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On influence of modulus of continuity of the coefficient $P_2(x)$ on uniform equiconvergence rate for an even order differential operator

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Abstract. *In the paper, an even order ordinary differential operator is considered on the interval $G = (0,1)$. Uniform equiconvergence on a compact of a spectral expansion in root functions of the given operator with trigonometric expansion is studied. Dependence of uniform equiconvergence rate on the modulus of continuity of the coefficient at $(n-2)-$th derivative in the given operators, is revealed.*

Keywords. spectral expansion, root functions, uniform equiconvergence rate.

Mathematics Subject Classification (2010): 34L10, 42A20

1 Introduction and formulation of results.

In the paper we study dependence of uniform equiconvergence on the modulus of continuity of the coefficient $P_2(x)$ of an even order ordinary differential operator. The research is conducted by V.A. Il'ins method [5], [6].

Uniform equiconvergence on a compact was comprehensively studied in the papers [2], [4], [7], [8] for Schrodinger operator with a summable potential. The influence of summability degree of coefficients of a differential operator on equiconvergence rate was studied in the papers [9], [11].

Note that estimates of uniform equiconvergence in terms of integral module of continuity of expanded functions were set up in the papers [3], [9].

Dependence of uniform equiconvergence rate on module of continuity of the potential of Schrodinger's one-dimensional operator was studied in the papers [7], [8]. In this paper we extend these results to the case of even order arbitrary differential operator.

On the interval $G = (0,1)$ we consider a formal differential operator

$$Lu = u^{(2m)} + P_2(x)u^{(2m-2)} + ... + P_{2m}(x)u$$

with complexvalued coefficients $P_\ell(x) \in L_1(G)$, $\ell = \overline{2, 2m}$.

Denote by $D(G)$ a class of functions absolutely continuous with their derivatives to $(2m-1)$-th order inclusively, on $\bar{G}$ $(D(G) = W_1^{2m}G))$.

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Under the eigenfunction of the operator L responding to the complex eigenvalue λ, we will understand any complex-valued function $\overset{0}{y}(x) \in D$ not identically equal to zero and satisfying almost everywhere in G the equation $L\overset{0}{y} + \lambda\overset{0}{y} = 0$. Similarly, under the associated function of this operator of order $m \quad (m \geq 1)$ responding to the same eigenvalue λ and eigenfunction $\overset{0}{y}(x)$ we will understand any complex-valued function $\overset{m}{y}(x) \in D(G)$, satisfying almost everywhere in G the equation $L\overset{m}{y} + \lambda\overset{m}{y} = \overset{m-1}{y}$ (see [5], [6]).

We will consider every eigenfunction an associated function of order 0. The highest order of root (associated) functions responding to the given eigenfunction will be said the rank of this eigenfunction.

Let us consider an arbitrary system $\{u_k(x)\}_{k=1}^{\infty}$ consisting of the root functions of the operator L, responding to the system of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ and require that together with each root function of order $\ell \geq 1$, this system includes appropriate root functions of order less than ℓ and the rank of eigenfunction be uniformly bounded. This means that $u_k \in D(G)$ and satisfies almost everywhere in G the equation $Lu_k + \lambda_k u_k = \theta_k u_{k-1}$, where θ_k equals either 0 (in this case $u_k(x)$ is an eigenfunction) or 1 (in this case we require $\lambda_k = \lambda_{k-1}$ and call $u_k(x)$an associated function) [4].

Denote $\mu_k = [(-1)^m(-\lambda_k)]^{\frac{1}{2m}}$, where $\left[re^{i\varphi}\right]^{\frac{1}{2m}} = r^{\frac{1}{2m}} e^{\frac{i\varphi}{2m}}, \quad -\pi < \varphi \leq \pi$.

