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On inverse boundary value problem for a second order hyperbolic equation with nonclassical boundary conditions

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Abstract. *In the paper an inverse value problem for the parabolic equations of the second order with non-classic boundary conditions is investigated. For this reason, first of all the initial problem is reduced to the equivalent problem, for which the theorem of existence and uniqueness is proved. Then using these facts the existence and uniqueness of the classical solution of initial problem is proved.*

Keywords. Inverse boundary problem, parabolic equation, method Fourier, classic solution.

Mathematics Subject Classification (2010): 35J;25, 35R30

1 Introduction

It is accepted to call the problem of definition of coefficients or the right hand side of differential equation simultaneously with its solution, the inverse problems of mathematical physics.

Inverse problems arise in different areas of human activity such as seismology, prospecting of mineral resources,biology, medical visualization, computer tomography, remote sounding of the Earth, spectral analysis, nondestructive control and so on. Solvability of various inverse problems for parabolic equations was studied in many papers (see e.i.[1]-[4])

In the paper the existence and uniqueness of the solution of a inverse boundary value problem is studied for second order parabolic equation with nonclassical boundary conditions.

2 Problem statement and reducing it to equivalent problem

Let us consider for the equation

$$a_1(t)u_t(x,t) + a_0(t)u(x,t) = u_{xx}(x,t) + f(x,t) \tag{2.1}$$

in the domain $D_T = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$ an inverse boundary value problem with the nonlocal condition

$$u(x,0) + \delta u(x,T) = \varphi(x) \quad (0 \le x \le 1)\ , \tag{2.2}$$

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the boundary condition

$$u(0,t) = 0 \quad (0 \le t \le T) \ , \tag{2.3}$$

nonlocal boundary condition

$$u_x(1,\ t) + du_{xx}(1,\ t) = 0 \quad (0 \le t \le T), \tag{2.4}$$

and extra condition

$$u(x_0, t) = h(t) \qquad (0 \le t \le T), \tag{2.5}$$

where $x_0 \in (0,1)$, $d > 0$, $\delta \ge 0$ are the given numbers, $a_1(t) > 0$, $f(x,t)$, $\varphi(x)$, $h(t)$ are the given functions, $u(x,t)$ and $a_0(t)$ are the sought for functions.

Definition 2.1 *Under the solution of inverse boundary value problem* (2.1)-(2.5) *we will understand the pair of functions* $\{u(x,t),\ a_0(t)\}$*, if* $u(x,t) \in C^{2,1}(D_T)$, $a_0(t) \in C\,[0,T]$ *and relations* (2.1)- (2.5) *are fulfilled.*

At first problem (2.1)-(2.5) will be reduced to the equivalent problem. To this end, we consider the following spectral problem

$$y''(x) + \lambda y(x) = 0\,(0 \le x \le 1),\ y(0) = 0,\ y'(1) = d\lambda\, y(1),\ d > 0. \tag{2.6}$$

This problem has only eigen functions $y_k(x) = \sqrt{2}\sin(\sqrt{\lambda_k}x)$, $k = 0,\ 1, 2, ...$, with positive eigen values λ_k from the equation $ctg\sqrt{\lambda} = d\sqrt{\lambda}$. We assign a zero index to any eigen function, and enumerate the remaining ones in the order of increase of eigen numbers.

The following lemma is valid.

Lemma 2.1 *Let* $f(x,t) \in C(D_T)$, $\varphi(x) \in C[0,\ 1]$, $h(t) \in C^1\,[0,T]$, $h\,(t) \neq 0$ *for* $t \in [0,T]$,

$$\varphi(1) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 \varphi(x)\sin(\sqrt{\lambda_0}x)dx = 0, \tag{2.7}$$

$$f(1,\ t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 f(x,t)\sin(\sqrt{\lambda_0}x)dx = 0,\ \ 0 \le t \le T, \tag{2.8}$$

and agreement conditions

$$\varphi\,(x_0) = h\,(0) + \delta h\,(T)\,. \tag{2.9}$$

be fulfilled. Then the problem of finding of classic solution of problem (2.1)-(2.5) *is equivalent to the problem of definition of functions* $u(x,t) \in C^{2,1}(D_T)$ *and* $a_0(t) \in C\,[0,T]$, *satisfying equation* (2.1)*, condition* (2.2)*,* (2.3) *and the conditions*

$$u(1,\ t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx = 0,\ 0 \le t \le T, \tag{2.10}$$

$$a_1(t)\,h'(t) + a_0(t)h(t) = u_{xx}(x_0,t) + f(x_0\,,t) \quad (0 \le t \le T) \tag{2.11}$$

