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An inversion theorem for Dunkl transform

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Abstract. *Using the same method as Alan L. Schwartz [11], we give an inversion formula for Dunkl transform associated with the Dunkl operator defined on* $\mathbb{R}$ *by* $\Lambda_\alpha u(x) = \frac{du(x)}{dx} + \frac{\alpha+\frac{1}{2}}{x}\big[u(x) - u(-x)\big]$, $\alpha > \frac{-1}{2}$.

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1 Introduction

For $\alpha > \frac{-1}{2}$, the differential-difference operators Λ_α is defined on $\mathbb{R}$ by

$$\Lambda_\alpha u(x) = \frac{du(x)}{dx} + \frac{\alpha + \frac{1}{2}}{x}\big[u(x) - u(-x)\big]. \tag{1.1}$$

This operator has been introduced by Dunkl and is called in the litterature Dunkl operator on $\mathbb{R}$ of index $\alpha + \frac{1}{2}$ associated with the reflection group $\mathbb{Z}_2$, see [2,3].

These operators are very important in mathematics and physics. They provide a useful tool in the study of special functions with root systems and they are closely related to certain representations of degenerate affine Hecke algebras [1,8], moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely Calogero-Sutherland-Moser models, which deal with systems of identical particles in a one dimensional space [7].

The search for inversion formula has been an active fields for many years. In particular in 1968 Alan L. Schwartz proved an inversion theorem for Hankel transform h_α defined on $(0, +\infty)$ by

$$h_\alpha(f)(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) d\mu_\alpha(t), \tag{1.2}$$

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where

$$d\mu_\alpha(t) = \frac{t^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} dt \tag{1.3}$$

and

$$j_\alpha(\lambda t) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(\lambda t)}{(\lambda t)^\alpha}, \tag{1.4}$$

where J_α is the Bessel function of the first kind of order α, see [13].

The inversion formula for integral transform plays a fundamental role in modern harmonic analysis, it is natural to exploit previous researches and results for studying inversion formula for the Fourier transform associated with Dunkl operator Λ_α defined on $\mathbb{R}$ by

$$\forall \lambda \in \mathbb{R}, \ \ \mathcal{F}_\alpha(f)(y) = \int_{\mathbb{R}} f(y) \psi_\lambda^\alpha(y) d\nu_\alpha(y), \tag{1.5}$$

where

$$d\nu_\alpha(y) = \frac{|y|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} dy. \tag{1.6}$$

and

$$\psi_\lambda^\alpha(y) = j_\alpha(\lambda y) + \frac{i\lambda y}{2(\alpha+1)} j_{\alpha+1}(\lambda y), \tag{1.7}$$

where j_α is the normalized Bessel function of index α given by relation (1.4).
This is the aim of this paper.

2 Preliminaries

In this section, we collect relevant material from the harmonic analysis associated with Λ_α we briefly cite some properties.
For more details we refer the reader to [4,5,9,10].
For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} \Lambda_\alpha u(x) = i\lambda u(x) \\ u(0) = 1, \end{cases} \tag{2.1}$$

has a unique solution on $\mathbb{R}$, denoted ψ_λ^α, given in the introduction by relation (1.7).
The function ψ_λ^α called Dunkl kernel, has a unique extension to $\mathbb{C} \times \mathbb{C}$ and satisfies the following properties,

$$\forall \lambda, \ z \in \mathbb{C}, \ \ \psi_\lambda^\alpha(z) = \psi_z^\alpha(\lambda)$$
$$\forall \lambda, \ z \in \mathbb{C}, \ \ \psi_{-\lambda}^\alpha(z) = \psi_\lambda^\alpha(-z).$$

For all $\lambda, \ z \in \mathbb{C}$

$$\psi_\lambda^\alpha(z) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} e^{-i\lambda z t} (1-t)^{\alpha-\frac{1}{2}} (1+t)^{\alpha+\frac{1}{2}} dt. \tag{2.2}$$

So, for all $n \in \mathbb{N}$ and $\lambda, \ x \in \mathbb{R}$ we have

$$\Big| \frac{d^n \psi_\lambda^\alpha(x)}{dx^n} \Big| \leq |\lambda|^n. \tag{2.3}$$