Let the system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy conditions A_p (V.A. Il'in conditions):

1) the system $\{u_k(x)\}_{k=1}^{\infty}$ is closed and minimal in $L_p(G)$ for fixed $p \geq 1$;

2) the Carleman and the "sum of units" conditions

$$|Im\mu_k| \leq const, \ k = 1, 2, ...; \quad \sum_{\tau \leq Re\mu_k \leq \tau+1} 1 \leq const\tau, \ \forall\tau \geq 0$$

are fulfilled;

3) for any compact $K \subset G$ there exists a constant $C_0(K)$ such that

$$\|u_k\|_{p,K} \|\vartheta_k\|_q \leq C_0(K),$$

where $p^{-1} + q^{-1} = 1, \quad (p = 1, \ q = \infty)$, $\{\vartheta_k(x)\}_{k=1}^{\infty}$ is a system biorthogonally adjoint to the system $\{u_k(x)\}_{k=1}^{\infty}$, $\|\cdot\|_{p,K} = \|\cdot\|_{L_p(K)}$, $\|\cdot\|_q = \|\cdot\|_{L_q(G)}$.

Denote by $S_v(x, f)$ partial sum of trigonometric series of the function $f(x) \in L_p(G)$ and introduce the partial sum of biorthogonal expansion of the function $f(x)$ in the system $\{u_k(x)\}_{k=1}^{\infty}$:

$$\sigma_v(x, f) = \sum_{\rho_k \leq v} f_k u_k(x), \quad \rho_k = Re\mu_k, \ v > 0,$$

where $f_k = \int_G f(x)\overline{\vartheta_k(x)}dx$.

Introduce the following denotation:

$$\Delta_v(K, f) = \|\sigma_v(\cdot, f) - S_v(\cdot, f)\|_{C(K)};$$

$$\Omega\left(f, \ \frac{v}{2}, \alpha\right) = v^{-1} \sum_{1 \leq \rho_k \leq \frac{v}{2}} \left|\hat{f}_k\right| \rho_k^{-\alpha}, \hat{f}_k = f_k \|\vartheta_k\|_q^{-1}, \quad 0 \leq \alpha \leq 1;$$

$$\Phi_p(f, v) = v^{-1}\|f\|_p + \max_{\rho_k \geq \frac{v}{2}} \left|\hat{f}_k\right| ; Q_p(f, v) = v^{-1}\|f\|_p + \max_{2\pi k \geq \frac{v}{2}} \left|\tilde{f}_k\right|,$$

where $\tilde{f}_k$ are the Fourier coefficients of the function $f(x)$ in the trigonometric system normalized in $L_q(G)$;

$$D(v) = \inf_{\substack{\alpha>1\\ n\geq 2}} \left\{ \Omega\left(f, \frac{v}{2}, 0\right) \omega_1(P_2,\, n^{-1}) + n^{2(1-\alpha^{-1})} \left\|P_2\right\|_1 \cdot \Omega\left(f, \frac{v}{2},\, 1-\alpha^{-1}\right) \right\}$$

$$+ \max_{\ell=\overline{3,2m}} \left\|P_\ell\right\|_1 \Omega\left(f, \frac{v}{2}, 1\right),$$

where $\omega_1(f,\, \delta)$ is the modulus of continuity of the function $f(x)$ in $L_1(G)$;

$$\mathrm{T}\left(f, \ell, \beta\right) = \sum_{i=1}^{\ell} i^{-\beta} \omega_1\left(f, i^{-1}\right), \quad \beta \geq 0;$$

$$N_1\left(f, \left[\frac{v}{2}\right], 1\right) = T\left(f, \left[\frac{v}{2}\right],\ 1\right) + \|f\|_1$$

$$N_p\left(f, \left[\frac{v}{2}\right],\ 1-s^{-1}\right) = T\left(f, \left[\frac{v}{2}\right],\ 1-s^{-1}\right) + \frac{s}{s-1}\left\|f\right\|_p, s>1;$$

$$\varphi_p\left(f, v\right) = \omega_1\left(f, v^{-1}\right) + v^{-1}\left\|f\right\|_p;$$

$$E\left(P_2, v\right) = \inf_{\substack{\alpha>1\\ n\geq 2}} \left\{ \ \omega_1(P_2, n^{-1}\left(T\left(f, \left[\frac{v}{2}\right], 0\right) + \ln v \left\|f\right\|_1\right)\right.$$

$$\left. + \left\|P_2\right\|_1 n^{2(1-\alpha^{-1})} \left(T\left(f, \left[\frac{v}{2}\right],\ 1-\alpha^{-1}\right) + \frac{\alpha}{\alpha-1}\left\|f\right\|_1\right)\right\} \ ,$$

where $\left[\frac{v}{2}\right]$ is the entire part of the number $\frac{v}{2}$.