Proof. Let $\{u(x,t), a_0(t)\}$ be any solution of problem (2.1)-(2.5). Then allowing for (2.8), from equation (2.1), we have:

$$a_1(t)\left[u_t(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 u_t(x,t)\sin(\sqrt{\lambda_0}x)dx\right]$$

$$+a_0(t)\left[u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]$$

$$= \left[u_{xx}(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_{xx}(x,t)\sin(\sqrt{\lambda_0}x)dx\right] \; (0 \le t \le T). \tag{2.12}$$

Integrating twice by parts, allowing for (2.3), by simple transformations we find:

$$\int_0^1 u_{xx}(x,t)\sin(\sqrt{\lambda_0}x)d\,x = -u_x(1,t)\sin\sqrt{\lambda_0} - \sqrt{\lambda_0}u(1,t)\cos\sqrt{\lambda_0}$$

$$-\lambda_0\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)d\,x \;\; (0 \le t \le T)\,.$$

Hence we have:

$$u_{xx}(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_{xx}(x,t)\sin(\sqrt{\lambda_0}x)dx = \frac{1}{d}\left(u_x\left(1,t\right) + du_{xx}\left(1,t\right)\right)$$

$$-\lambda_0\left[u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]. \tag{2.13}$$

Substituting (2.13) in (2.12) we get:

$$a_1(t)\left[u_t(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_t(x,t)\sin(\sqrt{\lambda_0}x)dx\right]$$

$$+a_0(t)\left[u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right] = \frac{1}{d}\left(u_x\left(1,t\right) + du_{xx}\left(1,t\right)\right)$$

$$-\lambda_0\left[u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]. \tag{2.14}$$

From (2.14) by (2.4) we find:

$$a_1(t)\omega'\left(t\right) + \left(a_0\left(t\right) + \lambda_0\right)\omega\left(t\right) = 0 \;\; (0 \le t \le T)\;, \tag{2.15}$$

where

$$\omega\left(t\right) \;\equiv \left[u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right] \;\; (0 \le t \le T)\;. \tag{2.16}$$

Further, by (2.2) and allowing for (4.8), we finf:

$$\omega\left(0\right) + \delta\omega\left(T\right) = \varphi(1) + \frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 \varphi(x)\sin(\sqrt{\lambda_0}x)dx = 0, \tag{2.17}$$

Obviously, the general solution of (2.15) has the form:

$$\omega(t) = ce^{-\int_0^t \frac{a_0(\tau)+\lambda_0}{a_1(\tau)}d\tau}\;(0 \le t \le T). \tag{2.18}$$

Hence, allowing for (2.12), we get:

$$c(1 + \delta e^{-\int_0^T \frac{a_0(\tau)+\lambda_0}{a_1(\tau)}d\tau}) = 0. \tag{2.19}$$

By $\delta \ge 0$, from (2.19) we get $c = 0$. Substituting the last one in (2.18) we deduce that $\omega(t) = 0$ $(0 \le t \le T)$. Consequently, from (2.16) it is clear that condition (2.10) is also fulfilled.

Further, from (2.5) it is

$$u_t(x_0,t) = h'(t)\ (0 \le t \le T). \tag{2.20}$$

Substituting $x = x_0$ in equation (2.1), we have

$$a_1(t)u_t(x_0,t) + a_0(t)u(x_0,t) = u_{xx}(x_0,t) + f(x_0,t)\,(0 \le t \le T). \tag{2.21}$$

Hence allowing for (2.5) and (2.20), we arrve at fulfillment of (2.11).

Now, suppose that $\{u(x,t), a_0(t)\}$ is the solution of problem (2.1)-(2.3), (2.10), (2.11), and the argeement condition (2.9) is fulfilled.

Then , allowing for (2.10), from (2.14) we arrive at fulfilment of (2.4).

Further, from (2.11) and (2.21), we get:

$$a_1(t)\frac{d}{dt}(u(x_0,t) - h(t)) + a_0(t)(u(x_0,t) - h(t)) = 0\ (0 \le t \le T). \tag{2.22}$$

By (2.2) agreement condition (2.9), we have:

$$u\,(x_0,t) - h(0) + \delta(u\,(x_0,T) - h(T)) = \varphi(x_0) - \delta\,(h(0) + \delta h(T)) = 0. \tag{2.23}$$

From (2.22) and (2.23) we deduce that condition (2.5) is fulfilled. The lemma is proved.

