17],[23],[1] and references therein.

Note that the Laplace- Bessel differential operator Δ_B is known as an important operator in analysis and its applications. The relevant harmonic analysis has been the research area for many mathematicicians as B. M. Levitan, B. Muckenhoupt, E. M. Stein, I. A. Kipriyanov, M. I. Klyunchantsev, J. Löfström, J. Peetre, A. D. Gadjiev, I. A. Aliev, V. S. Guliyev, B. Rubin, S. Uyhan, I. Ekincioglu, J. J. Hasanov, S. Keles, S. Bayrakci, M. N. Omarova and others. (see, [6], [7], [18], [19], [20], [22], [24], [2], [8], [21], [12], [3], [5], [9], [10], [13], [14], [15], [16]).

The structure of the paper is as follows. In section 2, we present some definitions and auxiliary results. In section 3, we give square function associated with Laplace- Bessel differential operator and prove its boundedness on $L_{p,\nu}\left(\mathbb{R}_{+}^{n}\right)$.

2 Some Definitions and Auxiliary

Let R^n is n- dimensional Euclidean space, $R_{+}^{n}=\{x=(x_1,...,x_n)\in\mathbb{R}^n : x_n>0\}$ and $|x|=\left(\sum\limits_{i=1}^{n}x_i^2\right)^{\frac{1}{2}}$ and define

$$L_{p,\nu}(\mathbb{R}_{+}^{n}):=L_p(\mathbb{R}_{+}^{n},x_n^{2\nu}dx)=\Big\{f:\|f\|_{L_{p,\nu}(\mathbb{R}_{+}^{n})}=\Big(\int_{\mathbb{R}_{+}^{n}}|f(x)|^p x_n^{2\nu}dx\Big)^{\frac{1}{p}}<\infty\Big\},$$

where $\nu>0$ is a fixed parameter, $1\le p<\infty$ and $dx=dx_1dx_2...dx_n$.

For $x\in\mathbb{R}_{+}^{n}$ and $r>0$, we denote by $E(x,r)=\{y\in\ \mathbb{R}_{+}^{n} : |x-y|<r\}$ the open ball centered at x of radius r, by ${}^{\complement}E(x,r)=\mathbb{R}_{+}^{n}\setminus E(x,r)$ denote its complement. For any $A\subset\mathbb{R}_{+}^{n}$, $|A|_\nu=\int_A x_n^{2\nu}dx$ and $|E(0,r)|_\nu=Cr^{n+2\nu}$. Denote by T^y the generalized shift operator (B- shift operator) acting according to the law

$$T^y f(x)=\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)\Gamma(\frac{1}{2})}\int_0^{\pi} f\left(x'-y',\sqrt{x_n^2-2x_ny_n\cos\alpha_n+y_n^2}\right)(\sin\alpha)^{2\nu-1}\ d\alpha$$

where $x=(x',x_n)$, $y=(y',y_n)$ and $x',y'\in\mathbb{R}^{n-1}$.

We remark that T^y is closely connected with the Bessel differential operator

$$B_{x_n}=\frac{\partial^2}{\partial x_n^2}+\frac{2\nu}{x_n}\frac{\partial}{\partial x_n},\quad x_n>0$$

(see [21] for details).

For $\varphi\in L_{p,\nu}(R_{+}^{n})$

$$\|T^y\varphi\|_{L_{p,\nu}(\mathbb{R}_{+}^{n})}\le C\,\|\varphi\|_{L_{p,\nu}(\mathbb{R}_{+}^{n})}\,.$$

The T^y shift operator generates the corresponding "$B-convolution$"

$$\left(\varphi\otimes\psi\right)(x)=\int_{\mathbb{R}_{n}^{+}}\varphi(y)T^y\psi(x)y_n^{2\nu}dy$$

for which the Young inequality holds:

$$\|\varphi\otimes\psi\|_{L_{r,\nu}(\mathbb{R}_{+}^{n})}\le\|\varphi\|_{L_{p,\nu}(\mathbb{R}_{+}^{n})}\|\psi\|_{L_{q,\nu}(\mathbb{R}_{+}^{n})};\quad 1\le p,q,r\le\infty,\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1.$$