Let $\omega\left(t\right)$ be a non-decreasing continuous function on $[0,\infty)$ and satisfy the conditions a) $\omega\left(0\right)=0$, $\omega(t)>0$ for $t>0$; b) $t^{-1}\omega\left(t\right)$ does not increase.

Denote by $H_p^{\omega}(G),\ p\geq 1$ the set of functions from $L_p(G)$ satisfying the condition $\omega_p\left(f, \delta\right) \leq C(f)\omega\left(\delta\right)$, where $C(f)$ is a constant dependent on $f(x)$. The norm in $H_p^{\omega}(G)$ is defined by the equality

$$\|f\|_p^{\omega} = \|f\|_p + \sup_{\delta>0} \frac{\omega_1(f,\delta)}{\omega(\delta)}.$$

Denote by $B_{p,\theta}^{\alpha}(G),\ \ 0<\alpha<1,\ \ 1\leq\theta\leq\infty$ the Besov class with the norm

$$\|f\|_{B_{p,\theta}^{\alpha}(G)} = \|f\|_p + \left(\int_0^{h_0} \left(t^{-\alpha-\frac{1}{\theta}}\omega_p(f,t)\right)^{\theta} dt\right)^{\frac{1}{\theta}},\ \ h_0>0.$$

Note that $B_{p,\infty}^{\alpha}(G) \equiv H_p^{\alpha}(G);\quad H_p^{\omega}(G) = H_p^{\alpha}(G)$ for $\omega\left(t\right)=t^{\alpha}$, $0<\alpha<1$ ($H_p^{\alpha}(G)$ is the Nikolskii class).

Theorem 1.1 *Let $P_2(x) \in L_r(G)$, $r \geq 1$, $P_\ell(x) \in L_1(G)$, $\ell = \overline{3, 2m}$ and the system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy conditions A_p. Then expansion of an arbitrary function $f(x) \in L_p(G)$ in biorthogonal series in the system $\{u_k(x)\}_{k=1}^{\infty}$ and in trigonometric Fourier series uniformly equiconverge on any compact $K \subset G$, i.e.*

$$\lim_{v \to +\infty} \Delta_v(K, f) = 0 \tag{1.1}$$

and the estimations

$$\Delta_v(K, f) \leq C_1(K) \left\{ \|P_2\|_r \, \Omega\left(f, \frac{v}{2}, \ 1 - r^{-1}\right) + \max_{\ell = \overline{3,2m}} \{\|P_\ell\|_1\} \right.$$

$$\left. \times \, \Omega\left(f, \frac{v}{2}, \ 1\right) + \Phi_p(f, \nu) \ + Q_p(f, v) \right\} \tag{1.2}$$

are valid for $r > 1$;

$$\Delta_v(K, f) \leq C_2(K) \{D(v) + \Phi_p(f, v) + Q_p(f, v)\} \tag{1.3}$$

for $r = 1$, where $C_1(K)$, $C_2(K)$ are positive constants independent of v and $f(x)$.

Theorem 1.2 *Let the conditions of theorem 1.1 be fulfilled for $p = 1$ and for the coefficients $\overset{\wedge}{f_k}$ of the function $f(x) \in L_1(G)$ the estimation*

$$\left| \overset{\wedge}{f_k} \right| \leq const \left\{ \omega_1(f, \rho_k^{-1}) + \rho_k^{-1} \|f\|_1 \right\}, \quad \rho_k \geq 1 \tag{1.4}$$

be fulfilled.