In particular, we have

$$\forall \lambda, \ x \in \mathbb{R}, \ \ \Big| \psi_\lambda^\alpha(x) \Big| \leq 1. \tag{2.4}$$

Definition 2.1 *The Dunkl transform $\mathcal{F}_\alpha$ is defined on $L^1_\alpha(\mathbb{R})$(the space of measurable function on $\mathbb{R}$ such that $\int_{\mathbb{R}} |f(x)|d\nu_\alpha(x)$ is finite) by*

$$\forall \lambda \in \mathbb{R}, \;\; \mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}} f(x)\psi^\alpha_\lambda(x)d\nu_\alpha(x), \tag{2.5}$$

where

$$d\nu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}dx. \tag{2.6}$$

Theorem 2.1 *Let f be in $L^1_\alpha(\mathbb{R})$ such that the function $\mathcal{F}_\alpha(f)$ belongs to $L^1_\alpha(\mathbb{R})$. Then we have the following inversion formula for the transform $\mathcal{F}_\alpha$:*

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\alpha(f)(\lambda)\psi^\alpha_{-\lambda}(x)d\nu_\alpha(\lambda), \quad a.e. \tag{2.7}$$

Theorem 2.2 *(i) Parseval formula for $\mathcal{F}_\alpha$: For all f, g in $\mathcal{D}(\mathbb{R})$(the space of C^∞-functions on $\mathbb{R}$, with compact support), we have*

$$\int_{\mathbb{R}} f(x)\overline{g(x)}d\nu_\alpha(x) = \int_{\mathbb{R}} \mathcal{F}_\alpha(f)(\lambda)\overline{\mathcal{F}_\alpha(g)(\lambda)}d\nu_\alpha(\lambda). \tag{2.8}$$

(ii) Plancherel formula for $\mathcal{F}_\alpha$: For all f in $\mathcal{D}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |f(x)|^2 d\nu_\alpha(x) = \int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(\lambda)|^2 d\nu_\alpha(\lambda). \tag{2.9}$$

(ii) Plancherel theorem for $\mathcal{F}_\alpha$: The Dunkl transform $\mathcal{F}_\alpha$ extends uniquely to an isometric isomorphism from $L^2_\alpha(\mathbb{R})$ onto itself.

Theorem 2.3 *[11] Suppose $f \in L^1_\alpha\big((0,\infty)\big)$ such that*

(i) f is of bounded variation in a neighborhood of $x > 0$.
(ii) $\int_0^1 |f(x)|x^{\alpha+\frac{1}{2}}dx < \infty$.

Then

$$\lim_{A\longrightarrow+\infty}\int_0^A \Big(\int_0^{+\infty} f(t)j_\alpha(\lambda t)d\mu_\alpha(t)\Big)j_\alpha(\lambda x)d\mu_\alpha(\lambda) = \frac{1}{2}\big[f(x+0)+f(x-0)\big]. \tag{2.10}$$

3 Inversion formula

In this section, we give an inversion formula for Dunkl transform associated with Dunkl operator Λ_α.

Theorem 3.1 *Suppose $f \in L^1_\alpha(\mathbb{R})$ such that*

(i) f is of bounded variation in a neighborhood of $x \neq 0$;
(ii) $\int_{-1}^1 |f(x)||x|^{\alpha+\frac{1}{2}}dx < +\infty$.

Then

$$\lim_{A\longrightarrow+\infty}\int_{-A}^A \mathcal{F}_\alpha(f)(\lambda)\psi^\alpha_{-\lambda}(x)d\nu_\alpha(\lambda) = \frac{1}{2}\big[f(x+0)+f(x-0)\big]. \tag{3.1}$$

Proof. Let $f = f_o + f_e$, where

$$f_e(x) = \frac{f(x) + f(-x)}{2} \tag{3.2}$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2}. \tag{3.3}$$