We note the following property (needed below) of the "$B - convolution$" :

$$\varphi \otimes \psi = \psi \otimes \varphi$$

is valid. The Fourier-Bessel transform is defined as follows [12]

$$F_\nu \varphi(z) = \int_{\mathbb{R}_+^n} \varphi(x) e^{-ix'z'} j_{\nu-\frac{1}{2}}(x_n z_n) x_n^{2\nu} dx$$

where $j_{\nu-\frac{1}{2}}$ is the normalized Bessel function that is defined as for $t > 0$, $\nu > \frac{-1}{2}$;

$$j_\nu(t) = 2^\nu \Gamma(\nu+1) \frac{J_\nu(t)}{t^\nu} = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)} \int_{-1}^{1} \left(1-u^2\right)^{\nu-\frac{1}{2}} \cos(tu)\, du.$$

Here $J_\nu(t)$ is the first kind of Bessel function. The normalized Bessel function j_ν is the eigenfunction of the Bessel differential operator satisfying the conditions for all $t \in \mathbb{R}$, $|j_\nu(t)| \le 1$, $j_\nu(0) = 1$, $j_\nu'(0) = 0$ and for $\lambda \in \mathbb{C}$

$$j_\nu(\lambda x)\, j_\nu(\lambda y) = T^x(j_\nu(\lambda\, \cdot))(y).$$

The influence of the Fourier- Bessel transform to B- convolution is as follows

$$F_\nu(\varphi \otimes \psi)(z) = (F_\nu \varphi)(z)\,(F_\nu \psi)(z).$$

3 Vector-valued function and its some properties

Let H be a separable Hilbert space. Then a function $f(x)$, from $\mathbb{R}_+^n$ to H is measurable if the scalar valued functions $\langle f(x), \varphi\rangle$ are measurable, where $\langle \cdot, \cdot \rangle$ denotes the inner product of H, and φ denotes an arbitrary vector of H. If $f(x)$ is such a measurable function, then $\|f(x)\|_H$ is also measurable (as a function with non-negative values), where $\|\cdot\|_H$ denotes the norm of H. Thus $L_{p,\nu}\left(\mathbb{R}_+^n, H\right)$ is defined as the equivalence classes of measurable functions $f(x)$ from $\mathbb{R}_+^n$ to H,with the property that the norm

$$\|f\|_{L_{p,\nu}\left(\mathbb{R}_+^n, H\right)} = \left(\int_{\mathbb{R}_+^n} \|f(x)\|_H^p\, x_n^{2\nu} dx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty$$

is finite. When $p = \infty$ there is a similiar definition, except

$$\|f\|_{L_\infty(\mathbb{R}_+^n, H)} = ess \sup_{x \in \mathbb{R}_+^n} \|f(x)\|_H.$$

Now, let H_1 and H_2 be two separable Hilbert spaces, and let $B(H_1, H_2)$ be the Banach space of bounded linear operators from H_1 to H_2, with the usual oeprator norm. We say that a function $f(x)$, from $\mathbb{R}_+^n$ to $B(H_1, H_2)$ is measurable if $f(x)\varphi$ is an H_2-valued measurable function for every $\varphi \in H_1$.In this case $\|f(x)\|_{B(H_1,H_2)}$ is also measurable and we can define the space $L_{p,\nu}\left(\mathbb{R}_+^n, B(H_1, H_2)\right)$, as before; (here again $\|\cdot\|_{B(H_1,H_2)}$ denotes the norm, this time in $B(H_1, H_2)$). The usual facts about B-convolution hold in this setting. For example, suppose $g \in L_{q,\nu}(\mathbb{R}_+^n, B(H_1, H_2))$ and $f \in L_{p,\nu}(\mathbb{R}_+^n, B(H_1, H_2))$, $1 \le p$, $q \le \infty$. Then

$$(f \otimes g)(x) = \int_{\mathbb{R}_+^n} g(y) T^y f(x) y_n^{2\nu} dy$$

converges in the norm of H_2 almost every x, and

$$\|(f \otimes g)(x)\|_{H_2} \leq \int_{\mathbb{R}^n_+ +} \|g(y)\|_{B(H_1,H_2)} \|T^y f(x)\|_{H_1} y_n^{2\nu} dy.$$

Also, when $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ with $1 \leq r \leq \infty$,

$$\|f \otimes g\|_{L_{r,\nu}(\mathbb{R}^n_+, H_2)} \leq \|g\|_{L_{q,\nu}(\mathbb{R}^n_+, B(H_1,H_2))} \|f\|_{L_{p,\nu}(\mathbb{R}^n_+, H_1)}.$$

Now, we suppose that $f \in L_{1,\nu}(\mathbb{R}^n_+, H)$. Then we can define its H-valued Fourier-Bessel transform

$$(F_\nu f)(x) = \int_{\mathbb{R}^n_+} f(y) e^{-ix'y'} j_{\nu-\frac{1}{2}}(x_n y_n) y_n^{2\nu} dy,$$

which is an element of $L_\infty(\mathbb{R}^n_+, H)$. If $f \in L_{1,\nu}(\mathbb{R}^n_+, H) \cap L_{2,\nu}(\mathbb{R}^n_+, H)$, then $F_\nu f \in L_{2,\nu}(\mathbb{R}^n_+, H)$. The Fourier- Bessel transform can then be extended by continuity to a unitary mapping of the Hilbert space $L_{2,\nu}(\mathbb{R}^n_+, H)$ to itself (see for details [25]; p. 45-46, [28]; p. 307-309).

