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The mixed problem for nonlinear Tymoshenko systems

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Abstract. *In this paper, a mixed problem is considered for a system of fourth order multidimensional semilinear hyperbolic equations with constant coefficients. Under suitable hypotheses we prove existence of a global solution.*

Keywords. system of fourth order wave equation · global solvability · Timoshenko system · mixed problem.

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1 Introduction

The linear theory of the vibration of the bar is described by a system of differential equations

$$\left.\begin{array}{l} EIy_{xxxx} + \rho A y_{tt} - \rho A e \theta_{tt} = g_1(t,x) \\ EC_w \theta_{xxxx} - GC\theta_{xx} - \rho A e y_{tt} + \rho\left(I + Ae^2\right)\theta_{tt} = g_2\left(t,x\right) \end{array}\right\}, \tag{1.1}$$

where $0 < x < l$, $0 < t < T$, $l > 0$, $T > 0$ are given numbers, $y\left(x,t\right)$ is a transverse displacement, $\theta\left(x,t\right)$ is an angle of cross-section of a bar, E is the Young modulus, I is a polar moment of inertia of the cross section with respect to its center of gravity, ρ is a density of the material of the bar, A is a cross-sectional area, e is a distance from center of gravity to center of torsion, C_w is a sectorial moment of inertia of the cross section, G is a shear modulus, C is a geometric rigidity of free torsion, EC_w is a stiffness of bending torsion, GC is a stiffness of free torsion [1,2].

2 Formulation of the problem and main results

In this work we study a mixed problem for a multidimensional semilinear analogue of the system (1.1). Let $\Omega \subset R^n$ be a bounded domain with smooth boundary Γ. In the domain

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$Q_T = [0,T] \times \Omega$ we consider the mixed problem:

$$\left.\begin{array}{l} EI\triangle^2 y + \rho A y_{tt} - \rho A e \theta_{tt} = f_1(t,x,y,\theta) \\ EC_w \triangle^2 \theta - GC\triangle\theta - \rho A e y_{tt} + \rho\left(I + Ae^2\right)\theta_{tt} = f_2\left(t,x,y,\theta\right) \end{array}\right\} \tag{2.1}$$

with boundary conditions

$$\left.\begin{array}{l} y\left(t,x\right) = \Delta y\left(t,\ x\right) = 0, \quad t \in [0,T], \quad x \in \Gamma \\ \theta\left(t,x\right) = \Delta\theta\left(t,\ x\right) = 0, \quad t \in [0,T], \quad x \in \Gamma \end{array}\right\} \tag{2.2}$$

and initial conditions

$$\left.\begin{array}{ll} y\left(0,x\right) = y_0\left(x\right), & y_t\left(0,x\right) = y_1\left(x\right) \\ \theta\left(0,x\right) = \theta_0\left(x\right), & \theta_t\left(0,x\right) = \theta_1\left(x\right) \end{array}\right\}, \quad x \in \Omega, \tag{2.3}$$

where $t \in [0,T]$, $x \in \Omega, \Delta$ is a Laplace operator. We assume that all the coefficients $A,\ C,\ E,\ G,\ I,\ C_w,\ e$ and ρ are positive constants. We denote by $\left\langle w^1, w^2\right\rangle$ the scalar product in $L_2(0,1)$. We will investigate the problem (2.1) - (2.3) in the Hilbert space $\mathcal{H} = L_2\left(0,1\right) \times L_2\left(0,1\right)$ with the scalar product of $\mathcal{H}$

$$\left\langle w^1, w^2\right\rangle = \frac{I}{C_w}\left\langle y^1, y^2\right\rangle_{L_2(0,1)} + \left\langle \theta^1, \theta^2\right\rangle_{L_2(0,1)},$$

where $w^k = \begin{pmatrix} y_k \\ \theta_k \end{pmatrix}$, $k = 1,2$.

Let us define the functional spaces

$$\widehat{H_0^{2s}} = \left\{u : u \in H^{2s} = W_2^{2s}\left(\Omega\right), \Delta^l u\left(x\right) = 0,\ x \in \Gamma,\ l = 0,1,\ \ldots,\ s-1\right\}.$$

We introduce the following notation: $\mathcal{H}_1 = \widehat{H_0^2} \times \widehat{H_0^2}$, $\mathcal{H}_2 = \widehat{H_0^4} \times \widehat{H_0^4}$.