Then for $r > 1$ the following estimation

$$\Delta_v(K, f) \leq C_3(K) v^{-1} \left\{ \|P_2\|_r \, N_1\left(f, \left[\frac{v}{2}\right], \ 1 - r^{-1}\right) + \max_{\ell = \overline{3,2m}} \{\|P_\ell\|\}_1 \right.$$

$$\left. \times N_1\left(f, \left[\frac{v}{2}\right], \ 1\right) + v\varphi_1(f, v) \right\}, \tag{1.5}$$

while for $r = 1$ the estimation

$$\Delta_v(K, f)$$

$$\leq C_4(K) v^{-1} \left\{ E(P_2, v) + \max_{\ell = \overline{3,2m}} \{\|P_\ell\|\}_1 N_1\left(f, \left[\frac{v}{2}\right], \ 1\right) + v\varphi_1(f, v) \right\}, \tag{1.6}$$

are valid, where the constants $C_3(K)$, $C_4(K)$ are independent of v and $f(x)$.

The following Corollary follows from theorem 1.2.

Corollary 1.1 *Under the conditions of theorem 1.2 the estimations*

$$\Delta_v(K, f) \leq C_5(K) v^{-\alpha} \|f\|_{B^{\alpha}_{p,\theta}(G)} \quad for \ r = 1, \ \ f \in B^{\alpha}_{p,\theta}(G); \tag{1.7}$$

$$\Delta_v(K, f) \leq C_6(K) \omega\left(v^{-1}\right) \begin{cases} \|f\|_p^{\omega} & for \ r > 1 \\ (1 + R(v)) \|f\|_p^{\omega} & for \ r = 1, \end{cases} \tag{1.8}$$

if $f \in H_p^{\omega}(G)$, where $R(v) = \inf_{n \geq 2} \left\{ \omega_1(P_2, n^{-1}) \ln v + \|P_2\|_1 \ln n \right\} + \max_{\ell = \overline{3,2m}} \{\|P_\ell\|\}_1$ are valid .

Corollary 1.2 *Let $r = 1$ and the conditions of theorem 1.2 be fulfilled. Then for any function $f(x) \in H_p^\omega(G)$ the following estimation is fulfilled*

$$\Delta_v(K,f) = O\left(\omega\left(v^{-1}\right)\ln v\right), \quad v \to \infty, \tag{1.9}$$

if in addition we require $\omega_1(P_2,\delta) = O(\ln^{-\gamma}\delta^{-1})$, $\delta \to +0$, $\gamma > 0$, then as $v \to +\infty$ the following estimation is fulfilled

$$\Delta_v(K,f) = O\left(\omega\left(v^{-1}\right)\ln^{\frac{1}{1+\gamma}} v\right) B\left(\gamma\right), \tag{1.10}$$

where $B(\gamma) = 2^\gamma \gamma^{-\frac{\gamma}{\gamma+1}} + 2\gamma^{\frac{1}{1+\gamma}}$, the symbol "O" is dependent on the function f. In particular, for $\gamma = 1$, $p = 1$, $f(x) \in W_1^1(G)$ the estimation

$$\Delta_v(K,f) = O\left(v^{-1}\ln^{\frac{1}{2}} v\right), \quad v \to +\infty \tag{1.11}$$

is valid.

2 Proof of results

Proof of theorem 1.1. Note that estimations (1.1) and (1.2) were set up for an arbitrary order differential operator (see [9]). We prove only estimation (1.3). Fix an arbitrary section $K = [a,b] \subset G$ and consider the function (see [5], [6])

$$W(t,v,R) = \begin{cases} \frac{\sin vt}{\pi t} & for\ t \le R \\ 0 & for\ \ t > R, \end{cases}$$

where $v > 0$, $t = |x-y|$, $y \in G$, $x \in K$, $R \in \left[\frac{R_0}{2}, R_0\right]$, $R_0 > 0$, $dist(K,\partial G) > 4C_0 R_0$, C_0 is a constant from the mean-value theorem (50) of the paper [10].
Denote by $\hat{W}(r,v,R_0)$ the averaging of the function $W(t,v,R)$ with respect to the argument R: $\hat{W}(r,v,R_0) = S_{R_0}[W]$, where $S_{R_0}[g] = \frac{2}{R_0}\int_{\frac{R_0}{2}}^{R_0} g(R)dR.$