Definition 3.1 *[16] We say that a function K on $\mathbb{R}^n_+$ whose values are bounded operators from H_1 to H_2 is vector valued $B-$ singular integral kernel provided that*

1) K is measurable and integrable on compacts sets not containing the origin,

2) There exists $M > 0$ for all $\varepsilon : 0 < \varepsilon < r$, $\left\| \int_{E(0,r)\backslash E(0,\varepsilon)} K(x)\, x_n^{2\nu} dx \right\|_{B(H_1,H_2)} \leq M$ and for each h in H_1, $\left[\int_{E(0,r)\backslash E(0,\varepsilon)} K(x)\, x_n^{2\nu} dx \right] h$ converges as $\varepsilon \to 0$,

3) For each h in H_1 with $\|h\|_{H_1} \leq 1$

$$\int_{E(0,4r)\backslash E(0,r)} |x| \, \|K(x)\, h\|_{H_2} x_n^{2\nu} dx \leq M,$$

4)

$$\int_{^c E(0,4|y|)} \|T^y K(x) - K(x)\|_{B(H_1,H_2)} x_n^{2\nu} dx \leq M; \quad |y| < \frac{1}{4}.$$

Theorem 3.1 *[16] Let $K \in L^{loc}_{1,\nu}(\mathbb{R}^n_+, B(H_1, H_2))$ be a vector valued $B-$ singular integral kernel. For $f \in L_{p,\nu}\left(\mathbb{R}^n_+, H_1\right)$, $1 < p < \infty$ suppose*

$$(T_\varepsilon f)(x) = \int_{^c E(0,\varepsilon)} K(y)\, T^y f(x)\, y_n^{2\nu} dy, \ \varepsilon > 0.$$

Then there exists a constant $C_{p,\nu} > 0$ such that for all $f \in L_{p,\nu}\left(\mathbb{R}^n_+, H_1\right)$ the inequality

$$\|T_\varepsilon f\|_{L_{p,\nu}\left(\mathbb{R}^n_+, H_2\right)} \leq C_{p,\nu} \|f\|_{L_{p,\nu}\left(\mathbb{R}^n_+, H_1\right)}$$

is valid and

$$Tf(x) = \lim_{\varepsilon \to 0^+} (T_\varepsilon f)(x)$$

exists in $L_{p,\nu}\left(\mathbb{R}^n_+, H_2\right)$ and also the following inequality is valid

$$\|Tf\|_{L_{p,\nu}\left(\mathbb{R}^n_+, H_2\right)} \leq C_{p,\nu} \|f\|_{L_{p,\nu}\left(\mathbb{R}^n_+, H_1\right)}$$

where the constant $C_{p,\nu} > 0$ is independent of f.

4 Boundedness of Square Functions

We let $H_1 = \mathbb{C}$, the complex numbers, and $H_2 = L_2\left(\mathbb{R}^+, \frac{dt}{t}\right)$the Hilbert space of square integrable functions on the positive half-line with respect to the measure $\frac{dt}{t}$ and norm

$$\|h\|_{H_2} = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}.$$

Definition 4.1 *We say that a scalar valued function φ on $\mathbb{R}^n_+$ is a Littlewood- Paley function provided it satisfies*

1)

$$\varphi \in L_{1,\nu}\left(\mathbb{R}^n_+\right) \quad \textit{and} \int_{\mathbb{R}^n_+} \varphi(x)\, x_n^{2\nu} dx = 0;$$

2)

$$|\varphi(x)| \leq C(1+|x|)^{-(Q+\alpha)}; \quad \exists\alpha > 0,\ Q = n + 2\nu;$$

3)

$$\int_{\mathbb{R}^n_+} \left|T^h\varphi(x) - \varphi(x)\right| x_n^{2\nu} dx \leq C|h|^\gamma; \ \exists\gamma > 0.$$

