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Kelley type Necessary Condition in Dynamic Systems with a Delay in Control

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Abstract. *In this paper, we consider an optimal control problem with delay in control which the admissible controls are not fixed at the initial set and study a class of singular (in the classical sense) controls. The Kelley and equality type optimality conditions are obtained. To prove our main results, we use the Legendre polynomials as variations of control.*

Keywords. Optimal control; singular control; variation transform method; Legendre polynomial; Kelley type optimality condition; equality type optimality condition.

Mathematics Subject Classification (2010): 39A12, 49K99, 90C29, 90C46

1 Introduction

In optimal control problems, the theory of high order necessary conditions is studied and developed mostly in two directions by many specialists. The first direction is about to generalise the idea of Kelley [10], the second one is to generalise the matrix impulse method introduced by Gabasov [5]. These directions were widely studied, and very important results were obtained in [1–4, 6–9, 11–13, 15–22, 24–26, and etc.].

In this paper, we study an optimal control problem with a delay in control which the admissible controls are not fixed at the initial set. Our aim is to obtain Kelley type condition and equality type optimality conditions for singular (in the classical sense) controls. For this, we apply the updated version of transformation method of variations and use the Legendre polynomials as variations of control. Note that the similar results were obtained in [17] but for the case when the admissible controls are fixed at the initial set.

The paper is organized as follows. In section 2 we introduce the control problem to be considered, some notations and basic assumptions. We evaluate the first and second variations of cost functional, and define a singular (in the classical sense) control in section 3. Also we find the transformation of the second order variation along a singular control, and prove Proposition 3.5 that is applied to create necessary optimality conditions in next section. In section 4 we obtain Kelley type condition and equality type optimality conditions for singular (in the classical sense) controls.

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2 Problem statement

In this paper, we consider the following optimal control problem with a control delay

$$S(u) = \varphi(x(t_1)) \to \min, \tag{2.1}$$

$$\dot{x}(t) = f(x(t), u(t), u(t-h), t), \quad t \in I := [t_0, t_1], x(t_0) = x_0, \tag{2.2}$$

$$u(t) \in V(t) \subseteq R^r, \;\; t \in I_h := [t_0 - h, t_1]. \tag{2.3}$$

Here R^r is r-dimensional Euclidean space with $R^1 := R := (-\infty, \infty)$; $x \in R^r$ is a state vector; $u \in R^r$ is a control vector; $x_0 \in R^n$ and $t_0, t_1, h \in R$ with $h > 0$ and $t_1 - t_0 > h$ are fixed points; $\varphi(x) : R^n \to R$ and $f(x, u, v, t) : Q \to R^n$, where $Q = R^n \times R^r \times R^r \times I$ are given functions; $U(t) = \begin{cases} V, & t \in I, \\ W, & t \in I_h \setminus I, \end{cases}$ where V and W are open subsets of R^r.

By $\tilde{C}^+([a, b], R^r)$ we denote the class of piecewise continuous on the right vector functions $c(t) : [a, b] \to R^r$, i.e. for every function $c(\cdot)$, it is either continuous or has finite number of discontinuity points of first kind, and at every discontinuity point $\tau \in (a, b)$ the value of $c(\tau)$ is defined as right limit: $c(\tau) = c(\tau + 0)$ (at points a and b they are continuous from right and left, respectively).

A function $u(\cdot)$ is called an *admissible control* if $u(\cdot)$ belongs to $\tilde{C}^+(I_h, R^r)$ and satisfies condition (2.3). Let $x(\cdot)$ be a solution of system (2.2) corresponding to $u(\cdot)$, then we call the pair $(u(\cdot), x(\cdot))$ an *admissible process*.

Note that if the function $f(\cdot)$ and partial derivative $f_x(\cdot)$ are continuous on Q, then by using the method of successive approximations in Sansone (1954) it is easy to show that for every admissible control $u(\cdot)$ there exists a unique absolutely continuous solution $x(\cdot)$ of the system (3.2) in $[t_0, t_0 + \alpha]$, where $\alpha > 0$ is some number. This solution will be assumed as defined everywhere on I.

An admissible control $u^0(t)$, $t \in I_h$, that solves problem (2.1)-(2.3) is called an *optimal control,* and the corresponding trajectory $x^0(t)$, $t \in I$, of system (2.2), (2.3) is called an *optimal trajectory.* The pair $(u^0(\cdot), x^0(\cdot))$ is said to be an *optimal process.*

In the study of problem (2.1)-(2.3), we use the following assumptions:

(A1) The functional $\varphi(\cdot) : R^n \to R$ is twice continuously differentiable in R^n, and the function $f(\cdot)$ and its partial derivatives $f_z(\cdot)$ and $f_{zz}(\cdot)$ are continuous in Q, where $z = (x, u, v)$.

(A2) The function $f(\cdot)$ is three times continuously differentiable with respect to its arguments in Q.

(A3) for the closures of the sets $U_0 = \{u = u^0(t) : t \in [t_0 - h, t_0]\}$, and $U_1 = \{u = u^0(t) : t \in I\}$ the inclusions $\overline{U}_0 \subset W$ and $\overline{U}_1 \subset V$ are valid;

(A4) An admissible control $u^0(\cdot)$ is piecewise smooth from right, i.e. $\dot{u}^0(\cdot) \in \tilde{C}^+(I_h, R^r)$.

It should be emphasized that more accurate assumptions concerning the analytical properties of $\varphi(\cdot)$, $f(\cdot)$, $u^0(\cdot)$ will follow directly from the representation for each of the optimality conditions obtained below.

3 Transformation of the second variation of a functional along a singular control

In this section, using the ideas of [11] and approach applied in [17] we show the transformation of the second variation of a functional along an optimal singular control. The results of this section are auxiliary and play an important role in the proof of theorems in the following section.

31 **Second variation of the cost functional and the definition of a singular (in classical sense) control**

Let assumptions (A1) and (A3) hold, and $(u^0(\cdot), x^(\cdot))$ is an admissible control. If $(u^0(\cdot), x^(\cdot))$ is an optimal process, then by known method (see for example [6, pp.51-53]), it is easy to obtain that

$$\delta^1 S\left(u^0(\cdot); \delta u(\cdot)\right) = 0, \ \ \forall \delta u(\cdot) \in \tilde{C}^+(I_h, R^r), \tag{3.1}$$

$$\delta^2 S\left(u^0(\cdot); \delta u(\cdot)\right) \geq 0, \ \ \forall \delta u(\cdot) \in \tilde{C}^+(I_h, R^r) \tag{3.2}$$