Then the coefficients of the Fourier expansion of the function $\hat{W}(r,v,R_0)$ in the system $\left\{\overline{\vartheta_k(y)}\right\}$ are calculated by the formula

$$\hat{W}_k = \hat{W}_k(x,v,R_0) = \frac{2}{\pi}S_{R_0}\left[\int_0^R \frac{\sin vt}{t}\frac{u_k(x-t)+u_k(x+t)}{2}dt\right].$$

Let us consider the difference $\sum_{k=1}^\infty \hat{W}_k f_k - \sigma_v(x,f)$, $f \in L_p(G)$.
From the paper [9] it is known that this difference is representable in the form

$$\sum_{k=1}^\infty \hat{W}_k f_k - \sigma_v(x,f) = \sum_{j=1}^{14}\gamma_j(x),$$

and for the addends $\gamma_j(x)$, $j = \overline{1,7}$; $j = \overline{12,14}$ the estimation

$$\|\gamma_j\|_{C(K)} \le C_7(K)\left\{v^{-1}\|f\|_p + \max_{\rho_k \ge \frac{v}{2}}\left|\hat{f}_k\right|\right\} = C_7(K)\Phi_p(f,v), \tag{2.1}$$

for $\gamma_j(x)$, $j=\overline{8,\,11}$ the estimation

$$\|\gamma_j\|_{C(K)} \le C_8(K)\Big\{\, \|P_2\|_r\,\Omega\left(f,\ \frac{v}{2},\ 1-r^{-1}\right) + \max_{3\le\ell\le 2m}\|P_\ell\|_1\,\Omega\left(f,\ \frac{v}{2},\ 1\right) + \max_{\rho_k\ge\frac{v}{2}}\big|\,\overset{\wedge}{f_k}\,\big|\Big\} \tag{2.2}$$

are fulfilled.

Let $r=1$. In this case we estimate the sum $\gamma_j(x)$, $j=\overline{8,\,11}$ in a different way. For this we represent $\gamma_8(x)$ in the form (see [9]):

$$\gamma_8(x)=\sum_{\rho_k\ge\rho_0} f_k\sum_{j=0}^{m_k}\mu_k^{-(2m-1)(j+1)}S_{R_0}\left[\int\limits_x^{x+R}P_2\left(\xi\right)u_{k-j}^{(2m-2)}\left(\xi\right)J_{1j}\left(\xi-x,\,R,\,\mu_k,v\right)d\xi\right]$$

$$+\sum_{\rho_k\ge\rho_0} f_k\sum_{j=0}^{m_k}\mu_k^{-(2m-1)(j+1)}\sum_{\ell=3}^{2m}S_{R_0}\left[\int\limits_x^{x+R}P_\ell\left(\xi\right)u_{k-j}^{(2m-\ell)}\left(\xi\right)J_{1j}\left(\xi-x,\,R,\,\mu_k,v\right)d\xi\right]$$

$$=\gamma_8^1(x)+\gamma_8^2(x)$$

where m_k is the order of the associated function $u_k(x)$.
For the sum $\gamma_8^2(x)$ the estimation (see [9]).

$$\left\|\gamma_8^2\right\|_{C(K)}\le C_9(K)\max_{3\le\ell\le 2m}\|P_\ell\|_1\left\{\Omega\left(f,\ \frac{v}{2},\ 1\right)+\max_{\rho_k\ge\frac{v}{2}}\left|\overset{\wedge}{f_k}\right|\right\} \tag{2.3}$$

is fulfilled.
For estimating the sum $\gamma_8^1(x)$ represent it in the form

$$\gamma_8^1(x)\equiv\gamma_8^1(x,\,P_2)=\gamma_8^1(x,\,P_2-Q_n)+\gamma_8^1(x,\,Q_n),$$

where $Q_n(x)$ is the algebraic polynomial of the best approximation of the function $P_2(x)$ in the metrics $L_1(G)$ of degree n.