3 Information from theory of spectral problems and introduction of some spaces

Solving a homogeneous problem corresponding to problem (2.1)-(2.3), (2.10), (2.11), by the method of separation of variables we arrive at the spectral problem

$$y''(x) + \lambda y(x) = 0\,(0 \le x \le 1),\ y(0)$$

$$= 0,\ y(1) + \frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 y(x)\sin(\sqrt{\lambda_0}x)dx = 0. \tag{3.1}$$

It is known that spectral problem [5] is equivalent to spectral problem (2.6) without an eigen function corresponding to the eigen value λ_0. Consequently spectral problem (3.1) has only eigen functions $y_k(x) = \sqrt{2}\sin\left(\sqrt{\lambda_k}x\right)$ $k = 1,2,...,$ with possitive eigen numbers λ_k, defined from the equation $ctg\sqrt{\lambda} = d\sqrt{\lambda}$, ennumerated in increasing order.

The following statements were formulated and justified in the papers [5,6].

Lemma 3.1 *Beginning with some number N it holds the estimation*

$$0 < \sqrt{\lambda_k} - \pi k < (d\pi k)^{-1}. \tag{3.2}$$

Corollary 3.1 *Let $v_k(x) = \sqrt{2}\sin(\sqrt{\mu_k}x)$, where $\sqrt{\mu_k} = \pi k$, $k = 1,2,3,...$. Then the following inequalities are valid:*

$$\|y_k(x) - v_k(x)\|_{C[0,1]} \le \sqrt{2}(d\pi k)^{-1},\ k \ge N,$$

$$\sum_{k=N}^{\infty}\|y_k(x) - v_k(x)\|^2_{L_2(0,1)} \le 1/(9d^2). \tag{3.3}$$

Lemma 3.2 *The system $\{z_k(x)\}$ biorthogonally conjugated to the system $\{y_k(x)\}$, $k = 1,2,3,...,$ is determined by the formula*

$$z_k(x) = \sqrt{2}(\sin(\sqrt{\lambda_k}x) - \sin\sqrt{\lambda_k}(\sin\sqrt{\lambda_0}x)/(\sin\sqrt{\lambda_0}))/(1 + d\sin^2\sqrt{\lambda_k}). \tag{3.4}$$

Theorem 3.1 *The systems* $\{y_k(x)\}$ $k=1,2,...,$ *form the Riesz basis in space* $L_2(0,1)$.

Now let $\eta_k(x) = \sqrt{2}\cos(\sqrt{\lambda_k}x),\ \xi_k(x) = \sqrt{2}\cos(\sqrt{\mu_k}x),\ k=1,2,3,....$ *then similar to* (3.3)*, the following inequalities are valid*

$$\|\eta_k(x) - \xi_k(x)\|_{C[0,1]} \leq \sqrt{2}(d\pi k)^{-1},\ \ k \geq N,$$

$$\sum_{k=N}^{\infty} \|\eta_k(x) - \xi_k(x)\|^2_{L_2(0,1)} \leq 1/(9d^2). \tag{3.5}$$

Assume that $g(x) \in L_2(0,1)$. Then allowing for (3.3), we get

$$\sum_{k=1}^{\infty}\left(\int_0^1 g(x)y_k(x)dx\right)^2 \leq \sum_{k=1}^{N}\int_0^1 g^2(x)dx \cdot \int_0^1 y_k^2(x)dx$$

$$+2\sum_{k=N}^{\infty}\left(\int_0^1 g(x)v_k(x)dx\right)^2 + 2\sum_{k=N}^{\infty}\int_0^1 g^2(x)dx \cdot \int_0^1 (y_k(x) - v_k(x))^2 dx$$

or

$$\left(\sum_{k=1}^{\infty}\left(\int_0^1 g(x)y_k(x)dx\right)^2\right)^{1/2} \leq M\ \|g(x)\|_{L_2(0,1)}\,, \tag{3.6}$$

where

$$M = \left[\sum_{k=1}^{N}\int_0^1 y_k^2(x)dx + 2/\left(9d^2\right) + 2\right]^{1/2}. \tag{3.7}$$