Let L be a linear operator defined in the space, where

$$D\left(L\right) = \mathcal{H}_2,$$

$$Lw = \begin{bmatrix} \frac{E(I+Ae^2)}{\rho A}\triangle^2 & \frac{eEC_w}{\rho I}\triangle^2 \\ \frac{eE}{\rho}\triangle^2 & \frac{EC_w}{\rho I}\triangle^2 \end{bmatrix} w, \qquad w = \begin{pmatrix} y \\ \theta \end{pmatrix} \in D(L).$$

We also define the linear operator L_1 as follows

$$D\left(L_1\right) = \mathcal{H}_1,$$

$$L_1 w = \begin{bmatrix} 0 & -\frac{eGC}{\rho I}\triangle \\ 0 & -\frac{GC}{\rho I}\triangle \end{bmatrix} w, \quad w = \begin{pmatrix} y \\ \theta \end{pmatrix} \in D\left(L_1\right).$$

Then problem (2.1), (2.3) can be written in the form

$$\left.\begin{array}{ll} w_{tt} + Lw + L_1 w = G\left(t,\ w\right) & \\ w\left(0\right) = w_0, & w_t\left(0\right) = w_1 \end{array}\right\}, \tag{2.4}$$

where $w(t) = \begin{pmatrix} y(t,x) \\ \theta(t,x) \end{pmatrix}$, $w_0 = \begin{pmatrix} y_0\left(x\right) \\ \theta_0\left(x\right) \end{pmatrix}$, $w_1 = \begin{pmatrix} y_1\left(x\right) \\ \theta_1\left(x\right) \end{pmatrix}$. Here $G(t,x)$ is a nonlinear operator defined as follows

$$G\left(t,\ w\right) = \begin{pmatrix} h_1(t,x,w) \\ h_2(t,x,w) \end{pmatrix},$$

where

$$h_1(t,x,w) = \frac{I+Ae^2}{\rho AI} f_1(t,x,y,\theta) + \frac{e}{\rho I} f_2(t,x,y,\theta)$$

$$h_2(t,x,w) = \frac{e}{\rho I} f_1(t,x,y,\theta) + \frac{1}{\rho I} f_2(t,x,y,\theta).$$

Suppose that the following conditions are satisfied:

i) $f_k(t,x,y,\theta) \in C^1\left([0,T]\times\Omega\times R^2\right),\ k=1,2;$

ii) if $n \ge 4$, then

$$|f_k(t,x,y,\theta)| \le c\left(1+|y|^p+|\theta|^q\right),$$

$$\left|{f'}_{k_t}(t,x,y,\theta)\right| \le c\left(1+|y|^p+|\theta|^q\right),$$

$$\left|{f'}_{k_y}(t,x,y,\theta)\right| \le c\left(1+|y|^{p-1}+|\theta|^{q-1}\right),$$

$$\left|{f'}_{k_\theta}(t,x,y,\theta)\right| \le c\left(1+|y|^{p-1}+|\theta|^{q-1}\right),$$

where $p,\ q \ge 1$ and $p,\ q \le \frac{n}{n-4}$ if $n > 4$.

Using the definitions of L, L_1 and $G(t,\ w)$ the following lemmas are proved.

Lemma 2.1 *L is a positive self-adjoint operator in $\mathcal{H}$.*

Lemma 2.2 *The linear operator L_1 is subjected to the operator $L^{\frac{1}{2}}$, i.e, $D(L^{\frac{1}{2}}) \subset D(L)$ and $\|L_1 w\|^2_{\mathcal{H}} \le c\langle Lw, w\rangle,\ c>0,\ w \in D\left(L^{\frac{1}{2}}\right)$,*

Lemma 2.3 *Let the conditions i) and ii) be satisfied. Then $G(t,\ w)$ acts from $\mathcal{H}_1$ to $\mathcal{H}$ and satisfies the local Lipschits condition, i.e. for any $t_1,\ t_2 \in [0,\ T]$ and $w^1, w^2 \in \mathcal{H}_1$ the following inequalities hold*

$$\left\|G\left(t_1,\ w^1\right) - G\left(t_2,\ w^2\right)\right\|_{\mathcal{H}} \le c\left(\left\|w^1\right\|_{\mathcal{H}_1},\ \left\|w^2\right\|_{\mathcal{H}_1}\right)\times\left[|t_1-t_2| + \left\|w^1-w^2\right\|_{\mathcal{H}_1}\right],$$

where $c(\cdot) \in C(R^2_+, R_+)$.