Proposition 4.1 *Suppose φ is a Littlewood- paley function. Then there exists a C constant for all $z \in \mathbb{R}^n_+$ such that*

$$\|F_\nu\varphi(\cdot\, z)\|_{H_2} \leq C.$$

Proof. We begin by showing that

$$|(F_\nu\varphi)(z)| \leq C\min\left\{|z|^{\frac{\alpha}{Q+\alpha+1}}, |z|^{-\gamma}\right\},$$

where C is independent of z. Now, we investigate $|(F_\nu\varphi)(z)|$.

$$\begin{aligned}(F_\nu\varphi)(z) &= \int_{\mathbb{R}^n_+} \varphi(x)\, e^{-ix'z'} j_{v-\frac{1}{2}}(x_n z_n)\, x_n^{2\nu} dx \\ &= \int_{\mathbb{R}^n_+} \varphi(x)\left(e^{-ix'z'} j_{v-\frac{1}{2}}(x_n z_n) - 1\right) x_n^{2\nu} dx.\end{aligned}$$

We get

$$\begin{aligned}|(F_\nu\varphi)(z)| &\leq \int_{\mathbb{R}^n_+} |\varphi(x)|\left|e^{-ix'z'} j_{v-\frac{1}{2}}(x_n z_n) - 1\right| x_n^{2\nu} dx \\ &\leq 2\int_{\mathbb{R}^n_+} |\varphi(x)| \min\{|x||z|, 1\}\, x_n^{2\nu} dx \\ &\leq 2\int_{E(0,\eta)} |\varphi(x)||x||z|\, x_n^{2\nu} dx + 2\int_{^cE(0,\eta)} |\varphi(x)|\, x_n^{2\nu} dx \\ &= I + J,\end{aligned}$$

where

$$I = 2\int_{E(0,\eta)} |\varphi(x)||x||z|\, x_n^{2\nu} dx \leq 2c|z|\eta^{Q+1}$$

and

$$J = 2\int_{{}^{c}E(0,\eta)} |\varphi(x)|\, x_n^{2\nu} dx \le 2c\int_{{}^{c}E(0,\eta)} |x|^{-(Q+\alpha)}\, x_n^{2\nu} dx \le 2c\eta^{-\alpha}.$$

Using I and J, we write

$$|(F_\nu \varphi)(z)| \le 2c\left(|z|\,\eta^{Q+1} + \eta^{-\alpha}\right).$$

Now, we take minimizing with respect to $\eta > 0$. Let

$$g(\eta) = |z|\,\eta^{Q+1} + \eta^{-\alpha}, \quad \eta > 0.$$

We have

$$\begin{aligned} g^{'}(\eta) &= |z|\,(Q+1)\,\eta^{Q} - \alpha\eta^{-\alpha-1} \\ &= \eta^{-\alpha-1}\left(|z|\,(Q+1)\,\eta^{Q+\alpha+1} - \alpha\right). \end{aligned}$$

For $g^{'}(\eta) = 0$, $\eta = \left(\frac{\alpha}{|z|(Q+1)}\right)^{\frac{1}{Q+\alpha+1}}$. Let

$$c_0 = \left(\frac{\alpha}{|z|\,(Q+1)}\right)^{\frac{1}{Q+\alpha+1}}.$$

For $0 < \eta < c_0$, $g^{'}(\eta) < 0$ and for $\eta > 0$, $g^{'}(\eta) > 0$. Then $\min\limits_{\eta>0} g(\eta) = g(c_0)$. We obtain that

$$\begin{aligned} g(c_0) &= |z|\left(\frac{\alpha}{|z|\,(Q+1)}\right)^{\frac{Q+1}{Q+\alpha+1}} + \left(\frac{\alpha}{|z|\,(Q+1)}\right)^{\frac{-\alpha}{Q+\alpha+1}} \\ &= |z|^{1-\frac{Q+1}{Q+\alpha+1}}\left(\frac{\alpha}{(Q+1)}\right)^{\frac{Q+1}{Q+\alpha+1}} + |z|^{\frac{\alpha}{Q+\alpha+1}}\left(\frac{\alpha}{(Q+1)}\right)^{\frac{-\alpha}{Q+\alpha+1}} \\ &= |z|^{\frac{\alpha}{Q+\alpha+1}}\, C(\alpha, Q). \end{aligned}$$

$$\min_{\eta>0} g(\eta) = |z|^{\frac{\alpha}{Q+\alpha+1}}\, C(\alpha, Q).$$

We get

$$|(F_\nu \varphi)(z)| \le |z|^{\frac{\alpha}{Q+\alpha+1}}\, C(\alpha, Q). \tag{4.1}$$