Allowing for (3.6), similar to (3.5), we find:

$$\left(\sum_{k=1}^{\infty}\left(\int_0^1 g(x)\eta_k(x)dx\right)^2\right)^{1/2} \leq M\ \|g(x)\|_{L_2(0,1)}\,. \tag{3.8}$$

As the functions $\{y_k(x)\}$, $k=1,2,3,...$, form the Riesz basis in space $L_2(0,1)$, then it is known [7] that for any function $g(x) \in L_2(0,1)$ it is valid

$$g(x) = \sum_{k=1}^{\infty} g_k \cdot y_k(x), \tag{3.9}$$

where

$$g_k = \int_0^1 g(x)z_k(x)dx.\ (k=1,2,....)$$

Further, it is easy to see that

$$g_k = \frac{\sqrt{2}}{\alpha_k}\left[\int_0^1 g(x)\sin(\sqrt{\lambda_k}x)dx - \frac{\cos\sqrt{\lambda_k}}{d\sqrt{\lambda_k}\ \sin\sqrt{\lambda_0}}\int_0^1 g(x)\sin\sqrt{\lambda_0}x dx\right], \tag{3.10}$$

where

$$\alpha_k = 1 + d\sin^2\sqrt{\lambda_k} > 1.$$

Hence, allowing for (3.6) we find:

$$\left(\sum_{k=1}^{\infty} g_k^2\right)^{1/2} \leq M_0\ \|g(x)\|_{L_2(0,1)}\,, \tag{3.11}$$

where

$$M_0 = \left[M + \frac{1}{d\left|\sin\sqrt{\lambda_0}\right|} \left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \right)^{1/2} \right] \sqrt{2}. \tag{3.12}$$

Assume that $g(x) \in C[0,1],\ g'(x) \in L_2(0,1)$, $g(0) = 0$ and

$$J(g) \equiv g(1) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 g(x)\sin(\sqrt{\lambda_0}x)dx = 0.$$

Then from (3.10), we have:

$$g_k = \frac{\sqrt{2}}{\alpha_k}\frac{1}{\sqrt{\lambda_k}} \int_0^1 g'(x)\cos\left(\sqrt{\lambda_k}x\right)dx. \tag{3.13}$$

Hence, allowing for (3.5) we find:

$$\left(\sum_{k=1}^{\infty} (\sqrt{\lambda_k}\,|g_k|)^2 \right)^{1/2} \le M\left\| g'(x) \right\|_{L_2(0,1)}. \tag{3.14}$$

Let $g(x) \in C^1[0,1]$, $g''(x) \in L_2(0,1)$, $g(0) = 0$ and $J(g) = 0$. Then from (3.13) we have:

$$g_k = -\frac{\sqrt{2}}{\alpha_k}\left[\frac{1}{\lambda_k}\int_0^1 g''(x)\sin(\sqrt{\lambda_k}x)dx - \frac{\cos\sqrt{\lambda_k}}{d\lambda_k\sqrt{\lambda_k}} g'\,(1) \right]. \tag{3.15}$$

Hence we find:

$$\left(\sum_{k=1}^{\infty} (\lambda_k\ |g_k|\,)^2 \right)^{1/2} \le m\left| g'(0) \right| + \sqrt{2}M\left\| g''(x) \right\|_{L_2(0,1)}, \tag{3.16}$$

where

$$m = \frac{\sqrt{2}}{d}\left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \right)^{1/2}.$$

Now suppose that $g(x) \in C^2[0,1],\ g'''(x) \in L_2(0,1), g(0) = 0, J(g) = 0, g''(0) = 0$ and $dg''(2.1) + g'(2.1) = 0$.. then from (3.15) , we have:

$$g_k = -\frac{\sqrt{2}}{\alpha_k}\frac{1}{\lambda_k\sqrt{\lambda_k}} \int_0^1 g'''(x)\cos(\sqrt{\lambda_k}x)dx.$$

Hence, allowing for (3.5) we get:

$$\left(\sum_{k=1}^{\infty} (\lambda_k\sqrt{\lambda_k}\ |g_k|\,)^2 \right)^{1/2} \le M\left\| g'''(x) \right\|_{L_2(0,1)}. \tag{3.17}$$

1 Denote by $B_{2,T}^{3/2}$ [8], the totality of all functions $u(x,t)$ of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)y_k(x),$$

considered in D_T, where each of the functions $u_k(t)$ is continuous on $[0,T]$ and

$$\left\{\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]})^2\right\}^{1/2}<+\infty.$$