Here

$$\delta^1 S\left(u^0(\cdot); \delta u(\cdot)\right) := -\int\limits_{t_0}^{t_1} \left[H_u^T(t)\,\delta u(t) + H_\upsilon^T(t)\,\delta u(t-h)\right] dt, \delta u(\cdot) \in \tilde{C}^+(I_h, R^r), \tag{3.3}$$

$$\delta^2 S\left(u^0(\cdot); \delta u(\cdot)\right) = \delta x^T(t_1)\,\varphi_{xx}\left(x^0(t_1)\right)\delta x(t_1) - $$
$$-\int\limits_{t_0}^{t_1} \left\{\delta x^T(t)\,H_{xx}(t)\,\delta x(t) + \delta u^T(t)\,H_{uu}(t)\,\delta u(t) + \right.$$
$$+\delta u^T(t-h)\,H_{\upsilon\upsilon}(t)\,\delta u(t-h) + 2\left[\delta x^T(t)\,H_{xu}(t)\,\delta u(t) + \right.$$
$$\left.\left. +\delta x^T(t)\,H_{x\upsilon}(t)\,\delta u(t-h) + \delta u^T(t)\,H_{u\upsilon}(t)\,\delta u(t-h)\right]\right\} dt,$$
$$\delta u(\cdot) \in \tilde{C}^+(I_h, R^r), \tag{3.4}$$

where $\delta^1 S(u^0(\cdot), \delta u(\cdot))$ and $\delta^2 S(u^0(\cdot), \delta u(\cdot))$ are, correspondingly, the first and second variation of functional $S(u(\cdot))$ at the point $u^0(\cdot)$; $H(\psi, x, u, \upsilon, t) = \psi^T f(x, u, \upsilon, t)$ - Pontryagin function; $H(t) := H\left(\psi^0(t), x^0(t), u^0(t), \upsilon^0(t), t\right)$,
$H_\mu(t) = H_\mu\left(\psi^0(t), x^0(t), u^0(t), \upsilon^0(t), t\right)$,
$H_{\mu\nu}(t) = H_{\mu\nu}\left(\psi^0(t), x^0(t), u^0(t), \upsilon^0(t), t\right), t \in I, \ \mu, \nu \in \{x, u, \upsilon\}$

$\delta u(\cdot)$-the variation of an admissible control $u^0(\cdot)$; $\delta x(\cdot)$-the corresponding variation of the trajectory $x^0(t), t \in I$, which is the solution of the system

$$\begin{aligned} &\delta\dot{x}(t) = f_x(t)\,\delta x(t) + f_u(t)\,\delta u(t) + f_\upsilon(t)\,\delta u(t-h)\ , \ t \in I, \\ &\delta x(t_0) = 0, \end{aligned} \tag{3.5}$$

where
$$f_\mu(t) := f_\mu\left(x^0(t), u^0(t), u^0(t-h), t\right) \ t \in I, \ \mu \in \{x, u, \upsilon\},$$

further, the vector-function $\psi^0(\cdot)$ is the solution of the system

$$\dot{\psi}^0(t) = -H_x(t), \ t \in I, \ \psi^0(t_1) = -\varphi_x\left(x^0(t_1)\right). \tag{3.6}$$

Note that taking into account (3.3) and (3.5), from optimality conditions (3.1) and (3.2) as immediate consequence, by using the traditional reasoning (see [6, p. 53]) it is not hard to prove the following:

Proposition 3.1 *Let the assumptions (A1) and (A3) hold. Then for the optimality of $u^0(t), t \in I_h$, it is necessary that the followings are satisfied*

$$\chi(t)H_u(t) + \chi(t+h)H_v(t+h) = 0,\ \forall t \in I_h, \tag{3.7}$$

$$\tilde{u}\left[\chi(t)H_{uu}(t) + \chi(t+h)H_{vv}(t+h)\right]\tilde{u} \leq 0,\ \forall t \in I_h,\ \forall \tilde{u} \in R^r, \tag{3.8}$$

where $\chi(t)$ is the characteristic function of the set $[t_0, t_1)$.

Here, conditions (3.7) and (3.8) are the analogues of Euler equation and the condition Legendre-Clebsch, respectively.

In many cases, for example, when the function $f(\cdot)$ is a linear with respect to u and v, then, as seen, condition (3.8) degenerates, i.e. the condition Legendre-Clebsch will be trivial ($0 \leq 0$). These cases were studied widely by firstly Kelley [10] and then other researchers [7, 1, 3, 4, 6, 8, 9, 11–13, 17–21, 24–26].

Definition 3.1 *An admissible control $u^0(t), t \in I_h$, satisfying conditions* (3.7) *and* (3.8) *is called* singular *(in the classical sense), if*

$$rang\left[\chi(t)H_{uu}(t) + \chi(t+h)H_{vv}(t+h)\right] = r_1 < r,\ \forall t \in I_h$$

In this case, the set I_h is called a singular interval for an admissible control $u^0(\cdot)$.

The main goal of this paper is to study such singular controls.

Let $u = (p, q)^T$, $v = (\tilde{p}, \tilde{q})^T$, where $p, \tilde{p} \in R^{r_0}$, $q, \tilde{q} \in R^{r_1}$, $r_0 + r_1 = r$. Without loss of generality [6, p. 138], suppose that the singularity in $u^0(\cdot)$ is due to the vector component $p \in R^{r_0}$; i.e.,

$$\chi(t)H_{pp}(t) + \chi(t+h)H_{\tilde{p}\tilde{p}}(t+h) = 0,\ t \in I_h. \tag{3.9}$$

In this case, for the singular (in the classical sense) control $u^0(\cdot)$, the general inequality (3.8) implies the equality type optimality condition

$$\chi(t)H_{pq}(t) + \chi(t+h)H_{\tilde{p}\tilde{q}}(t+h) = 0,\ t \in I_h. \tag{3.10}$$

32 Transformation of the second order variation of $\delta^2 S(u^0(\cdot); \delta u)$ along a singular control $u^0(\cdot)$

Let assumptions (A2) - (A4) hold. Moreover, let $u^0(\cdot)$ be a singular control satisfying condition (3.9). Applying the transformation of variation method [11] we will transform $\delta^2(u^0(\cdot); \delta u(\cdot))$ to the Kelley type variation.