$$\begin{aligned} F_\nu\left(T^h\varphi\right)(z) &= \int_{\mathbb{R}_+^n}\left(T^h\varphi\right)(x)\, e^{-ix'z'} j_{v-\frac{1}{2}}(x_n z_n)\, x_n^{2\nu} dx \\ &= \int_{\mathbb{R}_+^n}\varphi(x)\, T^h\left(e^{-ix'z'} j_{v-\frac{1}{2}}(x_n z_n)\right) x_n^{2\nu} dx \\ &= \int_{\mathbb{R}_+^n}\varphi(x)\, e^{-i(x'-h')z'} T^{h_n}\left(j_{v-\frac{1}{2}}(x_n z_n)\right) x_n^{2\nu} dx \\ &= e^{ih'z'}\int_{\mathbb{R}_+^n}\varphi(x)\, e^{-ix'z'} T^{h_n}\left(j_{v-\frac{1}{2}}(x_n z_n)\right) x_n^{2\nu} dx \\ &= -\int_{\mathbb{R}_+^n}\varphi(x)\, e^{-ix'z'} j_{v-\frac{1}{2}}(x_n z_n)\, x_n^{2\nu} dx \\ &= -F_\nu\varphi(z) \end{aligned}$$

for $h = (h', h_n)$, where $h' = \frac{\pi z'}{|z'|^2}$ and $h_n = 0$.

We can write

$$2(F_\nu \varphi)(z) = \int_{\mathbb{R}_+^n} \left[\varphi(x) - T^h \varphi(x)\right] e^{-ix'.z'} j_{\nu-\frac{1}{2}}(x_n z_n) x_n^{2\nu} dx$$

and

$$|(F_\nu \varphi)(z)| \le \frac{1}{2} \int_{\mathbb{R}_+^n} \left|T^h \varphi(x) - \varphi(x)\right| x_n^{2\nu} dx \le \frac{1}{2} |h|^\gamma .$$

Here $h = (h', h_n)$, $h' = \frac{\pi z'}{|z'|^2}$ and $h_n = 0$, we have

$$|(F_\nu \varphi)(z)| \le C |z|^{-\gamma} . \tag{4.2}$$

From 4.1 and 4.2, we obtain that.

$$|(F_\nu \varphi)(z)| \le C \min\left\{|z|^{\frac{\alpha}{Q+\alpha+1}}, |z|^{-\gamma}\right\}.$$

To complete the proof we estimate

$$\|F_\nu \varphi(\cdot z)\|_{H_2}^2 = \int_0^\infty |F_\nu \varphi(tz)|^2 \frac{dt}{t} \le C \int_0^\infty \min\left\{|tz|^{\frac{\alpha}{Q+\alpha+1}}, |tz|^{-\gamma}\right\}^2 \frac{dt}{t} \le C.$$

The proof is completed.

Let now $K(x) \in L\left(\mathbb{C}, L_2\left(\mathbb{R}^+, \frac{dt}{t}\right)\right)$ be given by

$$K(x) a = t^{-Q} \varphi\left(\frac{x}{t}\right) a = \varphi_t(x) a$$

where $x \in \mathbb{R}_+^n$, a is a complex scalar and φ is a Littlewood- Paley function. Corresponding to K we consider the singular integral operator

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{{}^c E(0,\varepsilon)} K(y) T^y f(x) y_n^{2\nu} dy = \lim_{\varepsilon \to 0} \int_{{}^c E(0,\varepsilon)} \varphi_t(y) T^y f(x) y_n^{2\nu} dy.$$

We want to show that T falls with Theorem 2.3. and thus obtain its $L_{p,\nu}$ continuity, $1 < p < \infty$. To get a feeling for the situation we do the $L_{2,\nu}$ case first.

Proposition 4.2 *Defined by*

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{{}^c E(0,\varepsilon)} K(y) T^y f(x) y_n^{2\nu} dy$$

T operator is bounded from $L_{2,\nu}\left(\mathbb{R}_+^n, H_1\right)$ into $L_{2,\nu}\left(\mathbb{R}_+^n, H_2\right)$.