Thus,

$$\mathcal{F}_\alpha(f)(\lambda) = \mathcal{F}_\alpha(f_e)(\lambda) + \mathcal{F}_\alpha(f_o)(\lambda). \tag{3.4}$$

$$\begin{aligned}
\mathcal{F}_\alpha(f_e)(\lambda) &= \int_{\mathbb{R}} f_e(t)\psi_\lambda^\alpha(t)d\nu_\alpha(t) \\
&= \int_{\mathbb{R}} f_e(t)\Big(j_\alpha(\lambda t) + \frac{i\lambda t}{2(\alpha+1)} j_{\alpha+1}(\lambda t)\Big)d\nu_\alpha(t) \\
&= \int_{\mathbb{R}} f_e(t) j_\alpha(\lambda t)d\nu_\alpha(t) + \frac{i\lambda}{2(\alpha+1)}\int_{\mathbb{R}} t f_e(t) j_{\alpha+1}(\lambda t)d\nu_\alpha(t) \\
&= h_\alpha(f_e)(\lambda).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathcal{F}_\alpha(f_o)(\lambda) &= \int_{\mathbb{R}} f_o(t)\Big(j_\alpha(\lambda t) + \frac{i\lambda}{2(\alpha+1)} j_{\alpha+1}(\lambda t)\Big)d\nu_\alpha(t) \\
&= i\lambda h_{\alpha+1}\Big(\frac{f_o(t)}{t}\Big)(\lambda).
\end{aligned}$$

Therefore

$$\mathcal{F}_\alpha(f)(\lambda) = h_\alpha(f_e)(\lambda) + i\lambda h_{\alpha+1}\Big(\frac{f_o(t)}{t}\Big)(\lambda) \tag{3.5}$$

Now, Let $A > 0$

$$\begin{aligned}
&\int_{-A}^{A} \mathcal{F}_\alpha(f)(\lambda)\psi_{-\lambda}^\alpha(x)d\nu_\alpha(\lambda) \\
&= \int_{-A}^{A} \Big(h_\alpha(f_e)(\lambda) + i\lambda h_{\alpha+1}\Big(\frac{f_o(t)}{t}\Big)(\lambda)\Big)\Big(j_\alpha(\lambda x) - \frac{i\lambda x}{2(\alpha+1)} j_{\alpha+1}(\lambda x)\Big)d\nu_\alpha(\lambda) \\
&= \int_{-A}^{A} h_\alpha(f_e)(\lambda) j_\alpha(\lambda x)d\nu_\alpha(\lambda) + i\int_{-A}^{A} \lambda h_{\alpha+1}\Big(\frac{f_o(t)}{t}\Big)(\lambda) j_\alpha(\lambda x)d\nu_\alpha(\lambda) \\
&+ x\int_{-A}^{A} \frac{\lambda^2}{2(\alpha+1)} h_{\alpha+1}\Big(\frac{f_o(t)}{t}\Big)(\lambda) j_{\alpha+1}(\lambda x)d\nu_\alpha(\lambda) \\
&- \frac{ix}{2(\alpha+1)}\int_{-A}^{A} \lambda h_\alpha(f_e)(\lambda) j_{\alpha+1}(\lambda x)d\nu_\alpha(\lambda).
\end{aligned}$$

From the fact that the functions

$$\lambda \longrightarrow \lambda h_\alpha(f_e)(\lambda),$$

$$\lambda \longrightarrow \lambda h_{\alpha+1}\Big(\frac{f_o(t)}{t}\Big)(\lambda)$$

are odd functions we get

$$\int_{-A}^{A} \lambda h_{\alpha+1}\Big(\frac{f_o(t)}{t}\Big)(\lambda) j_\alpha(\lambda x) d\nu_\alpha(\lambda) = 0 = \int_{-A}^{A} \lambda h_\alpha(f_e)(\lambda) j_{\alpha+1}(\lambda x) d\nu_\alpha(\lambda). \quad (3.6)$$

It follows, that

$$\begin{aligned}
&\int_{-A}^{A} \mathcal{F}_\alpha(f)(\lambda)\psi_{-\lambda}^{\alpha}(x) d\nu_\alpha(\lambda) \\
&= \int_{-A}^{A} h_\alpha(f_e)(\lambda) j_\alpha(\lambda x) d\nu_\alpha(\lambda) \\
&+ \frac{x}{2(\alpha+1)} \int_{-A}^{A} \lambda^2 h_{\alpha+1}\Big(\frac{f_o(t)}{t}\Big)(\lambda) j_{\alpha+1}(\lambda x) d\nu_\alpha(\lambda).
\end{aligned}$$