Proposition 3.2 *Let variation $\delta u = (\delta p(t), \delta q(t))^T \in \tilde{C}^+(I_h, R^r)$ be nonzero only on $[\theta, \theta+\varepsilon)$. Here $\delta p(t) \in \tilde{C}^+(I_h, R^{r_0})$, $\delta q(t) \in \tilde{C}^+(I_h, R^{r_1})$, $\theta \in [t_0 - h, t_1)$ and $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 \in (0, h)$ such that **(a)** if $\theta \in [t_0 - h, t_0)$ then $\varepsilon_0 < t_0 - \theta$; **(b)** if $\theta \in [t_0, t_1 - h)$, then $\varepsilon_0 < t_1 - h - \theta$; **(c)** if $\theta \in [t_1 - h, t_1)$, then $\varepsilon_0 < t_1 - \theta$. Then system* (3.5) *and the second variation $\delta^2 S(u^0(\cdot); \delta u)$ defined by* (3.4) *take new forms:*

$$\begin{aligned} \delta\dot{x}(t) &= f_x(t)\delta x(t) + \chi(\theta)[f_p(t)\delta p(t) + f_q(t)\delta q(t)] + \chi(\theta+h)[f_{\tilde{p}}(t)\delta p(t-h) \\ &\quad + f_{\tilde{q}}(t)\delta q(t-h)], t \in \left[\bar{\theta}, t_1\right], \\ \delta x(t) &= 0,\ t \in [t_0, \bar{\theta}]; \end{aligned} \tag{3.11}$$

$$\delta^2 S\left(u^0(\cdot); \delta u(\cdot)\right) = \delta x^T(t_1)\,\varphi_{xx}\left(x^0(t_1)\right)\delta x(t_1) -$$

$$- \int_{\bar{\theta}}^{t_1} \delta x^T(t) H_{xx}(t)\delta x(t)dt -$$

$$-\int_{\theta}^{\theta+\varepsilon}\left\{\chi(\theta)\left[\delta u^T(t)\,H_{uu}(t)+2\delta x^T(t)H_{xu}(t)\right]\delta u(t)+\chi(\theta+h)\times\right.$$

$$\left.\times\left[\delta u^T(t)H_{vv}(t+h)+2\delta x^T(t+h)H_{xv}(t+h)\right]\right\}\delta u(t)dt,\ \ \varepsilon\in(0,\varepsilon_0), \tag{3.12}$$

where $\overline{\theta}=\theta+\chi(\theta+t_2-t_0)h$.

Proof. Since $u=(p,q)^T$, $v=(\tilde{p},\tilde{q})^T$ and $\delta u(t)=(\delta p(t),\delta q(t))^T$, $t\in I_h$, $\delta v(t)=(\delta p(t-h),\delta q(t-h))^T$, $t\in I$, then system (3.5) can be written in form

$$\begin{gathered}\delta\dot{x}(t)=f_x(t)\delta x(t)+f_p(t)\delta p(t)+f_q(t)\delta q(t)+\\ +f_{\tilde{p}}(t)\delta p(t-h)+f_{\tilde{q}}(t)\delta q(t-h),\ \ t\in I,\\ \delta x(t_0)=0.\end{gathered} \tag{3.13}$$

Let us consider the following cases:

1. Let $\theta\in[t_0-h,t_0)$. Then by the hypothesis of the Proposition we have: $\delta u(t)=(0,0)^T$, $t\in I$, and $\delta u(t-h)=(0,0)^T$, for $t\in[t_0,\theta+h)\cup[\theta+h+\varepsilon,t_1]$, where $\varepsilon\in(0,\varepsilon_0)$. Considering these, from system (3.3) and (3.4) we get:

$$\begin{cases}\delta\dot{x}(t)=f_x(t)\,\delta x(t)+f_{\tilde{p}}(t)\delta p(t-h)+f_{\tilde{q}}(t)\delta q(t-h),\ t\in[\theta+h,t_1]\\ \delta x(t)=0,\ \ t\in[t_0,\theta+h],\end{cases} \tag{3.14}$$

$$\delta^2 S\left(u^0(\cdot);\delta u(\cdot)\right)=\delta x^T(t_1)\,\varphi_{xx}\left(x^0(t_1)\right)\delta x(t_1)-\int_{\theta+h}^{t_1}\delta x^T(t)H_{xx}(t)\delta x(t)dt-$$

$$-\int_{\theta}^{\theta+\varepsilon}\left[\delta u^T(t)H_{vv}(t+h)+2\delta x^T(t+h)H_{xv}(t+h)\right]\delta u(t)dt. \tag{3.15}$$

For this case, since $\chi(\theta)=0$, and $\chi(\theta+h)=\chi(\theta+t_1-t_0)=1$, we have $\overline{\theta}=\theta+h$, and from (3.14) system (3.11) is obtained. Also, from (3.15) we obtain the decomposition (3.12). In conclusion, if $\theta\in[t_0-h,t_0)$. then (3.11) and (3.12) are valid.

2. Let $\theta\in[t_0,t_1-h)$. Then $\chi(\theta)=\chi(\theta+h)=1$, $\overline{\theta}=\theta+h$, and since $\delta u(t)=(p(t),q(t))^T=(0,0)^T$ for $t\in[t_0-h,\theta)\cup[\theta+\varepsilon,t_1]$, $\varepsilon\in(0,\varepsilon_0)$ from (3.13) and (3.4) we obtain

$$\begin{cases}\delta\dot{x}(t)=f_x(t)\,\delta x(t)+f_p(t)\delta p(t)+f_q(t)\delta q(t)+\\ +f_{\tilde{p}}(t)\delta p(t-h)+f_{\tilde{q}}(t)\delta q(t-h),t\in[\theta,t_1],\ \ \delta x(t)=0,\ \ t\in[t_0,\theta],\end{cases} \tag{3.16}$$

$$\delta^2 S\left(u^0(\cdot);\delta u(\cdot)\right)=\delta x^T(t_1)\,\varphi_{xx}\left(x^0(t_1)\right)\delta x(t_1)-\int_{\theta}^{t_1}\delta x^T(t)H_{xx}(t)\delta x(t)dt-$$

$$-\int_{\theta}^{\theta+\varepsilon}\left\{\left[\delta u^T(t)H_{uu}(t)+2\delta x^T(t)H_{xu}(t)\right]\delta u(t)+\right.$$

$$\left.+[\delta u^T(t)H_{vv}(t+h)+2\delta x^T(t+h)H_{xv}(t+h)]\delta u(t)\right\}dt, \tag{3.17}$$

also, (3.16) and (3.17) coinside with (3.11) and (3.12), respectively, i.e. if $\theta \in [t_0, t_1 - h)$, then (3.11) and (3.12) are valid.

3. Let $\theta \in [t_1 - h, t_1)$. Then $\delta u(t) = 0,\ t \in [t_0 - h, 0) \cup [\theta + \varepsilon, t_1]$, $\delta v(t) = 0, t \in I_h$, $\chi(\theta) = 1,\ \chi(\theta + h) = 0, \overline{\theta} = \theta,$

Therefore, firstly, by the hypothesis of the Proposition from (3.13) and (3.4) we obtain:

$$\begin{cases} \delta \dot{x}\,(t) = f_x\,(t)\,\delta x\,(t) + f_p(t)\delta p(t) + f_q(t)\delta q(t), \\ t \in [\theta, t_1],\ \ \delta x(t) = 0,\ \ t \in [t_0, \theta] \end{cases} \tag{3.18}$$

$$\delta^2 S\left(u^0(\cdot); \delta u(\cdot)\right) = \delta x^T\,(t_1)\,\varphi_{xx}\left(x^0\,(t_1)\right)\delta x\,(t_1) - \int\limits_{\theta}^{t_1} \delta x^T(t) H_{xx}(t) \delta x(t) dt -$$

$$- \int\limits_{\theta}^{\theta+\varepsilon} \left[\delta u^T(t) H_{uu}(t) + 2\delta x(t) H_{xu}(t)\right] \delta u(t) dt \tag{3.19}$$

secondly, (3.18) and (3.19) coincide with (3.11) and (3.12), respectively, i.e. if $\theta \in [t_1 - h, t_1)$ then (3.11) and (3.12) are valid. Thus, Proposition 3.2 is proved.