Taking into account that $\gamma_8^1(x)$ is independent of the coefficients $P_\ell(x)$, $\ell=\overline{3,2m}$, we apply the estimation (2.2) at $r=1$ for the sum $\gamma_8^1(x,\,P_2-Q_n)$. As a result we get

$$\left\|\gamma_8^1(\cdot\ ,\ P_2-Q_n)\right\|_{C(K)}\le C_{10}(K)\left\{\|P_2-Q_n\|_1\,\Omega\left(f,\frac{v}{2},0\right)+\max_{\rho_k\ge\frac{v}{2}}\left|\overset{\wedge}{f_k}\right|\right\}. \tag{2.4}$$

As $Q_n(x)$ is a polynomial, it belongs to $L_\alpha(G)$ for any $\alpha>1$. Therefore, for the sum $\gamma_8^1(x,\,Q_n)$ we can use estimation (2.2) for $r=\alpha$. As a result we find

$$\left\|\gamma_8^1(\cdot,\ Q_n)\right\|_{C(K)}\le C_{11}(K)\Big\{\Big\|Q_n\Big\|_\alpha\Omega\Big(f,\frac{v}{2},\,1-\alpha^{-1}\Big)+\max_{\rho_k\ge\frac{v}{2}}\Big|\,\overset{\wedge}{f_k}\,\Big|\Big\}. \tag{2.5}$$

Taking into account the known inequalities (see [1]; [12], p. 251)

$$\|P_2-Q_n\|_1\le const\,\omega_1\left(P_2,n^{-1}\right)\ ,\qquad \|Q_n\|_\alpha\le const\ n^{2(1-\alpha^{-1})}\,\|Q_n\|_1$$

in estimations (2.4) and (2.5) we get

$$\left\|\gamma_8^1\right\|_{C(K)}\le C_{12}(K)\Big\{D(v)+\max_{\rho_k\ge\frac{v}{2}}\Big|\,\overset{\wedge}{f_k}\,\Big|\Big\}.$$

Consequently, for the sum $\gamma_8(x)$ the estimation

$$\left\|\gamma_8\right\|_{C(K)} \leq C_{13}(K)\Big\{D(v)+\max_{\rho_k\geq\frac{v}{2}}\Big|\hat{f}_k\Big|\Big\} \tag{2.6}$$

is fulfilled.

The sum $\gamma_j(x),\ j=\overline{9,\,11}$ is estimated just in the same way, and estimation (2.6) is fulfilled for it.

Consequently, allowing for estimations (2.1) and (2.6) we get:

$$\left\|\sum_{k=1}^{\infty}\hat{W}_k f-\sigma_v(\cdot,f)\right\|_{C(K)} \leq C_{14}(K)\left\{D(v)+\Phi_p\left(f,v\right)\right\}.$$

As in the metrics $C(K)$ the equality (see [9]).

$$\sum_{k=1}^{\infty}\hat{W}_k f_k=\int_G f(y)\,\hat{W}\left(\left|x-y\right|,v,R_0\right)dy,$$

is valid, then from the last inequality we get

$$\left\|\int_G f(y)\,\hat{W}\left(\left|\cdot-y\right|,v,R_0\right)dy-\sigma_v\left(\cdot,\,f\right)\right\|_{C(K)} \leq C_{15}(K)\left\{D(v)+\Phi_p\left(f,v\right)\right\}. \tag{2.7}$$