Proof. Observe that for f in $L_{2,\nu}\left(\mathbb{R}_{+}^{n}, H_1\right)$ and account of the Fubuni Theorem, Plancherel's identity and Proposition 3.2. we have

$$\begin{aligned}
&\int_{\mathbb{R}_{+}^{n}} \|K \otimes f(x)\|_{H_2}^{2} x_n^{2\nu} dx \\
&= \int_{\mathbb{R}_{+}^{n}} \left(\int_0^\infty |\varphi_t \otimes f(x)|^2 \frac{dt}{t}\right) x_n^{2\nu} dx \\
&= \int_0^\infty \int_{\mathbb{R}_{+}^{n}} |F_\nu (\varphi_t \otimes f)(x)|^2 x_n^{2\nu} dx \frac{dt}{t} \\
&= C \int_0^\infty \left(\int_{\mathbb{R}_{+}^{n}} |F_\nu \varphi_t (x)|^2 |F_\nu f(x)|^2 x_n^{2\nu} dx\right) \frac{dt}{t} \\
&\leq C \int_{\mathbb{R}_{+}^{n}} \left(\sup_{x\in\mathbb{R}_{+}^{n}} \int_0^\infty |F_\nu \varphi(tx)|^2 \frac{dt}{t}\right) |F_\nu f(x)|^2 x_n^{2\nu} dx \\
&\leq C \sup_{x\in\mathbb{R}_{+}^{n}} \|F_\nu \varphi(.z)\|_{H_2}^{2} \|F_\nu f\|_{L_{2,\nu}\left(\mathbb{R}_{+}^{n}, H_1\right)}^{2} \leq C \|f\|_{L_{2,\nu}\left(\mathbb{R}_{+}^{n}, H_1\right)}^{2}.
\end{aligned}$$

We obtain

$$\|Tf\|_{L_{2,\nu}\left(\mathbb{R}_{+}^{n}, H_2\right)} \leq C \|f\|_{L_{2,\nu}\left(\mathbb{R}_{+}^{n}, H_1\right)}.$$

Theorem 4.1 *Suppose T is given by*

$$Tf(x) = \lim_{\varepsilon\to 0} \int_{{}^{c}E(0,\varepsilon)} \varphi_t(y)\, T^y f(x)\, y_n^{2\nu} dy.$$

Then there exists a constant $C = C_p > 0$ such that for all $f \in L_{p,\nu}\left(\mathbb{R}_{+}^{n}, H_1\right)$ the inequality

$$\|Tf\|_{L_{p,\nu}\left(\mathbb{R}_{+}^{n}, H_2\right)} \leq C \|f\|_{L_{p,\nu}\left(\mathbb{R}_{+}^{n}, H_1\right)}, \qquad 1 < p < \infty$$

is valid.

Proof. We verify that $(1) - (4)$ in Definition 3.1. are satisfied. (1) is immediate. As for (2), observe that since

$$\int_{E(0,R)} \varphi(x)\, x_n^{2\nu} dx = -\int_{{}^{c}E(0,R)} \varphi(x)\, x_n^{2\nu} dx$$

by property (2) of Definition 3.1

$$\left|\int_{E(0,R)} \varphi(x)\, x_n^{2\nu} dx\right| \leq \frac{CR^Q}{(1+R)^{Q+\alpha}}.$$

Now, we must show that $\left\| \int_{E(0,R)} \varphi_t(x)\, x_n^{2\nu} \right\| dx_{H_2} \le C.$

$$\begin{aligned}
\left\| \int_{E(0,R)} \varphi_t(x) x_n^{2\nu} dx \right\|_{H_2} &= \Big(\int_0^\infty \Big| \int_{E(0,R)} \varphi_t(x) x_n^{2\nu} dx \Big|^2 \frac{dt}{t} \Big)^{\frac{1}{2}} \\
&= \Big(\int_0^\infty \Big| t^{-Q} \int_{E(0,R)} \varphi(\frac{x}{t}) x_n^{2\nu} dx \Big|^2 \frac{dt}{t} \Big)^{\frac{1}{2}} \\
&= \Big(\int_0^\infty \Big| \int_{E(0,\frac{R}{t})} \varphi(u) u_n^{2\nu} du \Big|^2 \frac{dt}{t} \Big)^{\frac{1}{2}} \\
&\le \Big(\int_0^\infty \Big(\frac{C(\frac{R}{t})^Q}{(1+\frac{R}{t})^{Q+\alpha}} \Big)^2 \frac{dt}{t} \Big)^{\frac{1}{2}} \\
&\le C \Big(\int_0^\infty \frac{R^{2Q} t^{2(Q+\alpha)}}{t^{2Q}(t+R)^{2(Q+\alpha)}} \frac{dt}{t} \Big)^{\frac{1}{2}} \\
&\le C R^Q \Big(\int_0^\infty \frac{t^{2\alpha-1}}{(t+R)^{2(Q+\alpha)}} dt \Big)^{\frac{1}{2}} \le C.
\end{aligned}$$