Proposition 3.3 *Let an admissible control $u^0(\cdot)$ satisfying condition* (3.9) *be singular optimal. Then for every $\theta \in [t_0 - h, t_1)$ and for all nonzero $\delta u(t) = (\delta p(t), \delta q(t))^T \in \tilde{C}^+(I_h, R^r)$ only on $[\theta, \theta + \varepsilon)$ the following is valid:*

$$\delta^2 S\left(u^0(\cdot); \delta u(\cdot)\right) = \delta x^T\,(t_1)\,\varphi_{xx}\left(x^0\,(t_1)\right)\delta x\,(t_1) - \int\limits_{\bar{\theta}}^{t_1} \delta x^T(t) H_{xx}(t) \delta x(t) dt -$$

$$- \int\limits_{\theta}^{\theta+\varepsilon} \left\{ \left[\chi(\theta)\delta x^T(t) H_{xp}(t) + \chi(\theta+h)\delta x^T(t+h) H_{x\tilde{p}}(t+h)\right]\delta p(t) + \right.$$

$$\left. + [\chi(\theta)\delta x^T(t) H_{xq}(t) + \chi(\theta+h)\delta x^T(t+h) H_{x\tilde{q}}(t+h)]\delta q(t) \right\} dt -$$

$$- \int\limits_{\theta}^{\theta+\varepsilon} \delta q^T(t)[\chi(\theta) H_{qq}(t) + \chi(\theta+h) H_{\tilde{q}\tilde{q}}(t+h)]\delta q(t) dt,\ \ \varepsilon \in (0, \varepsilon_0), \tag{3.20}$$

where $\overline{\theta} = \theta + \chi(\theta + t_1 - t_0)h$, $\delta p(\cdot) \in \tilde{C}^+(I_h, R^{r_0})$, $\delta x(t), t \in I$- is the solution of (3.11)*, $\delta q(\cdot) \in \tilde{C}^+(I_h, R^{r_1})$ and the number ε_0 is defined in Proposition 3.2.*

Proof. Firstly, using Proposition 3.3, let us rewrite the second variation (3.12) by components $p, q, \tilde{p}, \tilde{q}$ and by $\delta p(\cdot)$, $\delta q(\cdot)$. Then, considering (3.9) and (3.10), we obtain the proof of (3.20). Proposition 3.3 is proved.

We proceed to generalize and apply the method of transformation of variation [11]. We require that the variation $\delta u(\cdot) = (\delta p(\cdot), \delta q(\cdot))^T$ satisfies next condition

$$\int\limits_{\theta}^{\theta+\varepsilon} \delta p(t) dt = 0, \quad \delta p(t) = 0,$$

$$\delta q(t) = 0,\ \ t \in I_h \backslash [\theta, \theta + \varepsilon),\ \ \varepsilon \in (0, \varepsilon_0), \tag{3.21}$$

where $\theta \in [t_0 - h, t_1)$, the number ε_0 is defined in Proposition 3.2.

From the variation $\delta u(t) = (\delta p(t), \delta q(t))^T$, $t \in I_h$, satisfying (3.21), moves on a new variation $\delta_1 u(t) = (\delta_1 p(t), \delta q(t))^T$, $t \in I_h$, where

$$\delta_1 p(t) = \int_{\theta}^{t} \delta p(t) d\tau, \ \ t \in I_h \tag{3.22}$$

It is obvious that

$$\delta_1 p(t) = 0, \ \ t \in I_h \backslash (\theta, \theta + \varepsilon), \ \ \varepsilon \in (0, \varepsilon_0). \tag{3.23}$$

We also transform the variation of the trajectory: instead of the solution $\delta x(t), t \in I$, of system (3.11), we consider the function $\delta_1 x(t), t \in I$:

$$\delta_1 x(t) = \delta x(t) - \chi(\theta) g_0[p](t) \delta_1 p(t) - \chi(\theta + h) g_0[\tilde{p}](t) \delta_1 p(t - h), \ \ t \in I, \tag{3.24}$$

where

$$g_0[\mu](t) := f_\mu(t), \ \ t \in I, \ \ \mu \in \{p, \tilde{p}\}. \tag{3.25}$$

Here, considering (3.23), we have:

$$\delta x(t_1) = \delta_1 x(t_1). \tag{3.26}$$

Since (A2) and (A4) hold, the function $\delta_1 x(t), t \in I$, is continuous and $\delta_1 \dot{x}(t) \in \tilde{C}^+(I, R^n)$.

By direct differentiation and taking into account (A2), (A4), (3.11), (3.22) and (3.23) from (3.24) we obtain that $\delta_1 x(t), t \in I$, is the solution of the system

$$\delta_1 \dot{x}(t) = f_x(t) \delta_1 x(t) + \chi(\theta)[g_1[p](t) \delta_1 p(t) + f_q(t) \delta q(t)] +$$

$$+ \chi(\theta + h)[g_1[\tilde{p}](t) \delta_1 p(t - h) + f_{\tilde{q}}(t) \delta q(t - h)], \ \ \delta_1 x(t) = 0, \ \ t \in [t_0, \bar{\theta}], \tag{3.27}$$

where

$$g_1[\mu](t) := f_x(t) g_0[\mu](t) - \frac{d}{dt} g_0[\mu](t), \ \ t \in I, \ \ \mu \in \{p, \tilde{p}\}. \tag{3.28}$$

Let us rewrite the second variation (3.20) by considering new variables $\delta_1 p(\cdot), \delta_1 x(\cdot)$. Related with this, let us prove the next statement.