From Lemmas 2.1-2.3 and [3] theorem on local solvability follows:

Theorem 2.1 *Let the conditions i) and ii) be satisfied. Then for any $(y_0,\ \theta_0) \in \mathcal{H}_1, (y_1,\ \theta_1) \in \mathcal{H}$ there is a $T' > 0$, such that the problem (2.1)-(2.3) has a unique solution (y,θ), where*

$$(y,\theta) \in C(\left[0,T'\right], \mathcal{H}_1\times\mathcal{H}_1) \cap C(\left[0,T'\right], \mathcal{H}\times\mathcal{H}),$$

Moreover, if T_{max} is the length of the maximum interval of the existence of solutions, then one of the following alternatives is fulfilled:

i) $\lim\limits_{t\to T_{max}-0}\left[\|y_t(t,.)\|^2_{L_2(\Omega)} + \|\theta_t(t,.)\|^2_{L_2(\Omega)} + \|y(t,.)\|^2_{H^1} + \|\theta(t,.)\|^2_{H^1}\right] = +\infty$

or

ii) $T_{max} = T$.

Now consider the mixed problem

$$\left.\begin{array}{l}\rho A y_{tt} - \rho Ae\theta_{tt} + EI\triangle^2 y + |y|^p|\theta|^{p+2}y = g_1(t,x)\\ -\rho Ae y_{tt} + \rho\left(I+Ae^2\right)\theta_{tt} + EC_w\triangle^2\theta - GC\triangle\theta + |y|^{p+2}|\theta|^p\theta = g_2(t,x)\end{array}\right\} \qquad (2.5)$$

with boundary and initial conditions (2.2),(2.3).

We will suppose that, the following conditions are satisfied

1) $g_k(t,x) \in L_2((0,T); L_2(\Omega)),\ k=1,2,$

2) $p > 0$ and $0 < p < \frac{8}{n-4}$ if $n > 4$.

Theorem 2.2 *Let the conditions 1) and 2) be satisfied. Then for any*

$$y_0,\ \theta_0 \in \widehat{H_0^2},\quad y_1,\ \theta_1 \in L_2(\Omega).$$

There exist functions $y, \theta : (0,T) \to L_2(\Omega)$ such that

$$y, \theta \in L^\infty([0,T], \widehat{H_0^2}),$$
$$y', \theta' \in L^\infty([0,T], L_2(\Omega)),$$
$$y\theta \in L^\infty([0,T], L^{p+2}(\Omega)),$$

satisfying the nonlinear systems

$$\rho A y_{tt} - \rho A e \theta_{tt} + EI\triangle^2 y + |y|^p |\theta|^{p+2} y = g_1(t,x)$$

in $L_2(0,T;H^{-2}(\Omega)) \cap L^q(Q)$,

$$-\rho A e y_{tt} + \rho\left(I + Ae^2\right)\theta_{tt} + EC_w\triangle^2\theta - GC\triangle\theta + |y|^{p+2}|\theta|^p\theta = g_2(t,x)$$

in $L_2(0,T;H^{-2}(\Omega)) \cap L^q(Q)$.

3 Proof of Theorem 2.2.

We give the proof of Theorem 2.2 for $n > 4$. In the case $n \le 4$, Theorem 2.2 follows from Theorem 2.1 and the corresponding a priori estimate.

