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Operator Frames in the context of hypergroups

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Abstract. *In this paper, we study the concept of continuous O-frame on locally compact hypergroups and some sufficient and necessary coditions have been found for an operator has an O-frame.*

Keywords. locally compact hypergroup, Banach space, operator frame, continuous O-frame.

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1 Introduction and notations

In 2015, operator frames, as extensions of Schauder frames, were introduced and studied by O. Reinov in [7] and were called *O-frame*. In [8], some sufficient and necessary coditions have been found for an operator has an O-frame. In [9], the authors introduced and studied the concept continuous operator frames on locally compact groups. In this work we intend to generalize the main results of [8] and [9] to locally compact hypergroups which are known as extensions of locally compact groups.

Throughout this paper, K is a *locally compact hypergroup* (or simply a *hypergroup*) with a left Haar measure λ. For definition and basic properties of hypergroups we refer to the book [2] and the paper [6] in which hypergroups are called *convo*. Our special usage from the structure of hypergroup in this paper is via a key result of [1] which holds for hypergroups. We have used this fact for giving some sufficient conditions for a continuous O-frame of an operator to be also a boundedly complete continuous O-frame. Given a topological space Z, the set of all continuous functions from K into Z is denoted by $C(K, Z)$. Also, for Banach spaces X and Y, the dual of X is denoted by X^*, and the set of all bounded linear operators from X to Y is denoted by $B(X, Y)$.

2 Main results

Definition 2.1 *Let X, Y be infinite dimensional separable Banach spaces, $F \in C(K, X^*)$ and $G \in C(K, Y)$. The pair (F, G) is called continuous O-frame for an operator $S \in$*

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$B(X,Y)$ *if for every* x *in* X,

$$S(x) = \int_K F(t)(x)G(t)\, d\lambda(t).$$

Proposition 2.1 *Let* X, M, N *and* Y *be Banach spaces,* $H \in B(M,X)$, $S \in B(X,Y)$ *and* $R \in B(Y,N)$ *and let* (F,G) *be a continuous O-frame for the operator* S*, then the operator* RSH *has a continuous O-frame.*

Proof. Since (F,G) is a continuous O-frame for S, for every x in X we have

$$S(x) = \int_K F(t)(x)G(t)d\lambda(t).$$

Let H^* be the adjoint operator of H,then for every m in M we get

$$\begin{aligned} R(S(H(m))) &= \int_K F(t)(H(m))R(G(t))\, d\lambda(t) \\ &= \int_K H^*(F(t))(m)R(G(t))d\lambda(t). \end{aligned}$$

Cosequently (H^*F, RG) is a continuous O-frame for RSH.

Theorem 2.1 *Let* X *and* Y *be Banach spaces with* $\dim X = m < \infty$*, and let there exists* m *disjoint non-trivial open sets* $U_1, \ldots, U_m$ *in* K*. Then, every* S *in* $B(X,Y)$ *has a continuous O-frame.*

Proof. Since $\dim X = m$, we can choose $x_1, \ldots, x_m \in X$ and $\xi_1, \ldots, \xi_m \in X^*$ such that for every x in X, $x = \sum_{i=1}^m \xi_i(x)x_i$. Let $S \in B(X,Y)$ so we have

$$S(x) = \sum_{i=1}^m \xi_i(x)S(x_i).$$

Since K is locally compact, by Urysohn's Lemma we can pick continuous functions $\psi_1, \ldots, \psi_m$ from K into $\mathbb{C}$ such that

$$\text{supp}(\psi_i) \subseteq U_i \ \text{ and } \ M_i := \int_K |\psi_i(t)|^2\, d\lambda(t) > 0,$$

for all $i = 1, \ldots, m$. For every t in K the function $F(t)$ in X^* is defined by,

$$F(t)(\sum_{i=1}^m \eta_i x_i) := \sum_{i=1}^m \frac{\eta_i}{M_i}\overline{\psi_i(t)},$$

where $\eta_1, \ldots, \eta_m \in \mathbb{C}$ and the function $G \in C(K,Y)$ is defined by,

$$G(t) := \psi_1(t)S(x_1) + \ldots + \psi_m(t)S(x_m), \qquad (t \in K).$$

Therefore,

$$\begin{aligned} \int_K F(t)(x_r)G(t)\, d\lambda(t) &= \sum_{i=1}^m \left(\int_K F(t)(x_r)\psi_i(t)\, d\lambda(t) \right) S(x_i) \\ &= \sum_{i=1}^m \left(\int_K \frac{1}{M_r}\overline{\psi_r(t)}\psi_i(t)\, d\lambda(t) \right) S(x_i) \\ &= \left(\int_K \frac{1}{M_r}\overline{\psi_r(t)}\psi_r(t)\, d\lambda(t) \right) S(x_r) \\ &= S(x_r). \end{aligned}$$

Consequently, for every x in X we get

$$\begin{aligned} S(x) &= \sum_{i=1}^{m} \xi_i(x) \int_K F(t)(x_i)G(t)\, d\lambda(t) \\ &= \int_K F(t)(\sum_{i=1}^{m} \xi_i(x)x_i)G(t)\, d\lambda(t) \\ &= \int_K F(t)(x)G(t)\, d\lambda(t). \end{aligned}$$

This shows that (F, G) is a continuous O-frame for the operator S.