As the trigonometric system is a system of eigenfunctions of the operator $L_1u=u^{(2)}$, then for $S_v(x,f)$ the following inequalities (see [9]) are valid

$$\left\|\int_G f(y)\,\hat{W}\left(\left|\cdot\ -y\right|,v,R_0\right)dy-S_v\left(\cdot,\ f\right)\right\|_{C(K)} \leq C_{16}(K)Q_p(f,v); \tag{2.8}$$

$$\left\|\int_G f(y)\,\hat{W}\left(\left|\cdot\ -y\right|,v,R_0\right)dy-S_v\left(\cdot,f\right)\right\|_{C(K)}$$

$$\leq C_{17}\left\{\omega_1(f,v^{-1})+v^{-1}\left\|f\right\|_p\right\}. \tag{2.9}$$

Estimation (1.3) follows from inequalities (2.7) , (2.8) and triangular inequality. Theorem 1.1 is proved.

Proof of theorem 1.2. By theorem 1.1 estimation (1.2) is fulfilled for $r>1$. Derive estimation (1.5) from estimation (1.2) allowing for condition (1.4). By condition A_1, (1.4) and monotonicity of the modulus of continuity $\omega_1(f,t)$, we get

$$\Phi_1(f,v)=v^{-1}\left\|f\right\|_1+\max_{\rho_k\geq\frac{v}{2}}\left|\hat{f}_k\right| \leq v^{-1}\left\|f\right\|_1$$

$$+const\max_{\rho_k\geq\frac{v}{2}}\left\{\omega_1(f,\rho_k^{-1})+\rho_k^{-1}\left\|f\right\|_1\right\}$$

$$\leq v^{-1}\left\|f\right\|_1+C\left(\omega_1\left(f,\frac{2}{v}\right)+\frac{2}{v}\left\|f\right\|_1\right)\leq const\,\varphi_1(f,v); \tag{2.10}$$

$$Q_1(f,v) = v^{-1}\|f\|_1 + \max_{2\pi k \ge \frac{v}{2}} \left|\overset{\sim}{f_k}\right| \le v^{-1}\|f\|_1$$

$$+const \max_{2\pi k \ge \frac{v}{2}} \left\{\omega_1\left(f, \frac{1}{2\pi k}\right) + \frac{1}{2\pi k}\|f\|_1\right\} \le const\,\varphi_1(f,v)\,; \qquad (2.11)$$

$$\|P_2\|_r\,\Omega\left(f, \frac{v}{2},\ 1-r^{-1}\right) = \|P_2\|_r\,v^{-1} \sum_{1\le\rho_k\le\frac{v}{2}} \left|\overset{\wedge}{f_k}\right| \rho_k^{-1+r^{-1}}$$

$$\le const\,\|P_2\|_r\,v^{-1} \sum_{1\le\rho_k\le\frac{v}{2}} \left\{\omega_1\left(f,\rho_k^{-1}\right) + \rho_k^{-1}\|f\|_1\right\}\cdot\rho_k^{-1+r^{-1}}$$

$$\le const\ \|P_2\|_r\,v^{-1}\left\{T\left(f,\ \left[\frac{v}{2}\right],\,1-r^{-1}\right)\right.$$

$$\left.+\frac{r}{r-1}\|f\|_1\right\} = constv^{-1}\ \|P_2\|_r\,N_1\left(f,\ \left[\frac{v}{2}\right],\,1-r^{-1}\right); \qquad (2.12)$$

$$\max_{3\le\ell\le 2m}\|P_\ell\|_1\,\Omega\left(f,\frac{v}{2},\ 1\right)$$

$$\le const\ \max_{3\le\ell\le 2m}\{\|P_\ell\|_1\}\,v^{-1}\sum_{1\le\rho_k\le\frac{v}{2}}\left\{\omega_1\left(f,\rho_k^{-1}\right)+\rho_k^{-1}\|f\|_1\right\}\rho_k^{-1}$$

$$\le const\ \max_{3\le\ell\le 2m}\{\|P_\ell\|_1\}\,v^{-1}\left\{T\left(f,\ \left[\frac{v}{2}\right],\,1\right)+\|f\|_1\right\}$$

$$\le constv^{-1}\max_{3\le\ell\le 2m}\{\|P_\ell\|_1\}\,N_1\left(f,\ \left[\frac{v}{2}\right],\,1\right). \qquad (2.13)$$