Proposition 3.4 *Let assumptions (A2)-(A4) hold. Moreover, let the functions $g_0[\mu](\cdot)$ and $g_1[\mu](\cdot)$ be defined by* (3.25) *and* (3.28)*, $\delta_1 x(t), t \in I$, be a solution of system* (3.27)*. Then along a singular optimal control $u^0(t)$, $t \in I_h$, satisfying the condition* (3.9) *and on the variation $\delta u(t) = (\delta p(t), \delta q(t))^T, t \in I_h$, satisfying the conditions* (3.21) *and* (3.22)*, the representation*

$$\delta^2 S(u^0(\cdot); \delta u(\cdot)) = \Delta_1^{(2)} S(u^0(\cdot); \delta_1 p(\cdot), \delta q(\cdot), \delta_1 x(\cdot), \theta, \varepsilon) +$$

$$+ \Delta_2^{(2)} S(u^0(\cdot); \delta p(\cdot), \delta_1 p(\cdot), \delta q(\cdot), \theta, \varepsilon), \ \ \theta \in [t_0 - h, t_1), \ \ \varepsilon \in (0, \varepsilon_0), \tag{3.29}$$

is valid. Here, the number ε_0 is defined in Proposition 3.3;

$$\Delta_1^2 S(u^0(\cdot); \delta_1 p(\cdot), \delta q(\cdot), \delta_1 x(\cdot), \theta, \varepsilon) = \delta_1 x^T(t_1) \varphi_{xx}(x^0(t_1)) \delta_1 x(t_1) -$$

$$- \int_{\bar{\theta}}^{t_1} \delta_1 x^T(t) H_{xx}(t) \delta_1 x(t) dt - 2\chi(\theta) \int_{\theta}^{\theta + \varepsilon} \delta_1 x^T(t) [G_1[p](t) \delta_1 p(t) + H_{xq}(t) \delta q(t)] dt -$$

$$-2\chi(\theta+h)\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 x^T(t+h)\left[G_1[\tilde p](t+h)\delta_1 p(t)+H_{x\tilde q}(t+h)\delta q(t)\right]dt, \qquad (3.30)$$

where $\bar\theta=\theta+\chi(\theta+t_1-t_0)h,\ \ \theta\in I_h\backslash\{t_1\}$;

$$\Delta_2^{(2)}S(u^0(\cdot);\delta p(\cdot),\delta_1 p(\cdot),\delta q(\cdot),\theta,\varepsilon)=$$

$$=\chi(\theta)\int\limits_{\theta}^{\theta+\varepsilon}\big\{\delta_1 p^T(t)L_1[p](t)\delta_1 p(t)+$$

$$+2\delta_1 p^T(t)P_1[p,q](t)\delta q(t)+\delta p^T(t)Q_0[p](t)\delta_1 p(t)-\ \delta q^T(t)H_{qq}(t)\delta q(t)\big\}\,dt+$$

$$+\chi(\theta+h)\int\limits_{\theta}^{\theta+\varepsilon}\big\{\delta_1 p^T(t)L_1[\tilde p](t+h)\delta_1 p(t)+2\delta_1 p^T(t)P_1[\tilde p,\tilde q](t+h)\delta q(t)+$$

$$+\delta p^T(t)Q_0[\tilde p](t+h)\delta_1 p(t)-\ \delta q^T(t)H_{\tilde q\tilde q}(t+h)\delta q(t)\big\}\,dt; \qquad (3.31)$$

$$G_1[\mu](\tau):=H_{xx}(\tau)g_0[\mu](\tau)-f_x^T(\tau)H_{x\mu}(\tau)-$$
$$-\frac{d}{dt}H_{x\mu}(\tau), \qquad (3.32)$$

$$L_1[\mu](\tau):=-g_0^T[\mu](\tau)H_{xx}(\tau)g_0[\mu](\tau)+2g_1^T[\mu](\tau)H_{x\mu}(\tau)+$$
$$+\frac{d}{dt}(g_0^T[\mu](\tau)H_{x\mu}(\tau)), \qquad (3.33)$$

$$P_1[p,q](t):=H_{xp}(t)f_q(t)-g_0^T[p](t)H_{xq}(t), \qquad (3.34)$$

$$P_1[\tilde p,\tilde q](t+h):=H_{x\tilde p}(t+h)f_{\tilde q}(t+h)-g_0^T[\tilde p](t+h)H_{x\tilde q}(t+h), \qquad (3.35)$$

$$Q_0[\mu](\tau):=g_0^T[\mu](\tau)H_{x\mu}(\tau)-H_{x\mu}^T(\tau)g_0[\mu](\tau). \qquad (3.36)$$

where $\tau\in\{t,t+h\},\ \ \mu\in\{p,\tilde p\}$.

Proof. Under the conditions of Proposition, by (3.21)-(3.28), let us rewrite (3.20) taking into account new variables $\delta_1 p(\cdot),\delta_1 x(\cdot)$. Let us consider following cases:

Case 1: Let $\theta\in[t_0-h,t_0)$. Then $\chi(\theta)=0$, $\chi(\theta+h)=1$, $\overline{\theta}=\theta+h$, and therefore by (3.23) and (3.25), the formula (3.20), taking into account $t_1-h>t_0$ and $\varepsilon_0<t_0-\theta$, takes the form:

$$\delta^2 S(u^0(\cdot);\delta u(\cdot))=\delta_1 x^T(t_1)\varphi_{xx}(x^0(t_1))\delta_1 x(t_1)-$$

$$-\int\limits_{\theta+h}^{t_1}\delta_1 x^T(t)H_{xx}(t)\delta_1 x(t)dt-2\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 x^T(t+h)H_{xx}(t+h)g_0[\tilde p](t+h)\delta_1 p(t)dt-$$

$$-\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)g_0^T[\tilde p](t+h)H_{xx}(t+h)g_0[\tilde p](t+h)\delta_1 p(t)dt-$$

$$-2\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 x^T(t+h)H_{x\tilde q}(t+h)\delta q(t)dt-$$

$$-2\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)g_0^T[\tilde{p}](t+h)H_{x\tilde{q}}(t+h)\delta q(t)dt-\int\limits_{\theta}^{\theta+\varepsilon}\delta q^T(t)H_{\tilde{q}\tilde{q}}(t+h)\delta q(t)dt-$$

$$-2\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 x^T(t+h)H_{x\tilde{p}}(t+h)\delta p(t)dt-$$

$$-2\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)g_0^T[\tilde{p}](t+h)H_{x\tilde{p}}(t+h)\delta p(t)dt,\ \ \varepsilon\in(0,\varepsilon_0). \tag{3.37}$$

Let us transform last two integral by applying integration by parts. Then by (3.22) and (3.27) and taking into account $\chi(\theta)=0,\ \chi(\theta+h)=1$ and (3.28) we obtain:

$$\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 x^T(t+h)H_{x\tilde{p}}(t+h)\delta p(t)dt=$$

$$=-\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 x^T(t+h)\left[f_x^T(t+h)H_{x\tilde{p}}(t+h)+\frac{d}{dt}H_{x\tilde{p}}(t+h)\right]\delta_1 p(t)dt-$$

$$-\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)g_1^T[\tilde{p}](t+h)H_{x\tilde{p}}(t+h)\delta_1 p(t)dt-$$

$$-\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)H_{x\tilde{q}}^T(t+h)f_{\tilde{q}}(t+h)\delta q(t)dt; \tag{3.38}$$

$$2\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)g_0^T[\tilde{p}](t+h)H_{x\tilde{p}}(t+h)\delta p(t)dt=\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)g_0^T[\tilde{p}](t+h)H_{x\tilde{p}}(t+h)\delta p(t)dt+$$

$$+\int\limits_{\theta}^{\theta+\varepsilon}\delta p^T(t)H_{x\tilde{p}}^T(t+h)g_0[\tilde{p}](t+h)\delta_1 p(t)dt=$$

$$=\int\limits_{\theta}^{\theta+\varepsilon}\delta p^T(t)\left[H_{x\tilde{p}}^T(t+h)g_0[\tilde{p}](t+h)-g_0^T[\tilde{p}](t+h)H_{x\tilde{p}}(t+h)\right]\delta_1 p(t)dt-$$

$$-\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)\frac{d}{dt}\left(g_0^T[\tilde{p}]H_{x\tilde{p}}(t+h)\right)\delta_1 p(t)dt. \tag{3.39}$$