We determine the norm on this set as:

$$\|u(x,t)\|_{B^{3/2}_{2,T}}=\left\{\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]})^2\right\}^{1/2}.$$

2 Denote by $E^{3/2}_T$ a space consisting of topological product

$$B^{3/2}_{2,T}\times C[0,T].$$

The norm of the element $z=\{u,a\}$ is determined by the formula

$$\|z\|_{E^{3/2}_T}=\|u(x,t)\|_{B^{3/2}_{2,T}}+\|a_0(t)\|_{C[0,T]}\,.$$

It is known $B^{3/2}_{2,T}$ and $E^{3/2}_T$ are Banach spaces.

4 Studying the existence and uniqueness of classic solution to inverse boundary value problem

Taking into account lemma 3 and theorem 1, we will look for the first component $u(x,t)$ of the solution $\{u(x,t),a(t)\}$ of problem (2.1)-(2.3), (2.10), (2.11) in the form:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)y_k(x), \tag{4.1}$$

where

$$u_k(t)=\int_0^1 u(x,t)z_k(x)dx\ \ (k=1,2,...).$$

Apply the method of separation of variables for determining the sought -for functions $u_k(t)$ $(k=1,2,...;)$. Then, from (2.1) and (2.2) we have:

$$a_1(t)u_k'(t)+\lambda_k u_k(t)=F_k(t;u,a_0)\ \ (k=1,2,...;\ 0\le t\le T), \tag{4.2}$$

$$u_k(0)+\delta u_k(T)=\varphi_k(k=1,2,...), \tag{4.3}$$

where

$$F_k(t;u,a_0)=f_k(t)-a_0(t)u_k(t),\ \ f_k(t)=\int_0^1 f(x,t)z_k(x)dx,$$

$$\varphi_k=\int_0^1\varphi(x)z_k(x)dx\ \ (k=1,2,...).$$

Solving problem (4.2), (4.3), we find:

$$u_k(t) = \frac{\varphi_k e^{-\int\limits_0^t \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}} + \int\limits_0^t \frac{F_k(\tau;u,a_0)}{a_1(\tau)} e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau$$

$$-\frac{\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}} \int_0^T \frac{F_k(\tau;u,a_0)}{a_1(\tau)} e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau \; (k=1,2,...). \tag{4.4}$$

After substituting the expressions $u_k(t)$ $(k=1,2,\ldots)$ from (4.4) in (4.1) we have:

$$u(x,t) = \sum_{k=1}^{\infty} \left\{ \frac{\varphi_k e^{-\int\limits_0^t \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}} + \int_0^t \frac{F_k(\tau;u,a_0)}{a_1(\tau)} e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau \right.$$

$$\left. -\frac{\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}} \int\limits_0^T \frac{F_k(\tau;a_0,u)}{a_1(\tau)} e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau \right\} y_k(x). \tag{4.5}$$

Now, allowing for (4.1), from (2.11),we get :

$$a_0(t) = h^{-1}(t) \left\{ h'(t) - f(x_0,t) - \sum_{k=1}^{\infty} \lambda_k u_k(t) y_k(\, x_0) \right\} . \tag{4.6}$$

In order to obtain an equation for the second component $a_0(t)$ of the solution $\{u(x,t), a(t)\}$ of problem (2.1)-(2.3), (2.10) , (2.11) we substitute the expression (4.4) in (4.6):

$$a_0(t) = h^{-1}(t) \left\{ a_1(t)h'(t) - f(x_0,t) - \sum_{k=1}^{\infty} \lambda_k \left(\frac{\varphi_k e^{-\int_0^t \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}} + \int_0^t \frac{F_k(\tau;u,a_0)}{a_1(\tau)} \times \right.\right.$$

$$\left.\left. \times e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau - \frac{\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}} \int_0^T \frac{F_k(\tau;a_0,u)}{a_1(\tau)} e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau \right) y_k(x_0) \right\} . \tag{4.7}$$

Thus, the solution of problem (2.1)-(2.3), (2.10) , (2.11) is reduced to the solution of system (4.5), (4.7) with respect to the unknown functions $u(x,t)$ and $a_0(t)$.