We define the numbers $q > 1$ and $q' > 1$ in the following way:

$$q = \frac{2n}{(n-4)(p+2) + 2n(p+1)}, \quad q' = \frac{2n(p+2)}{(n+4)(p+2) - 2n(p+1)}$$

It is obvious that $\frac{1}{q} + \frac{1}{q'} = 1$. We choose $\rho > 1$ and $\rho' > 1$ in the following way

$$\rho = \frac{p+2}{p+1}\cdot\frac{1}{q} \quad , \quad \rho' = \frac{\rho}{\rho - 1} \quad .$$

Using the Holder inequality, we have

$$\int_\Omega \left(|y|^{p+2}|\theta|^{p+1}\right)^q dx \le \left(\int_\Omega |y\theta|^{(p+1)qp} dx\right)^{1/p} \cdot \left(\int_\Omega |y|^{q\rho'} dx\right)^{1/\rho'}$$

Taking into account (2.5) and embedding theorems, we obtain that

$$\int_\Omega \left(|y|^{p+2}|\theta|^{p+1}\right)^q dx \le \|y\theta\|^{(p+1)q}_{L_{p+2}(\Omega)} \cdot \|y\|^q_{L_{\frac{2n}{n-4}}} \le$$
$$\le c\ \|y\theta\|^{(p+1)q}_{L_{p+2}(\Omega)} \cdot \|y\|^q_{L_{\widehat{H_0^2}}}. \tag{3.1}$$

We will solve the problem (2.5), (3.1) by the Galerkin method :

Let $\{w_k\}$ be a basis in $\widehat{H_0^2} \cap L^{q_1}(\Omega)$, where $q_1 = max\{p+2,\ q'\}$. We denote by V_m the subspace spanned by the first m vectors $\{w_1,\ w_2,\ \ldots,\ w_m\}$

Let

$$y_m = \sum_{j=1}^m u_{j_m}(t)\, w_j,$$

$$\theta_m = \sum_{j=1}^{m} \upsilon_{j_m}(t)\, w_j,$$

where $(u_{j_m}(t),\ \upsilon_{j_m}(t))$ is defined from the following system of equations:

$$\rho A\left(y''_m, w_j\right) - \rho Ae\left(\theta''_m,\ w_j\right) + EI\left(\triangle y,\ \triangle w_j\right) + \left(|y_m|^p|\theta_m|^{p+2}y_m, w_j\right)$$

$$= \left(g_1\left(t, x\right), w_j\right) \tag{3.2}$$

$$-\rho Ae\left(y''_m,\ w_j\right) + \rho\left(I + Ae^2\right)\left(\theta''_m,\ w_j\right) + EC_w\left(\triangle\theta,\ \triangle w_j\right) + GC\left(\nabla_x\theta,\ \nabla w_j\right)$$

$$+\left(|y_m|^p|\theta_m|^{p+2}\theta_m, w_j\right) = \left(g_2\left(t,\ x\right), w_j\right). \tag{3.3}$$

The system (3.2), (3.3) is considered with the initial conditions

$$\left.\begin{array}{ll} y_m\left(0\right) = y_{0m}, & y'_m\left(0\right) = y_{1m} \\ \theta_m\left(0\right) = \theta_{0m}, & \theta'_m\left(0\right) = \theta_{1m} \end{array}\right\}. \tag{3.4}$$

Here $y_{0m},\ \ y_{1m},\ \theta_{0m},\ \theta_{1m} \in V_m$ and

$$y_{0m} \to y_0 \quad \text{in} \quad \widehat{H^2} \cap L^p\left(\Omega\right),$$

$$y_{1m} \to y_1 \quad \text{in} \quad L_2\left(\Omega\right),$$

$$\theta_{0m} \to \theta_0 \quad \text{in} \quad \widehat{H^2} \cap L^p\left(\Omega\right),$$

$$\theta_{1m} \to \theta_1 \quad \text{in} \quad L_2\left(\Omega\right).$$

By the theorem on the existence of the solutions of the Cauchy problem for the system of differential equations, there exist numbers $T_m > 0$, such that the problem (3.2) - (3.4) has a solution

$$(u_{j_m}(t),\ \upsilon_{j_m}(t)) \in C^1\left[0, T_m\right]\ ,\ \ j = 1, \ldots\ ,\ m\ \ ,\ \ m = 1,\ 2, \ldots\ .$$