We can write

$$\begin{aligned}
&\Big\| \int_{E(0,R)} \varphi_t(x) x_n^{2\nu} dx \Big\|_{H_2} \\
&\le \begin{cases} C \left(\int_R^\infty \left(\frac{R}{t}\right)^{2Q} dt \right)^{\frac{1}{2}}; R \le t \\ C\Big(\int_0^R \left(\frac{R-t}{t^{\alpha-1}}\right)^2 t^{2\alpha-1} dt \Big)^{\frac{1}{2}}; R > t \end{cases} \\
&\le C.
\end{aligned}$$

We obtain

$$\left\| \int_{E(0,R)} \varphi_t(x)\, x_n^{2\nu} dx \right\|_{H_2} \le C$$

which gives (2). For (3), since

$$|\varphi(x)| \le C(1+|x|)^{-(Q+\alpha)}; \quad \exists \alpha > 0,\ Q = n + 2\nu$$

$$|\varphi_t(x)| \le \frac{C t^\alpha}{(t+|x|)^{Q+\alpha}}$$

we have

$$\int_{E(0,4r)\setminus E(0,r)} |x|\, \|\varphi_t(x)\|_{H_2} x_n^{2\nu} dx \le Cr.$$

Finally to show that

$$\int_{{}^cE(0,4|y|)} \|T^y \varphi_t(x) - \varphi_t(x)\|_{H_2} x_n^{2\nu} dx \le M; \quad |y| < \frac{1}{4}.$$

Let $0<\varepsilon<\min\{\alpha,\gamma,Q\}$ and observe that

$$\begin{aligned}
&\int_{^cE(0,4|y|)} \|T^y\varphi_t(x)-\varphi_t(x)\|_{H_2} x_n^{2\nu}dx\\
&=\int_{^cE(0,4|y|)} |x|^{-\left(\frac{Q+\varepsilon}{2}\right)}\left(\|T^y\varphi_t(x)-\varphi_t(x)\|_{H_2}|x|^{\left(\frac{Q+\varepsilon}{2}\right)}\right)x_n^{2\nu}dx\\
&\le\Big(\int_{^cE(0,4|y|)} |x|^{-(Q+\varepsilon)}x_n^{2\nu}dx\Big)^{\frac12}\Big(\int_{^cE(0,4|y|)}\|T^y\varphi_t(x)-\varphi_t(x)\|_{H_2}^2|x|^{(Q+\varepsilon)}x_n^{2\nu}dx\Big)^{\frac12}\\
&\le C|y|^{\frac{-\varepsilon}{2}}\Big(\int_{^cE(0,4|y|)}|x|^{(Q+\varepsilon)}\Big(\int_0^\infty|T^y\varphi_t(x)-\varphi_t(x)|^2\frac{dt}{t}\Big)x_n^{2\nu}dx\Big)^{\frac12}\\
&\le C|y|^{\frac{-\varepsilon}{2}}\Big(\int_{^cE(0,4|y|)}|x|^{(Q+\varepsilon)}\Big(\int_0^\infty\Big|T^{\frac{y}{t}}\varphi\left(\frac{x}{t}\right)-\varphi\left(\frac{x}{t}\right)\Big|^2\frac{dt}{t^{2Q+1}}\Big)x_n^{2\nu}dx\Big)^{\frac12}\\
&\le C|y|^{\frac{-\varepsilon}{2}}\Big(\int_0^\infty t^{-2Q}\Big(\int_{^cE(0,4|y|)}|x|^{(Q+\varepsilon)}\Big|T^{\frac{y}{t}}\varphi\left(\frac{x}{t}\right)-\varphi\left(\frac{x}{t}\right)\Big|^2x_n^{2\nu}dx\Big)\frac{dt}{t}\Big)^{\frac12}.
\end{aligned}$$