The following lemma lays an important role for studying the uniqueness of the solution of problem (2.1)-(2.3), (2.10) , (2.11)

Lemma 4.1 *If* $\{u(x,t), a_0(t)\}$ *is any solution of problem* (2.1)-(2.3), (2.10) , (2.11)*, then the functions*

$$u_k(t) = \int_0^1 u(x,t) z_k(x) dx (k=1,2,...)$$

satisfy on $[0,T]$ *the system* (4.4).

Obviously, if $u_k(t) = \int\limits_0^1 u(x,t)z_k(x)\,dx$ $(k = 1,2,...)$ *is the solution of the system* (4.4)*, then the pair* $\{u(x,t), a_0(t)\}$ *of functions* $u(x,t) = \sum\limits_{k=0}^{\infty} u_k(t)y_k(x)$ *and* $a_0(t)$ *is the solution to the system* (4.5)*,* (4.7).

The following corollary follows from lemma 4.1

Corollary 4.1 *Let the system* (4.5)*,* (4.7) *have a unique solution. Then problem* (2.1)-(2.3)*,* (2.10) *,* (2.11) *may have at most one solution i.e. if problem* (2.1)-(2.3)*,* (2.10) *,* (2.11) *has a solution, then this solution is unique.*

Now let us consider in the space $E_T^{3/2}$ *the operator*

$$\Phi(u,a_0) = \{\Phi_1(u,a_0), \Phi_2(u,a_0)\},$$

where

$$\Phi_1(u,a_0) = \tilde{u}(x,t) \equiv \sum_{k=1}^{\infty} \tilde{u}_k(t)y_k(x),$$

$$\Phi_2(u,a_0) = \tilde{a}_0(t),$$

while $\tilde{u}_k(t)\,(k = 1,2,...)$ *and* $\tilde{a}_0(t)$ *are equal to the right hand sides of* (4.4) *and* (4.7)*, respectively.*

By means of simple transformations, we find that the following inequalities are valid:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\left\|\tilde{u}_k(t)\right\|_{C[0,T]}\right)^2\right)^{1/2} \le \sqrt{3}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,|\varphi_k|\right)^2\right)^{1/2}$$

$$+\sqrt{3}\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}(1+\delta)\times\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right.$$

$$\left. + T\left\|a_0(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right], \tag{4.8}$$

$$\left\|\tilde{a}_0\,(t)\right\|_{C[0,T]} \le \left\|h^{-1}(t)\right\|_{C[0,T]}\left\{\left\|a_1(t)h'(t) - f(x_0,t)\right\|_{C[0,T]} + \sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\right.$$

$$\times\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\,|\varphi_k|\right)^2\right)^{\frac{1}{2}} + \left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}\right.$$

$$\times(1+\delta)\left(\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,|f_k(\tau)|)^2\,d\tau\right)^{1/2}\right.$$

$$\left.\left.\left. + T\left\|a_0\,(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right)\right]\right\}. \tag{4.9}$$

Suppose that the data of problem (2.1)-(2.3), (2.10) , (2.11) satisfy the following conditions:

1. $\varphi(x) \in C^2\,[0,1]\,, \varphi'''(x) \in L_2(0,1),\ \varphi(0) = 0,\ J(\varphi) = 0,\ \varphi''(0) = 0\,,$
   $d\varphi''(1) + \varphi'(1) = 0.$
2. $f(x,t), f_x(x,t),\ f_{xx}(x,t) \in C(D_T), f_{xxx}(x,t) \in L_2(D_T\,), f\,(0,t) = 0, J\,(f) = 0,$
   $f_{xx}(0,t) = 0,\ ,.df_{xx}(1,t) + f_x(1.t) = 0\,(0 \leq t \leq T)$
3. $\delta \geq 0, 0 < a_1(t) \in C[0,T],\ h(t) \in C^1[0,T],\ h(t) \neq 0(0 \leq t \leq T).$