A priori estimates

We multiply the equation (3.2) by $y'_{j_m}(t)$ and sum from $j = 0\ \ to\ \ \ j = m.$ Then we have

$$\frac{\rho A}{2}\frac{d}{dt}\int_\Omega \left|y'_m(t,x)\right|^2 dx - \rho Ae\int_\Omega \theta''_m\left(t,x\right) y'_m\left(t,x\right)dx + \frac{EI}{2}\frac{d}{dt}\int_\Omega \left|\triangle y_m\left(t,x\right)\right|^2 dx$$

$$+\int_\Omega |y_m\left(t,x\right)|^p|\theta_m\left(t,x\right)|^{p+2}y_m\left(t,x\right)y'_m\left(t,x\right)dx = \int_\Omega g_1\left(t,x\right)y'_m\left(t,x\right)dx. \tag{3.5}$$

In the similar way, multiplying the equation (3.3) by $\theta'_{j_m}(t)$ and summing from $j = 0$ to $j = m$, we have :

$$-\rho Ae\int_\Omega y''_m\left(t,x\right)\theta'_m\left(t,x\right)dx$$

$$+\frac{\rho\left(I + Ae^2\right)}{2}\frac{d}{dt}\int_\Omega \left|\theta'_m\left(t,x\right)\right|^2 dx + \frac{EC_w}{2}\frac{d}{dt}\int_\Omega \left|\triangle\theta_m\left(t,x\right)\right|^2 dx$$

$$+\frac{GC}{2}\int_\Omega \left|\nabla\theta_m\left(t,x\right)\right|^2 dx + \int_\Omega |y_m\left(t,x\right)|^{p+2}|\theta_m\left(t,x\right)|^p\theta_m\left(t,x\right)\theta'_m\left(t,x\right)dx$$

$$= \int_{\Omega} g_2 (t, x)\, \theta'_m (t, x)\, dx. \tag{3.6}$$

Summing (3.5) and (3.6) , we get

$$\frac{d}{dt}\left[\frac{\rho A}{2}\int_{\Omega}\left|y'_m(t,x)\right|^2 dx\right.$$

$$+\frac{EI}{2}\int_{\Omega}|\triangle y_m(t,x)|^2 dx+\frac{\rho\left(I+Ae^2\right)}{2}\int_{\Omega}\left|\theta'_m(t,x)\right|^2 dx$$

$$+\frac{EC_w}{2}\int_{\Omega}|\triangle \theta_m(t,x)|^2 dx$$

$$\left.-\frac{GC}{2}\int_{\Omega}|\nabla \theta_m(t,x)|^2 dx-\rho A\int_{\Omega}\theta'_m(t,x)\, y'_m(t,x)\, dx\right]$$

$$+\frac{1}{p+2}\frac{d}{dt}\quad\int_{\Omega}|y_m(t,x)|^{p+2}|\theta_m(t,x)|^{p+2}dx$$

$$=\int_{\Omega} f_1(t,x)\, y'_m(t,x)\, dx+\int_{\Omega} f_2(t,x)\,\theta'_m(t,x)\, dx. \tag{3.7}$$

Using the Holder and Young inequality we have

$$\rho Ae\int_{\Omega}\theta'_m(t,x)\, y'_m(t,x)\, dx\le\frac{\rho Ae}{2\varepsilon}\int_{\Omega}\left|\theta'_m(t,x)\right|^2 dx+\frac{\rho A}{2}\varepsilon\int_{\Omega}\left|y'_m(t,x)\right|^2 dx, \tag{3.8}$$

where $0<\varepsilon<1$ and

$$\frac{\rho Ae}{2\varepsilon}<\frac{\rho\left(I+Ae^2\right)}{2}\ , \tag{3.9}$$

i.e.

$$0<\varepsilon<\frac{Ae}{I+Ae^2}.$$

It is obvious that $0<\frac{Ae}{I+Ae^2}<1$.