Taking into account estimations (2.10)-(2.13) in (1.2), we get

$$\Delta_v(K,f) \le C_3(K)v^{-1}\left\{\|P_2\|_1\,N_1\left(f,\ \left[\frac{v}{2}\right],\,1-r^{-1}\right)\right.$$

$$\left.+\max_{3\le\ell\le 2m}\|P_\ell\|_1\,N_1\left(f,\ \left[\frac{v}{2}\right],\,1\right)+\varphi_1(f,v)\right\}.$$

Estimation (1.5) is set up.

Let $r = 1$. Prove the validity of estimation (1.6). For that it suffices to estimate the expression $D(\nu)$ in (1.3) because for $\Phi_1(f,v)$, $Q_1(f,v)$ estimations (2.10) and (2.11) were already established. From conditions A_1, (1.4) and monotonicity of the function $\omega_1(f,t)$ we get

$$D(v) = \inf_{\substack{\alpha>1\\ n\ge 2}} \left\{\Omega\left(f,\frac{v}{2},0\right)\omega_1(P_2,n^{-1}) + n^{2(1-\alpha^{-1})}\|P_2\|_1\,\Omega\left(f,\frac{v}{2},1-\alpha^{-1}\right)\right\}$$

$$+\max_{3\le\ell\le 2m}\{\|P_\ell\|_1\}\,\Omega\left(f,\frac{v}{2},1\right)$$

$$\le const\ \inf_{\substack{\alpha>1\\ n\ge 2}} \left\{\omega_1(P_2,n^{-1})\left[v^{-1}\mathrm{T}\left(f,\ \left[\frac{v}{2}\right],\ 0\right)+v^{-1}\|f\|_1\sum_{1\le\rho_k\le\frac{v}{2}}\rho_k^{-1}\right]\right.$$

$$+n^{2(1-\alpha^{-1})}\|P_2\|_1\left[v^{-1}\mathrm{T}\left(f,\left[\frac{v}{2}\right],1-\alpha^{-1}\right)+v^{-1}\|f\|_1\sum_{1\le\rho_k\le\frac{v}{2}}\rho_k^{-2+\alpha^{-1}}\right]\Bigg\}$$

$$+const\max_{3\le\ell\le 2m}\{\|P_\ell\|_1\}\,v^{-1}N_1\left(f,\left[\frac{v}{2}\right],1\right)$$

$$\le constv^{-1}\inf_{\substack{\alpha>1\\ n\ge 2}}\left\{\omega_1(P_2,n^{-1})\left[\mathrm{T}\left(f,\left[\frac{v}{2}\right],0\right)+\|f\|_1\ln v\right]\right.$$

$$\left.+\|P_2\|_1\,n^{2(1-\alpha^{-1})}\left[\mathrm{T}\left(f,\left[\frac{v}{2}\right],1-\alpha^{-1}\right)+\frac{1}{1-\alpha^{-1}}\|f\|_1\right]\right\}$$

$$+constv^{-1}\max_{3\le\ell\le 2m}\{\|P_\ell\|_1\}\,N_1\left(f,\left[\frac{v}{2}\right],1\right)$$

$$=constv^{-1}E(P_2,v)+constv^{-1}\max_{3\le\ell\le 2m}\{\|P_\ell\|_1\}\,N_1\left(f,\left[\frac{v}{2}\right],1\right).$$

Consequently, for $D(\nu)$ the estimation

$$D(\nu)\le constv^{-1}\left\{E(P_2,v)+\max_{3\le\ell\le 2m}\{\|P_\ell\|_1\}\,N_1\left(f,\left[\frac{v}{2}\right],1\right)\right\}\tag{2.14}$$

is fulfilled.
Taking into account (2.10), (2.11) and (2.14) in (1.3), we get estimation (1.6).
Theorem 1.2 is proved.

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