The from (4.8) and (4.9), allowing for (3.17), we get

$$\|\tilde{u}(x,t)\|_{B^{3/2}_{2,T}} \leq A_1(T) + B_1(T)T\,\|a_0\,(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^{3/2}_{2,T}}\,, \tag{4.10}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T)T\,\|a_0\,(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^{3/2}_{2,T}}\,, \tag{4.11}$$

where

$$A_1(T) = \sqrt{3}M\,\|\varphi'''(x)\|_{L_2(0,1)} + \left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}(1+\delta)\sqrt{3T}M\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\,,$$

$$B_1\,(T) = \sqrt{3}\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}(1+\delta),$$

$$A_2(T) = \left\|h^{-1}(t)\right\|_{C[0,T]}\Big\{\left\|a_1(t)h'(t) - f\,(x_0,t)\right\|_{C[0,T]}$$

$$+\sqrt{2}(\sum_{k=1}^{\infty}\lambda_k^{-1})^{1/2}\Big[M\,\|\varphi'''(x)\|_{L_2(0,1)}$$

$$+\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}(1+\delta)\sqrt{T}M\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\Big]\Big\}\,,$$

$$B_2(T) = \left\|h^{-1}(t)\right\|_{C[0,T]}\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}(1+\delta)\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}.$$

From inequlities (4.10), (4.11) we deduce:

$$\|\tilde{u}(x,t)\|_{B^2_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \leq A(T) + B(T)T\,\|a_0\,(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^2_{2,T}}\,, \tag{4.12}$$

where

$$A(T) = A_1(T) + A_2(T),\ \ B(T) = B_1(T) + B_2(T).$$

Theorem 4.1 *Let conditions 1-3 be fulfilled, and*

$$(A(T)+2)^2 B(T) < 1. \tag{4.13}$$

Then problem (2.1)-(2.3), (2.10) , (2.11)*has in the ball* $K = K_R(||z||_{E^{3/2}_T} \leq R = A(T)+2)$ *of the space* $E^{3/2}_T$ *a unique solution.*

Proof. In space $E_T^{3/2}$ we consider the equation

$$z = \Phi z, \tag{4.14}$$

where $z = \{u, a\}$, the components $\Phi_i(u, a_0)(i = 1, 2)$ of the operator (u, a_0) are determined by the right hand sides of equations (4.5), (4.7), respectively. Let us consider the operator (u, a_0) in the ball $K = K_R(\|z\|_{E_T^2} \leq R = A(T) + 2)$ from $E_T^{3/2}$.

Similar to (4.12), we get that for any $z, z_1, z_2 \in K_R$ the following estimations are valid:

$$\|\Phi z\|_{E_T^{3/2}} \leq A(T) + B(T)T \|a_0(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^{3/2}}, \tag{4.15}$$

$$\|\Phi_{z_1} - \Phi_{z_2}\|_{E_T^{3/2}} \leq B(T)TR$$

$$\times \left(\|a_{0_1}(t) - a_{0_2}(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^{3/2}} \right) \tag{4.16}$$

Then, allowing for (4.13), from the estimations (4.15) and (4.16), it follows that the operator Φ acts in the ball $K = K_R$ and is contractive. Therefore, in the ball $K = K_R$ the operator Φ has a unique fixed point $\{u, a_0\}$, that is the unique solution of equation (4.14), i.e. it is a unique solution in the ball $K = K_R$ of system (4.5), (4.7).

The function $u(x,t)$, as an element of the space $B_{2,T}^{3/2}$, is continuous and has continuous derivatives $u_x(x,t)$, $u_{xx}(x,t)$ in D_T.

From (4.2) ,by (3.14), it is easy to see that

$$\left(\sum_{k=1}^{\infty} (\sqrt{\lambda_k} \|u_k'(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq \left\| \frac{1}{a_1(t)} \right\|_{C[0,T]} \left\{ \left(\sum_{k=1}^{\infty} (\lambda_k \sqrt{\lambda_k} \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right.$$

$$\left. + M \left\| \|f_x(x,t)\|_{C[0,T]} \right\|_{L_2(0,1)} + \|a_0(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k \sqrt{\lambda_k} \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right\}.$$

Hence it follows that $u_t(x,t)$ is continuous in D_T.

It is easy to verify that equation (2.1) conditions (2.2)-(2.3), (2.10) and (2.11) are satisfied in the ordinary sense.

Therefore, $\{u(x,t), a_0(t)\}$ is the solution of problem (2.1)-(2.3), (2.10), (2.11).By Corollary 4.1 of lemma 4.1 it is unique in the ball $K = K_R$. The theorem is proved

Theorem 4.2 *Let all the conditions of theorem 4.1 and agreement conditions*

$$\varphi(x_0) = h(0) + \delta h(T).$$

be fulfilled.

Then problem (2.1)-(2.5) *has in the ball* $K = K_R(\| z \|_{E_T^{3/2}} \leq R = A(T) + 2)$ *of the space* $E_T^{3/2}$ *a unique classic solution.*

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