Hence, since

$$\begin{aligned}
\int_0^1 |f_e(x)| x^{\alpha+\frac{1}{2}} dx &= \frac{1}{2}\int_0^1 |f(x)+f(-x)| x^{\alpha+\frac{1}{2}} dx \\
&\le \frac{1}{2}\int_0^1 |f(x)||x|^{\alpha+\frac{1}{2}} dx + \frac{1}{2}\int_0^1 |f(-x)||x|^{\alpha+\frac{1}{2}} dx \\
&\le \frac{1}{2}\int_0^1 |f(x)||x|^{\alpha+\frac{1}{2}} dx + \frac{1}{2}\int_{-1}^0 |f(x)||x|^{\alpha+\frac{1}{2}} dx \\
&\le \frac{1}{2}\int_{-1}^1 |f(x)||x|^{\alpha+\frac{1}{2}} dx < +\infty.
\end{aligned}$$

Then by Theorem 2.3, we have

$$\begin{aligned}
\lim_{A\longrightarrow+\infty} \int_{-A}^{A} h_\alpha(f_e)(\lambda) j_\alpha(\lambda x) d\nu_\alpha(\lambda) &= 2 \lim_{A\longrightarrow+\infty} \int_0^A h_\alpha(f_e)(\lambda) j_\alpha(\lambda x) \frac{|\lambda|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} d\lambda \\
&= \lim_{A\longrightarrow+\infty} \int_0^A h_\alpha(f_e)(\lambda) j_\alpha(\lambda x) d\mu_\alpha(\lambda) \\
&= \frac{1}{2}\big[f_e(x+0)+f_e(x-0)\big].
\end{aligned} \quad (3.7)$$

On the other hand,

$$\begin{aligned}
&\lim_{A\longrightarrow+\infty} \frac{x}{2(\alpha+1)} \int_{-A}^{A} \lambda^2 h_{\alpha+1}\Big(\frac{f_o(t)}{t}\Big)(\lambda) j_{\alpha+1}(\lambda x) d\nu_\alpha(\lambda) \\
&= \lim_{A\longrightarrow+\infty} \frac{x}{2^{\alpha+2}(\alpha+1)\Gamma(\alpha+1)} \int_{-A}^{A} h_{\alpha+1}\Big(\frac{f_o(t)}{t}\Big)(\lambda) j_{\alpha+1}(\lambda x)|\lambda|^{2(\alpha+1)+1} d\lambda \\
&= \lim_{A\longrightarrow+\infty} \frac{x}{2^{\alpha+2}\Gamma(\alpha+2)} \int_{-A}^{A} h_{\alpha+1}(g)(\lambda) j_{\alpha+1}(\lambda x)|\lambda|^{2(\alpha+1)+1} d\lambda \\
&= x \lim_{A\longrightarrow+\infty} \int_0^A h_{\alpha+1}(g)(\lambda) j_{\alpha+1}(\lambda x) d\mu_{\alpha+1}(\lambda),
\end{aligned}$$

where $g(t) = \frac{f_o(t)}{t}$.
From the fact that

$$\int_0^1 |g(t)|^{\alpha+\frac{3}{2}} dt = \int_0^1 |f_o(t)| t^{\alpha+\frac{1}{2}} dt \leq \frac{1}{2} \int_{-1}^1 |f(t)| t^{\alpha+\frac{1}{2}} dt < \infty,$$

we get

$$\begin{aligned} x \lim_{A \longrightarrow +\infty} \int_0^A h_{\alpha+1}(g)(\lambda) j_{\alpha+1}(\lambda x) d\mu_{\alpha+1}(\lambda) \\ = \frac{1}{2} x \big[g(x+0) + g(x-0) \big] = \frac{1}{2} \big[f_o(x+0) + f_o(x-0) \big]. \end{aligned} \tag{3.8}$$

Therefore relations (3.7) and (3.8) allow us to get

$$\lim_{A \longrightarrow +\infty} \int_{-A}^A \mathcal{F}_\alpha(f)(\lambda) \psi_{-\lambda}^\alpha(x) d\nu_\alpha(\lambda) = \frac{1}{2} \big[f(x+0) + f(x-0) \big]. \tag{3.9}$$