Theorem 2.2 *Let $S \in B(X, Y)$ and (F, G) be an O-frame for the operator S, and for every θ in Y^* the integral $\int_K F(t)G(t)(\theta)d\lambda(t)$ exits. Then (G, F) is an O-frame for the operator S^* and moreover S is weakly compact.*

Proof. First, we show that (G, F) is an O-frame for S^*. For every x in X and θ in Y^*,

$$\begin{aligned} S^*(\theta)(x) &= \theta(S(x)) \\ &= \theta \left(\int_K F(t)(x)G(t)d\lambda(t) \right) \\ &= \left(\int_K F(t)(x)G(t)(\theta)d\lambda(t) \right) \\ &= \left(\int_K F(t)G(t)(\theta)d\lambda(t) \right)(x). \end{aligned}$$

Therefore,

$$S^*(\theta) = \left(\int_K F(t)G(t)(\theta)d\lambda(t) \right), \tag{2.1}$$

this implies that (G, F) is an O-frame for S^*. By 2.1, in the same manner we can see for every ξ in X^{**},

$$S^{**}(\xi) = \int_K \xi(F(t))G(t)d\lambda(t).$$

Hence , for every ξ in X^{**} we have $S^{**}(\xi) \in Y$. So, by [3, Theorem VI.5.5], S is weakly compact.

Definition 2.2 *Let $F \in C(K, X^*)$, $G \in C(K, Y)$ and (F, G) be a continuous O-frame for an operator $S \in B(X, Y)$. The pair (F, G) is called boundedly complete if for each $\psi \in X^{**}$, the integral $\int_K \psi(F(t))G(t)d\lambda(t)$ exists.*

In following theorem, we will use the symbol $\mathcal{K}(K)$ to denote the set of all compact subsets of K.

Theorem 2.3 *Let $S \in B(X, Y)$ and (F, G) be a continuous O-frame for the operator S, such that*

$$\sup_{U \in \mathcal{K}(K), x \in X} \| \int_U F(t)(x)G(t)d\lambda(t) \| < \infty,$$

and let for every x in X and Borel measurable function $T : K \to \mathbb{C}$, if

$$\sup_{U \in \mathcal{K}(K)} \left\| \int_U T(t)F(t)(x)G(t)d\lambda(t) \right\| < \infty,$$

then the function

$$t \mapsto T(t)F(t)(x)G(t), \qquad (t \in K)$$

is compact supported. Then, (F, G) *is a boundedly complete continuous O-frame for the operator* S.

Proof. For every U in $\mathcal{K}(K)$, the function $S_U : X \to Y$ defined by $S_U(x) := \int_U F(t)(x)G(t)d\lambda(t), \quad (x \in X)$ is bounded. Let S_U^* be the adjoint operator of S_U. By [1], for every x in X and η in Y^* we have

$$\begin{aligned} S_U^*(\eta)(x) &= \eta(S_U(x)) \\ &= \eta\left(\int_U F(t)(x)G(t)d\lambda(t)\right) \\ &= \int_U F(t)(x)\eta(G(t))d\lambda(t) \\ &= \left(\int_U \eta(G(t))F(t)d\lambda(t)\right)(x). \end{aligned}$$

Consequently, for every $U \in \mathcal{K}(K)$ and η in Y^* we get

$$S_U^*(\eta) = \int_U \eta(G(t))F(t)d\lambda(t).$$

Again, by [1], for every η in Y^* and ψ in X^{**} we have

$$\begin{aligned} S_U^{**}(\psi)(\eta) &= \psi(S_U^*(\eta) \\ &= \psi\left(\int_U \eta(G(t))F(t)d\lambda(t)\right) \\ &= \int_U \eta(G(t))\psi(F(t))d\lambda(t) \\ &= \eta\left(\int_U G(t)\psi(F(t))d\lambda(t)\right). \end{aligned}$$

So, for every U in $\mathcal{K}(K)$ and ψ in X^{**}

$$S_U^{**}(\psi) = \Phi\left(\int_U G(t)\psi(F(t))d\lambda(t)\right),$$

where $\Phi : Y \to Y^{**}$ is the canonical mapping. Since Φ is an isometry, we get

$$\begin{aligned} \left\|\int_U G(t)\psi(F(t))d\lambda(t)\right\| &= \left\|\Phi\left(\int_U G(t)\psi(F(t))d\lambda(t)\right)\right\| \\ &= \|S_U^{**}(\psi)\| \\ &\le \|S_U\|\,\|\psi\|. \end{aligned}$$

Consequently,

$$\sup_{U \in \mathcal{K}(K)} \left\|\int_U G(t)\psi(F(t))d\lambda(t)\right\| < \infty. \tag{2.2}$$

Let for every t in K, $F(t) \neq 0$. Then, there exists a non-zero x in X such that for every t in K, $F(t)(x) \neq 0$. Let $x \in X$ and ψ in X^{**} , define

$$T(t) := \frac{\psi(F(t))}{F(t)(x)}, \qquad (t \in K).$$

Hence, by (2.2)

$$\sup_{U\in\mathcal{K}(K)}\left\|\int_U T(t)F(t)(x)G(t)d\lambda(t)\right\| < \infty.$$

So, by hypothesis, the integral

$$\int_K \psi(F(t))G(t)d\lambda(t)$$

exists. Therefore, (F, G) is a boundedly complete continuous O-frame for the operator S.

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