Taking into account (3.38) and (3.39) in (3.37). Further, based on (3.30)-(3.36), considering $\chi(\theta)=0,\ \chi(\theta+h)=1$ and $\theta+h=\overline{\theta}$, we get the proof of (3.28) for the case 1.

Case 2. Let $\theta \in [t_0, t_1 - h)$. Then $\chi(\theta) = \chi(\theta + h) = 1,\ \theta = \overline{\theta}$. Therefore, similarly to [17] let us do following reasonings. Taking into account (3.22)-(3.26), $t_1 - h > t_0$ and $\varepsilon_0 < t_1 - h - \theta$, the formula (3.20) takes the following form:

$$\delta^2 S(u^0(\cdot); \delta u(\cdot)) = \sum_{i=1}^{4} \Delta_i, \tag{3.40}$$

where

$$\Delta_1 := \delta_1 x^T(t_1)\varphi_{xx}(x^0(t_1))\delta_1 x(t_1) - \int_{\theta}^{t_1} \delta_1 x(t) H_{xx}(t)\delta_1 x(t)dt-$$

$$-2\int_{\theta}^{\theta+\varepsilon} \left[\delta_1 x^T(t)H_{xx}(t)g_0[p](t) + \delta_1 x^T(t+h)H_{xx}(t+h)g_0[\tilde{p}](t+h)\right]\delta_1 p(t)dt-$$

$$-2\int_{\theta}^{\theta+\varepsilon} \left[\delta_1 x^T(t)H_{xq}(t) + \delta_1 x(t+h)H_{x\tilde{q}}(t+h)\right]\delta q(t)dt;$$

$$\Delta_2 := \int_{\theta}^{\theta+\varepsilon} \left\{\delta_1 p^T(t)\left[g_0^T[p](t)H_{xx}(t)g_0[p](t)\ +\right.\right.$$

$$\left.+g_0^T[\tilde{p}](t+h)H_{xx}(t+h)g_0[\tilde{p}](t+h)\right]\delta_1 p(t)+$$

$$+2\delta_1 p^T(t)\left[g_0^T[p](t)H_{xq}(t) + g_0^T[\tilde{p}](t+h)H_{x\tilde{q}}(t+h)\right]\delta q(t)+$$

$$\left.+\delta q^T(t)\left[H_{qq}(t) + H_{\tilde{q}\tilde{q}}(t+h)\delta q(t)\right]\right\}dt;$$

$$\Delta_3 := -2\int_{\theta}^{\theta+\varepsilon} \left[\delta_1 x^T(t)H_{xp}(t) + \delta_1 x^T(t+h)H_{x\tilde{p}}(t+h)\right]\delta p(t)dt;$$

$$\Delta_4 := -2\int_{\theta}^{\theta+\varepsilon} \delta_1 p^T(t)\left[g_0^T[p](t)H_{xp}(t) + g_0^T[\tilde{p}](t+h)H_{x\tilde{p}}(t+h)\right]\delta p(t)dt.$$

Considering (A2) and (A4), taking into account (3.21)-(3.28) let us transform Δ_3, Δ_4 by applying integration by parts. Then by (3.22), (3.27), (3.28), (3.36), $\chi(\theta) = \chi(\theta + h) = 1$ we have:

$$\Delta_3 = +2\int_{\theta}^{\theta+\varepsilon} \left\{\delta_1 x^T(t)\left[\frac{d}{dt}(H_{xp}(t)) + f_x^T(t)H_{xp}(t)\right] +\right.$$

$$\left.+\ \delta_1 x^T(t+h)\left[\frac{d}{dt}(H_{x\tilde{p}}(t+h)) + f_x^T(t+h)H_{x\tilde{p}}(t+h)\right]\right\}\delta_1 p(t)dt+$$

$$+2\int_{\theta}^{\theta+\varepsilon} \delta_1 p^T(t)\left[g_1^T[p](t)H_{xp} + g_1^T[\tilde{p}](t+h)H_{x\tilde{p}}(t+h)\right]\delta_1 p(t)dt+$$

$$+2\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)\left[H_{xp}(t)f_q(t)+H_{x\tilde{p}}^T(t+h)f_{\tilde{q}}(t+h)\right]\delta q(t)dt,$$

$$\Delta_4=\int\limits_{\theta}^{\theta+\varepsilon}\delta p^T(t)\left[Q_0[p](t)+Q_0[\tilde{p}](t+h)\right]\delta_1 p(t)dt+$$

$$+\int_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)\left[\frac{d}{dt}(g_0^T[p](t)H_{xp}(t))+\frac{d}{dt}(g_0^T[\tilde{p}](t+h)H_{x\tilde{p}}(t+h))\right]\delta_1 p(t)dt,$$

where $g_1[p](\cdot)$ and $g_1[\tilde{p}](\cdot)$ are defined by (3.28); $Q_0[p](\cdot)$ and $Q_0[\tilde{p}](\cdot)$ are defined by (3.36). Let us substitute these obtained expressions for $\Delta_i,\ i=\overline{1,4}$, in (3.40). Considering (3.32)-(3.36), we do some grouping. Then by (3.30) and (3.31), taking into account $\theta=\overline{\theta}, \chi(\theta)=\chi(\theta+h)=1$, we obtain the proof of (3.29) for Case 2.