Estimating the right-hand side from above, we obtain

$$\int_{\Omega} f_1(t,x)\, y'_m(t,x)\, dx+\int_{\Omega} f_2(t,x)\,\theta'_m(t,x)\, dx$$

$$\le c+\int_{\Omega}\left|y'_m(t,x)\right|^2 dx+\int_{\Omega}\left|\theta'_m(t,x)\right|^2 dx. \tag{3.10}$$

From (3.5)-(3.10), we obtain that

$$\frac{\rho A}{2}(1-\varepsilon)\int_{\Omega}\left|y'_m(t,x)\right|^2 dx+\left(\frac{\rho\left(I+Ae^2\right)}{2}-\frac{\rho Ae}{2\varepsilon}\right)\int_{\Omega}\left|\theta'_m(t,x)\right|^2 dx$$

$$+\frac{EI}{2}\int_{\Omega}|\triangle y_m(t,x)|^2 dx+\frac{1}{p+2}\int_{\Omega}|y_m(t,x)\,\theta_m(t,x)|^{p+2}dx$$

$$\le\frac{1}{2}\int_0^t\left[\int_{\Omega}\left|y'_m(t,x)\right|^2 dx+\int_{\Omega}\left|\theta'_m(t,x)\right|^2 dx\right] ds\ \ .$$

Using Gronwall's Lemma, from here we get:

$$\int_{\Omega} \left|{y'}_m(t,x)\right|^2 dx \le c, \tag{3.11}$$

$$\int_{\Omega} \left|{\theta'}_m(t,x)\right|^2 dx \le c, \tag{3.12}$$

$$\int_{\Omega} \left|\triangle y_m(t,x)\right|^2 dx \le c, \tag{3.13}$$

$$\int_{\Omega} \left|\triangle \theta_m(t,x)\right|^2 dx \le c, \tag{3.14}$$

$$\int_{\Omega} \left|y_m(t,x)\,\theta_m(t,x)\right|^{p+2} dx \le c. \tag{3.15}$$

On the other hand, using (3.11), (3.13) - (3.15), embedding theorem [4], we obtain that

$$\int_{\Omega} \left(|y_m(t,x)|^{p+2}|\theta_m(t,x)|^{p+1}\right)^q dx$$

$$\le \|y_m(t,x)\,\theta_m(t,x)\|^{(p+1)q}_{L_{p+2}(\Omega)}\|y_m\|^q_{\widehat{H_0^2}} \le c. \tag{3.16}$$

Hence we have the following estimate

$$\int_{\Omega} \left(|y_m(t,x)|^{p+1}|\theta_m(t,x)|^{p+2}\right)^q dx \le c. \tag{3.17}$$

In the similar way, we obtain

$$\int_{\Omega} \left(|y_m(t,x)|^{p+2}|\theta_m(t,x)|^{p+1}\right)^q dx \le c \quad . \tag{3.18}$$

By (3.11)-(3.15) from the sequence $\{(y_m, \theta_m)\}$ we can select a sequence $\{(y_\mu, \theta_\mu)\}$, such that

$$y_\mu \rightharpoonup y \quad \text{weakly star in in} \quad L_\infty\left(0,T;\ H_0^2\right), \tag{3.19}$$

$$\theta_\mu \rightharpoonup \theta \quad \text{weakly star in} \quad L_\infty\left(0,T;\ H_0^2\right), \tag{3.20}$$

$$y'_\mu \rightharpoonup y' \quad \text{weakly star in} \quad L_\infty\left(0,T;\ L_2(\Omega)\right), \tag{3.21}$$

$$\theta'_\mu \rightharpoonup \theta' \quad \text{weakly star in} \quad L_\infty\left(0,T;\ L_2(\Omega)\right), \tag{3.22}$$

$$y_\mu\theta_\mu \rightharpoonup \chi \quad \text{weakly star in} \quad L_\infty\left(0,T;\ L_{p+2}(\Omega)\right). \tag{3.23}$$

It follows from (3.19)-(3.22) that

$$y_\mu \longrightarrow y \quad \text{strongly in} \quad L_2(Q_T), \tag{3.24}$$

$$\theta_\mu \longrightarrow \theta \quad \text{strongly in} \quad L_2(Q_T), \tag{3.25}$$

$$y_\mu \longrightarrow y \quad \text{a.e. in} \quad Q_T, \tag{3.26}$$

$$\theta_\mu \longrightarrow \theta \quad \text{a.e. in}\ Q_T. \tag{3.27}$$

It follows from (3.26), (3.27) that

$$|y_\mu|^p|\theta_\mu|^{p+2}y_\mu \longrightarrow |y|^p|\theta|^{p+2}y \quad \text{a.e} \quad Q_T \tag{3.28}$$

and

$$|y_\mu|^{p+2}|\theta_\mu|^p\theta_\mu \longrightarrow |y|^{p+2}|\theta|^p\theta \quad \text{a.e} \quad Q_T. \tag{3.29}$$