In the following we will prove that the exponent $\alpha + \frac{1}{2}$ cannot be increased. Indeed suppose that the result in Theorem 3.1 holds when we have

$$\int_{-1}^1 |f(t)| |t|^{\alpha+\epsilon+\frac{1}{2}} dt < \infty, \;\; \epsilon > 0.$$

Let

$$f(t) = \begin{cases} |t|^{-(\alpha+\frac{3}{2})}, & |t| < 1, \;\; t \neq 0 \\ 0, & |t| > 1. \end{cases} \tag{3.10}$$

Thus,

(i) it is clear that f is of bounded variation.
(ii) $\int_{-1}^1 |f(t)| |t|^{\alpha+\epsilon+\frac{1}{2}} dt = \int_{-1}^1 \frac{dt}{t^{1-\epsilon}} < \infty$, for $\epsilon > 0$.

Therefore, Theorem 3.1 implies that

$$\begin{aligned} \lim_{A \longrightarrow +\infty} I_A(x) = \lim_{A \longrightarrow +\infty} \int_{-A}^A \mathcal{F}_\alpha(f)(\lambda) \psi_{-\lambda}^\alpha(x) d\nu_\alpha(\lambda) \\ = \frac{1}{2} \big[f(x+0) + f(x-0) \big] = 0, \;\; x > 1. \end{aligned} \tag{3.11}$$

On the other hand, let $z_1, z_2, ...$ be the positive real zero of $J_\alpha(x)$ in ascending order and $a_n = \frac{z_n}{x}$, $x > 1$, such that $\lim_{n\longrightarrow+\infty} a_n = +\infty$. In addition,

$$\begin{aligned}
\lim_{n\longrightarrow+\infty} I_{a_n}(x) &= \lim_{n\longrightarrow+\infty} \int_{-a_n}^{a_n} \mathcal{F}_\alpha(f)(\lambda)\psi^\alpha_{-\lambda}(x)d\nu_\alpha(\lambda) \\
&= \lim_{n\longrightarrow+\infty} \int_{-a_n}^{a_n} h_\alpha(f)(\lambda)\psi^\alpha_{-\lambda}(x)d\nu_\alpha(\lambda) \\
&= \lim_{n\longrightarrow+\infty} \int_{-a_n}^{a_n} h_\alpha(f)(\lambda)\Big(j_\alpha(\lambda x) - \frac{i\lambda x}{2(\alpha+1)} j_{\alpha+1}(\lambda x)\Big)d\nu_\alpha(\lambda) \\
&= \lim_{n\longrightarrow+\infty} \int_{-a_n}^{a_n} h_\alpha(f)(\lambda)j_\alpha(\lambda x)d\nu_\alpha(\lambda) \\
&= \lim_{n\longrightarrow+\infty} \int_{0}^{a_n} h_\alpha(f)(\lambda)j_\alpha(\lambda x)d\mu_\alpha(\lambda) \\
&= \lim_{n\longrightarrow+\infty} \int_{0}^{a_n} \Big(\int_0^1 f(t)j_\alpha(\lambda t)d\mu_\alpha(t)\Big)j_\alpha(\lambda x)d\mu_\alpha(\lambda) \\
&= \lim_{n\longrightarrow+\infty} \int_{0}^{a_n} \Big(\int_0^1 \frac{1}{t^{\alpha+\frac{3}{2}}} j_\alpha(\lambda t)\frac{t^{2\alpha+1}dt}{2^\alpha\Gamma(\alpha+1)}\Big)j_\alpha(\lambda x)\frac{\lambda^{2\alpha+1}d\lambda}{2^\alpha\Gamma(\alpha+1)}.
\end{aligned}$$

By Fubini theorem, we get

$$\lim_{n\longrightarrow+\infty} I_{a_n}(x) = x^{-\alpha} \lim_{n\longrightarrow+\infty} \int_0^1 \Big(\int_0^{a_n} J_\alpha(\lambda t)J_\alpha(\lambda x)\lambda d\lambda\Big)\frac{dt}{\sqrt{t}}. \tag{3.12}$$