Case 3. Let $\theta\in[t_1-h,t_1)$. Then $\chi(\theta)=1,\ \chi(\theta+h)=0$ and $\overline{\theta}=\theta$ for $t_1-h>t_0$. Therefore, taking into account (3.23)-(3.26) the formula (3.20), considering $t_1-h>t_0$ and $\varepsilon_0<t_1-\theta$, takes the form:

$$\delta^2 S(u^0(\cdot);\delta u(\cdot))=\delta_1 x^T(t_1)\varphi_{xx}(x^0(t_1))\delta_1 x(t_1)-$$

$$-\int\limits_{\theta}^{t_1}\delta_1 x^T(t)H_{xx}(t)\delta_1 x(t)dt-2\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 x^T(t)\left[H_{xx}(t)g_0[p](t)\delta_1 p(t)dt+H_{xq}(t)\delta q(t)\right]dt-$$

$$-\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)g_0^T[p](t)H_{xx}(t)g_0[p](t)\delta_1 p(t)dt-$$

$$-2\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)g_0^T[p](t)H_{xq}(t)\delta q(t)dt-$$

$$-\int\limits_{\theta}^{\theta+\varepsilon}\delta q^T(t)H_{qq}(t)\delta q(t)dt+2I_1(\theta;\varepsilon)+I_2(\theta;\varepsilon), \qquad (3.41)$$

where

$$I_1(\theta;\varepsilon):=-\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 x^T(t)H_{xp}(t)\delta p(t)dt,$$

$$I_2(\theta;\varepsilon):=-2\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 p^T(t)g_0^T[p](t)H_{xp}(t)\delta p(t)dt.$$

We transform $I_1(\cdot)$ and $I_2(\cdot)$ by integration by parts. Then by (3.22), (3.27), (3.28) and (3.36), taking into account $\chi(\theta)=1,\ \chi(\theta+h)=0$ we get:

$$I_1(\theta;\varepsilon)=\int\limits_{\theta}^{\theta+\varepsilon}\delta_1 x^T(t)\left[\frac{d}{dt}(H_{xp}(t))+f_x^T(t)H_{xp}(t)\right]\delta_1 p(t)dt+$$

$$+\int_{\theta}^{\theta+\varepsilon} \delta_1 p^T(t) g_1^T[p] H_{xp}(t)\delta_1 p(t)dt + \int_{\theta}^{\theta+\varepsilon} \delta_1 p^T(t) H_{xp}^T(t) f_q(t)\delta q(t)dt, \qquad (3.42)$$

$$I_2(\theta;\varepsilon) = \int_{\theta}^{\theta+\varepsilon} \delta p^T Q_0[p](t)\delta_1 p(t)dt +$$

$$+\int_{\theta}^{\theta+\varepsilon} \delta_1 p^T(t)\frac{d}{dt}(g_0^T[p](t)H_{xp}(t))\delta_1 p(t)dt. \qquad (3.43)$$

In conclusion, since $\overline{\theta} = \theta$, $\chi(\theta) = 1$, $\chi(\theta + h) = 0$, substituting (3.42) and (3.43) in (3.41) and considering (3.30)-(3.36) we obtain the proof of representation (3.36) for Case 3. Proposition 3.4 is proven.

4 Necessary optimality conditions

In this section, based on Proposition 3.5 and using the proposed approach of [17], we prove the following theorem.

Theorem 4.1 *Let assumptions (A2)-(A4) hold, and matrix functions $L_1[\mu](\cdot)$, $P_1[p,q](\cdot)$, $P_1[\tilde{p},\tilde{q}](\cdot)$ and $Q_0[\mu](\cdot)$ (with $\mu \in \{p,\tilde{p}\}$) be defined by* (3.33)-(3.36). *Moreover, let along a singular (in the classical sense) control $u^0(\cdot)$ the condition* (3.9) *be satisfied. Then for the optimality of the singular control $u^0(\cdot)$ it is necessary that the relations*

$$\chi(t)Q_0[p](t) + \chi(t+h)Q_0[\tilde{p}](t+h) = 0, \qquad (4.1)$$

$$\begin{aligned}&\xi^T(\chi(t)L_1[p](t) + \chi(t+h)L_1[\tilde{p}](t+h))\xi +\\ &+2\xi\left(\chi(t)P_1[p,q](t) + \chi(t+h)P_1[\tilde{p},\tilde{q}](t+h)\right)\eta -\\ &-\eta^T\left(\chi(t)H_{qq}(t) + \chi(t+h)H_{\tilde{q}\tilde{q}}(t+h)\right)\eta \geq 0\end{aligned} \qquad (4.2)$$

hold for all $t \in [t_0 - h, t_1),\ \xi \in R^{r_0}, \eta \in R^{r_1}$.

Proof. Let the variation $\delta u(\cdot) = (\delta p(\cdot), \delta q(\cdot))^T$ satisfy the conditions (3.21) and (3.22). Then, under the hypothesis of this theorem Proposition 3.4 holds. Therefore, using statement (3.29) and Proposition 3.4 we prove (4.1) and (4.2).

Firstly, let us prove (4.1). We put [17]

$$\delta p_m(t) = 0,\ \forall t \in I_h,\ \forall m \in \{1,2,...,r_0\}\backslash\{i,j\},\ i,j \in \{1,2,...,r_0\},\ i \neq j$$

$$\delta p_m(t) = \begin{cases}\alpha l_1\left(\frac{2(t-\theta)}{\varepsilon} - 1\right), & t \in [\theta,\theta+\varepsilon),\ \varepsilon \in (0,\varepsilon_0),\\ 0, & t \in I_h\backslash[\theta,\theta+\varepsilon),\end{cases}$$

$$\delta p_j(t) = \begin{cases}\beta l_2\left(\frac{2(t-\theta)}{\varepsilon} - 1\right), & t \in [\theta,\theta+\varepsilon),\ \varepsilon \in (0,\varepsilon_0),\\ 0, & t \in I_h\backslash[\theta,\theta+\varepsilon),\end{cases} \qquad (4.3)$$

$$\delta q(t) = 0,\ t \in I_h,$$

where i,j $(i \neq j)$ are arbitrary fixed elements of the set $\{1,2,...,r_0\}$; $\delta p_k(\cdot)$-kth coordinate vector of $\delta p(\cdot)$; $\alpha, \beta \in R$ and $\theta \in [t_0 - h, t_1)$ are any fixed, functions $l_1(s) = s,\ l_2(s) = 3/2s^2 - 1/2,\ s \in [-1,1]$ are Legendre polynomials; ε is defined by Proposition 3.2.

It is easy to say that variation $\delta u(\cdot) = (\delta p(\cdot), \delta q(\cdot))^T$ is defined by (4.3) satisfying condition (3.21) and according to (4.3), we are convinced that function $\delta_1 p(t), t \in I_h$, is defined by (3.22) with of order ε and the solution $\delta_1 x(t)$ of the system (3.27) is of order ε^2, and also according to (3.36) it is easy to see that for every $t \in I$ the matrix $\chi(t)Q_0[p](t) + \chi(t+h)Q_0[\tilde{p}](t+h)$ is skew-symmetric .