We use the following inequality

$$\begin{aligned}
&\Big|T^{\frac{y}{t}}\varphi\left(\frac{x}{t}\right)-\varphi\left(\frac{x}{t}\right)\Big|\le\Big|T^{\frac{y}{t}}\varphi\left(\frac{x}{t}\right)\Big|+\Big|\varphi\left(\frac{x}{t}\right)\Big|\\
&\le c_\nu\int_0^\infty\left|\varphi\left(\frac{x'-y'}{t},\frac{\sqrt{x_n^2+y_n^2-2x_ny_n}}{t}\right)\right|(\sin\alpha)^{2\nu-1}d\alpha+\Big|\varphi\left(\frac{x}{t}\right)\Big|\\
&\le K\left(\frac{t}{|x|}\right)^{Q+\varepsilon},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_{^cE(0,4|y|)}\Big\|T^y\varphi_t(x)-\varphi_t(x)\Big\|_{H_2}x_n^{2\nu}dx\\
&\le C|y|^{\frac{-\varepsilon}{2}}\Big(\int_0^\infty t^{-2Q}\Big(\int_{^cE(0,4|y|)}|x|^{Q+\varepsilon}\Big|T^{\frac{y}{t}}\varphi\left(\frac{x}{t}\right)-\varphi\left(\frac{x}{t}\right)\Big|^2x_n^{2\nu}dx\Big)\frac{dt}{t}\Big)^{\frac12}\\
&\le C|y|^{\frac{-\varepsilon}{2}}\Big(\int_0^\infty t^{-Q+\varepsilon}\Big(\int_{^cE(0,4|y|)}\Big|T^{\frac{y}{t}}\varphi\Big(\frac{x}{t}\Big)-\varphi\left(\frac{x}{t}\right)\Big|x_n^{2\nu}dx\Big)\frac{dt}{t}\Big)^{\frac12}\\
&\le C|y|^{\frac{-\varepsilon}{2}}\left(\int_0^\infty t^{-Q+\varepsilon}\Big(\int_{\mathbb{R}_+^n}\Big|T^{\frac{y}{t}}\varphi\Big(\frac{x}{t}\Big)-\varphi\left(\frac{x}{t}\right)\Big|x_n^{2\nu}dx\Big)\frac{dt}{t}\right)^{\frac12}\\
&\le C|y|^{\frac{-\varepsilon}{2}}\left(\int_0^\infty t^{-Q+\varepsilon}\min\left\{2t^Q\|\varphi\|_{L_{1,\nu}(\mathbb{R}_+^n)},t^QC_1\left(\frac{|y|}{t}\right)^\gamma\right\}\frac{dt}{t}\right)^{\frac12}\\
&\le C|y|^{\frac{-\varepsilon}{2}}\left(\int_0^{|y|}t^{-Q+\varepsilon}2t^Q\|\varphi\|_{L_{1,\nu}(\mathbb{R}_+^n)}\frac{dt}{t}+\int_{|y|}^\infty t^{-Q+\varepsilon}t^QC_1\left(\frac{|y|}{t}\right)^\gamma\frac{dt}{t}\right)^{\frac12}\\
&\le C|y|^{\frac{-\varepsilon}{2}}|y|^{\frac{\varepsilon}{2}}=C
\end{aligned}$$

and (4) also holds. From Theorem 2.3. in section 2, we have

$$\|Tf\|_{L_{p,\nu}\left(\mathbb{R}_+^n,H_2\right)}\le C\|f\|_{L_{p,\nu}\left(\mathbb{R}_+^n,H_1\right)}.$$

Definition 4.2 *For $\varphi-$ Littlewood- Paley function and $1 \leq p < \infty$, $f \in L_{p,\nu}(\mathbb{R}^n_+, H_1)$ put*

$$F(x,t) = (f \otimes \varphi_t)(x)$$

and let

$$g(F)(x) = \left(\int_{[0,\infty)} |F(x,t)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

denote the square function generated by Littlewood- Paley φ functions.

Corollary 4.1 *Let $\varphi-$ Littlewood- Paley function, $f \in L_{p,\nu}\left(\mathbb{R}^n_+, H_1\right)$ $(1 \leq p < \infty)$ and*

$$g(F)(x) = \left(\int_{[0,\infty)} |F(x,t)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

where $F(x,t) = (f \otimes \varphi_t)(x)$. Then $g(F)$ is bounded from $L_{p,\nu}\left(\mathbb{R}^n_+\right)$ into $L_{p,\nu}\left(\mathbb{R}^n_+, H_1\right)$.

Proof. The proof of corollary is obviously from Theorem 3.4. Since $F(x,t) = (f \otimes \varphi_t)(x)$ and $F(\cdot,\cdot) \in L_{p,\nu}\left(\mathbb{R}^n_+, H_2\right)$, $\|F(x,\cdot)\|_{H_2} = g(F)(x)$. So we have

$$\begin{aligned} \|g(F)\|_{L_{p,\nu}\left(\mathbb{R}^n_+\right)} &= \left(\int_{\mathbb{R}^n_+} |g(F)(x)|^p x_n^{2\nu} dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\mathbb{R}^n_+} \|F(x,\cdot)\|_{H_2}^p x_n^{2\nu} dx \right)^{\frac{1}{p}} \\ &= \|F(\cdot,\cdot)\|_{L_{p,\nu}\left(\mathbb{R}^n_+, H_2\right)} \leq c \|f\|_{L_{p,\nu}\left(\mathbb{R}^n_+, H_1\right)} . \end{aligned}$$

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