On the other hand, it follows from (3.17) that $\left\{|y_\mu|^p|\theta_\mu|^{p+2}y_\mu\right\}$ is bounded in $L_\infty\left(0,T;\ L^q\left(\Omega\right)\right)$. Then in view of (3.28)

$$|y_\mu|^{p+2}|\theta_\mu|^p y_\mu \rightharpoonup |y|^{p+2}|\theta|^p y \quad \text{weakly star in} \quad L_\infty\left(0,T;\ L^q\left(\Omega\right)\right).$$

Similarly, it follows from (3.18) and (3.29) that

$$|y_\mu|^p|\theta_\mu|^{p+2}\theta_\mu \rightharpoonup |y|^p|\theta|^{p+2}\theta \ \text{ weakly star in} \quad L_\infty\left(0,T;\ L^q\left(\Omega\right)\right).$$

It follows from (3.24), (3.25) that

$$y_\mu\theta_\mu \longrightarrow y\theta \quad \text{in} \ \ L^1\left(Q\right). \tag{3.30}$$

By (3.23), (3.30)

$$\chi = y\theta.$$

Passing to the limit is carried out as follows:

Let $\eta \in D(0,T)$. Multiplying both sides of (3.3) by η and integrating parts, we obtain that

$$-\rho A\int_0^T \langle y_m',\ \eta' w_j\rangle dt + \rho Ae\int_0^T \langle \theta_m',\ \eta' w_j\rangle dt + EI\int_0^T \langle \triangle y,\ \eta w_j\rangle dt +$$

$$+\int_0^T \left\langle |y_m|^p|\theta_m|^{p+2}y_m, \eta w_j\right\rangle dt = \int_0^T \langle g_1, \eta w_j\rangle dt.$$

Passing to the limit, hence we get :

$$-\rho A\int_0^T \langle y',\ \eta' w_j\rangle dt + \rho Ae\int_0^T \langle \theta',\ \eta' w_j\rangle dt + EI\int_0^T \langle \triangle y,\ \eta w_j\rangle dt$$

$$+\int_0^T \left\langle |y|^p|\theta|^{p+2}y, \eta w_j\right\rangle dt = \int_0^T \langle g_1, \eta w_j\rangle dt.$$

The set of functions $\{\eta w_1,\ \eta w_2, \dots ,\ \eta w_n\}$ is dense in $H_0^1(0,T;H_0^1\left(\Omega\right)) \cap L^{p_1}\left(Q\right)$, therefore

$$-\rho A\int_0^T \langle y',\ z'\rangle dt + \rho Ae\int_0^T \langle \theta',\ \eta' w_j\rangle dt + EI\int_0^T \langle \triangle y_{xx},\ \triangle z\rangle dt$$

$$+\int_0^T \left\langle |y|^p|\theta|^{p+2}y, z\right\rangle dt = \int_0^T \langle g_1, z\rangle dt.$$

It is true for any $z \in H_0^1(0,T;H_0^2\left(\Omega\right)) \cap L^{p_1}\left(Q\right)$.

Similarly, we get that, for any $z \in H_0^1(0,T;H_0^2\left(\Omega\right)) \cap L^{p_1}\left(Q\right)$

$$-\rho Ae\int_0^T \langle y',\ z'\rangle dt + \rho\left(I + Ae^2\right)\int_0^T \langle \theta',\ z'\rangle dt + EC_w\int_0^T \langle \triangle\theta, \triangle z\rangle dt$$

$$+GC\int_0^T \langle \triangle\theta,\ \nabla w_j\rangle dt + \int_0^T \left\langle |y|^{p+2}|\theta|^p\theta, z\right\rangle dt = \int_0^T \langle g_2, z\rangle dt.$$

The uniqueness result via the standard energy method (see[5], pp.32-33).

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