But,

$$\begin{aligned}
\int_0^{a_n} J_\alpha(\lambda t)J_\alpha(\lambda x)\lambda d\lambda &= a_n(x^2-t^2)^{-1}\big(xJ_{\alpha+1}(a_n x)J_\alpha(a_n t) - tJ_{\alpha+1}(a_n t)J_\alpha(a_n x)\big) \\
&= a_n(x^2-t^2)^{-1}\big(xJ_{\alpha+1}(a_n x)J_\alpha(a_n t)\big) \\
&= z_n(x^2-t^2)^{-1}J_{\alpha+1}(z_n)J_\alpha(a_n t).
\end{aligned}$$

Therefore

$$\lim_{n\longrightarrow+\infty} \int_{-a_n}^{a_n} h_\alpha(f)(\lambda)\psi^\alpha_{-\lambda}(x)d\nu_\alpha(\lambda) = \lim_{n\longrightarrow+\infty} a_n x^{1-\alpha} J_{\alpha+1}(z_n) \int_0^1 \frac{J_\alpha(a_n t)}{(x^2-t^2)}\frac{dt}{\sqrt{t}}.$$

By change of variable $u = a_n t$, we get

$$\begin{aligned}
\int_0^1 \frac{J_\alpha(a_n t)}{(x^2-t^2)} a_n \frac{dt}{\sqrt{t}} &= \int_0^{a_n} \frac{J_\alpha(u)}{\Big(x^2 - \big(\frac{u}{a_n}\big)^2\Big)\big(\frac{u}{a_n}\big)^{\frac{1}{2}}}du \\
&= a_n^{\frac{1}{2}} \int_0^{a_n} \frac{J_\alpha(u)}{\Big(x^2 - \big(\frac{u}{a_n}\big)^2\Big)u^{\frac{1}{2}}}du \\
&= a_n^{\frac{1}{2}} x^{-2} \int_0^{a_n} \frac{J_\alpha(u)}{\Big(1 - \big(\frac{u}{z_n}\big)^2\Big)u^{\frac{1}{2}}}du.
\end{aligned}$$

It can be easily shown that

$$\lim_{n\longrightarrow+\infty} x^{-2} \int_0^{a_n} \frac{J_\alpha(u)}{\Big(1 - \big(\frac{u}{z_n}\big)^2\Big)u^{\frac{1}{2}}}du = x^{-2} \int_0^{+\infty} J_\alpha(u)u^{\frac{-1}{2}}du. \tag{3.13}$$

Therefore

$$\lim_{n\longrightarrow+\infty}\int_{-a_n}^{a_n} h_\alpha(f)(\lambda)\psi^\alpha_{-\lambda}(x)d\nu_\alpha(\lambda)$$

$$=\lim_{n\longrightarrow+\infty} a_n^{\frac{1}{2}}x^{1-\alpha}J_{\alpha+1}(z_n)x^{-2}\int_0^{+\infty} J_\alpha(u)u^{\frac{-1}{2}}du.$$

This last expression (3.13) has the value

$$\Gamma\big((2\alpha+1)/4\big)/(2^{\frac{1}{2}}\Gamma\big((2\alpha+3)/4\big)).$$

For more details of the previous result see [6].
From [12], it follows that

$$\begin{aligned}J_{\alpha+1}(x) &= (2/\pi x)^{\frac{1}{2}}\cos(x-\beta-(\pi/2))+O(x^{\frac{-3}{2}})\\ &= (2/\pi x)^{\frac{1}{2}}\sin(x-\beta)+O(x^{\frac{-3}{2}}),\end{aligned}$$

where $\beta=(2\alpha+1)\pi/4$ and

$$J_\alpha(x)=(2/\pi x)^{\frac{1}{2}}\cos(x-\beta)+O(x^{\frac{-3}{2}}).$$

Since $J_\alpha(z_n)=0$, we see that

$$|J_{\alpha+1}(z_n)|\geq(\pi z_n)^{\frac{-1}{2}}$$

for n sufficiently large. Thus for some constant C we have

$$|I_{a_n}(\lambda)|\geq Cx^{\frac{1}{2}-\alpha},\ \ for\ all\ \ n\in\mathbb{N}.$$

Then

$$\lim_{n\longrightarrow+\infty}|I_{a_n}(\lambda)|=0\geq Cx^{\frac{1}{2}-\alpha}.$$

Thus, it follows that the exponent $\alpha+\frac{1}{2}$ cannot be increased, for which the theorem holds.

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