Therefore, since condition (3.9) and Proposition 3.4 hold, along a singular optimal control $u^0(\cdot)$, by (3.2), (3.21), (3.22), (3.29)-(3.31), (4.3) we have

$$\delta^2 S\left(u^0(\cdot); \delta u\left(\cdot\right)\right) =$$

$$= \int\limits_{\theta}^{\theta+\varepsilon} \delta p^T(t)\left[\chi(\theta)Q_0[p](t) + \chi(\theta+t)Q_0[\tilde{p}](t+h)\right]\delta_1 p(t)dt + o(\varepsilon^2) =$$

$$= \int\limits_{\theta}^{\theta+\varepsilon} \left[q_{ij}^{(0)}\left(t\right)\delta p_i\left(t\right)\delta_1 p_j\left(t\right) + q_{ji}^{(0)}\left(t\right)\delta p_j\left(t\right)\delta_1 p_i\left(t\right)\right]dt =$$

$$= \frac{\varepsilon^2}{4}\alpha\beta\left[q_{ij}^{(0)}\left(\theta\right) - q_{ji}^{(0)}\left(\theta\right)\right]\int\limits_{-1}^{1} l_1\left(s\right)\int\limits_{-1}^{s} l_2\left(\tau\right)d\tau ds + o\left(\varepsilon^2\right) =$$

$$= -\frac{\varepsilon^2\alpha\beta}{30}\left[q_{ij}^{(0)}\left(\theta\right) - q_{ji}^{(0)}\left(\theta\right)\right] + o\left(\varepsilon^2\right) \geq 0, \ \ \forall\,\varepsilon \in \left(0, \varepsilon^*\right),$$

where $q_{ij}^0(t)$ and $q_{ji}^0(t)$ are elements of the matrix $\chi(\theta)Q_0[p](\theta) + \chi(\theta+h)Q_0[\tilde{p}](\theta+h)$.

Here, considering arbitrariness of $\alpha, \beta \in R$, $\theta \in [t_0 - h, t_1)$ and $i, j \in \{1, 2, ..., r_0\}$, $i \neq j$ we conclude that skew-symmetric matrix $\chi(\theta)Q_0[p](\theta) + \chi(\theta+h)Q_0[\tilde{p}](\theta+h)$ is symmetric. Hence, for each $t \in [t_0 + h, t_1)$ the equality $\chi(\theta)Q_0[p](\theta) + \chi(\theta+h)Q_0[\tilde{p}](\theta+h) = 0$. Thus, we get the proof of optimality condition (4.1).

Then, let us prove the validity of (4.2). For this, considering (3.21) and (3.22) let us set the vector components of the variation $\delta u(\cdot) = (\delta p(\cdot), \delta q(\cdot))^T$ in the following form [17]:

$$\delta p\left(t\right) = \begin{cases} \xi l_1\left(\frac{2(t-\theta)}{\varepsilon} - 1\right), & t \in [\theta, \theta+\varepsilon), \\ 0 & t \in I_h \backslash \left[\theta, \theta+\varepsilon\right), \ \varepsilon \in \left(0, \varepsilon_0\right), \end{cases}$$

$$\delta q\left(t\right) = \begin{cases} \eta \int\limits_{\theta}^{t} l_1\left(\frac{2(s-\theta)}{\varepsilon} - 1\right)ds, & t \in [\theta, \theta+\varepsilon), \\ 0, & t \in I_h \backslash \left[\theta, \theta+\varepsilon\right), \varepsilon \in \left(0, \varepsilon_0\right), \end{cases} \tag{4.4}$$

where $l_1(\tau) = \tau$, $\tau \in [-1, 1]$, is Legendre polynomial, and $\xi \in R^{r_0}$, $\eta \in R^{r_1}$, $\theta \in [t_0 - h, t_1)$ are arbitrary fixed points.

Taking into account (3.21), (3.22), (3.27) and (4.4), easily obtain:

$$\delta_1 p\left(t\right) \sim \varepsilon, \ \ \delta q\left(t\right) \sim \varepsilon, \ \ t \in I_h, \ \ \delta_1 x\left(t\right) \sim \varepsilon^2, \ \ t \in I.$$

Using the last relations and taking into account the proof of (4.1) from (3.29), considering (3.2), (3.22), (3.30) and (3.31) along a singular optimal control $u^0(\cdot)$ and the variation in the form of(4.4), we get

$$\delta^2 S\left(u^0(\cdot); \delta u(\cdot)\right) = \int\limits_{\theta}^{\theta+\varepsilon} \left\{\delta_1 p^T\left(t\right)\left[\chi(\theta)L_1[p](t) + \chi(\theta+t)L_1[\tilde{p}](t+h)\right]\delta_1 p(t) + \right.$$

$$+2\delta_1 p^T(t)\left[\chi(\theta)P_1[p,q](t)+\chi(\theta+t)P_1\left[\tilde{p},\tilde{q}\right](t+h)\right]\delta q(t)-$$
$$-\delta q^T(t)\left[\chi(\theta)H_{qq}(t)+\chi(\theta+t)H_{\tilde{q}\tilde{q}}(t+h)\right]\delta q(t)\big\}\,dt+o\left(\varepsilon^3\right)=$$
$$=\frac{\varepsilon^3}{8}\left\{\xi^T\left[\chi(\theta)L_1[p](\theta)+\chi(\theta+h)L_1[\tilde{p}](\theta+h)\right]\xi+\right.$$
$$+2\xi^T\left[\chi(\theta)P_1[p,q](\theta)+\chi(\theta+h)P_1[\tilde{p},\tilde{q}](\theta+h)\right]\eta-$$
$$\left.-\eta^T\left[\chi(\theta)H_{qq}(\theta)+\chi(\theta+h)H_{\tilde{q}\tilde{q}}(\theta+h)\right]\eta\right\}\int_{-1}^{1}\left(\int_{-1}^{t}l_1(\tau)d\tau\right)^2dt+$$
$$+o(\varepsilon^3)\geq 0,\ \ \forall\varepsilon\in(0,\varepsilon_0).$$

Here, considering the arbitrariness of $\theta\in[t_0+h,t_1)$, $\xi\in R^{r_0}$ and $\eta\in R^{r_1}$, we easily conclude the validity of optimality condition (4.2). Theorem is proved.

5 Conclusion

It is easy to see that problem (2.1)-(2.3) is not the most general among all the problems with retarded control. We have chosen it only for definiteness, just to demonstrate the essentials of our approach. Nevertheless, the optimality conditions (4.1) and (4.2) can be generalised to the cases of more general problem with retarded control. For example in the case when $f(\cdot)=f(x(t),u(t),u(t-h_1),...,u(t-h_m),t),\ t\in I$, where $0<h_1<h_2<...<h_m, t_1-h_m>t_0$.

It should be noted that the conditions (4.1) and (4.2) are actually the analogs of the equality type optimality condition and the Kelley condition for the problem (2.1) - (2.3), respectively. Similar results were obtained in [17] for the case when the set of admissible controls are fixed at initial set $[t_0-h,t_0]$. The optimality conditions of type ((4.1) and (4.2) for systems with delayed in state were obtained in [3,19,22], while for systems without retardation were treated in [1,5,4,6–8,12,13,24,25].

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