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On the error of approximation by radial basis functions with fixed centers

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Abstract. *We consider the problem of approximation of a continuous multivariate function by sums of two radial basis functions in the uniform norm. We obtain a formula for the approximation error in terms of functionals generated by closed paths.*

Keywords. radial basis function · path · extremal element · approximation error

Mathematics Subject Classification (2010): 41A30, 41A50, 41A63

1 Introduction

In modern approximation theory, radial basis functions play an essential role. A *radial basis function* is a multivariate function of the form

$$F(\mathbf{x}) = r(\|\mathbf{x}-\mathbf{c}\|),$$

where $r : \mathbb{R} \to \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_d)$ is the variable, $\mathbf{c} \in \mathbb{R}^d$ and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. The point $\mathbf{c}$ is called the center of F. In other words, a radial basis function is a multivariate function constant on the spheres $\|\mathbf{x}-\mathbf{c}\| = \alpha,\ \alpha \in \mathbb{R}$. These functions and their linear combinations arise naturally in many fields, especially in RBF (radial basis function) neural networks (see, e.g., [8, 16, 17, 20–25]).

Consider the following set of functions

$$\mathcal{D} = \left\{ r_1(\|\mathbf{x}-\mathbf{c}_1\|) + r_2(\|\mathbf{x}-\mathbf{c}_2\|) :\ r_i \in C(\mathbb{R}),\ i = 1, 2 \right\}.$$

That is, we fix centers $\mathbf{c}_1$ and $\mathbf{c}_2$ and consider linear combinations of radial basis functions with these centers.

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Let $f(\mathbf{x})$ be a given continuous function on some compact subset Q of $\mathbb{R}^d$. In this paper, we will obtain a formula for computation of the approximation error

$$E(f) = E(f, \mathcal{D}) \stackrel{def}{=} \inf_{r\in\mathcal{D}} \|f - r\|.$$

Note that if there exists $r_0 \in \mathcal{D}$ such that

$$\|f - r_0\| = E(f),$$

then r_0 is called an extremal element.

2 The approximation error formula

Suppose Q is a compact set in $\mathbb{R}^d$ and $\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}^d$ are fixed points.

Definition 2.1. *A finite or infinite ordered set $p = (\mathbf{p}_1, \mathbf{p}_2, ...) \subset Q$ with $\mathbf{p}_i \neq \mathbf{p}_{i+1}$, and either $\|\mathbf{p}_1 - \mathbf{c}_1\| = \|\mathbf{p}_2 - \mathbf{c}_1\|, \|\mathbf{p}_2 - \mathbf{c}_2\| = \|\mathbf{p}_3 - \mathbf{c}_2\|, \|\mathbf{p}_3 - \mathbf{c}_1\| = \|\mathbf{p}_4 - \mathbf{c}_1\|, ...$ or $\|\mathbf{p}_1 - \mathbf{c}_2\| = \|\mathbf{p}_2 - \mathbf{c}_2\|, \|\mathbf{p}_2 - \mathbf{c}_1\| = \|\mathbf{p}_3 - \mathbf{c}_1\|, \|\mathbf{p}_3 - \mathbf{c}_2\| = \|\mathbf{p}_4 - \mathbf{c}_2\|, ...$is called a path with respect to the centers $\mathbf{c}_1$ and $\mathbf{c}_2$.*

In the above definition, we alternate distances from two fixed points. Paths have many different variations. For example, instead of points, one can take two hyperplanes $\mathbf{a}^i \cdot \mathbf{x} = \alpha_i$, $i = 1, 2$, where " $\cdot$ " denotes the standard scalar product in $\mathbb{R}^d$, and alternate distances from these two hyperplanes. Certainly, in $\mathbb{R}^2$, hyperplanes turn into straight lines, thus one can talk about distances from straight lines. Paths with respect to two straight lines in $\mathbb{R}^2$ were first considered by Braess and Pinkus [5]. They showed that paths give geometric means of deciding if a set of points $\{\mathbf{x}^i\}_{i=1}^m \subset \mathbb{R}^2$ has the "non-interpolation property" for so called ridge functions (for this terminology see [5]). Ismailov and Pinkus [10] used paths with respect to two hyperplanes in $\mathbb{R}^d$ to solve the problem of interpolation on straight lines by ridge functions. If straight lines are fixed as the coordinate lines in $\mathbb{R}^2$, then the corresponding set of points $(\mathbf{p}_1, \mathbf{p}_2, ...)$ turn into "bolts of lightning" (see, e.g., [1,9,19]). It is well known that the idea of bolts was first introduced by Diliberto and Straus [6] and played an essential role in problems of approximation by sums of univariate functions (see, e.g., [6,7,9,14,15,18,19]). Note that the name "bolt of lightning" is due to Arnold [1]. Ismailov [11,12] generalized paths to those with respect to a finite set of functions. Paths with respect to n arbitrarily fixed functions turned out to be very useful in problems of representation by linear superpositions.

In the sequel, we use the term "path" instead of the long expression "path with respect to the centers $\mathbf{c}_1$ and $\mathbf{c}_2$". A finite path $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{2n})$ is said to be closed if $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{2n}, \mathbf{p}_1)$ is also a path.

We associate a closed path $p = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{2n})$ with the functional

$$G_p(f) = \frac{1}{2n} \sum_{k=1}^{2n} (-1)^{k+1} f(\mathbf{p}_k).$$

This functional has the following obvious properties:

(a) If $r \in \mathcal{D}$, then $G_p(r) = 0$.

(b) $\|G_p\| \leq 1$ and if $p_i \neq p_j$ for all $i \neq j$, $1 \leq i, j \leq 2n$, then $\|G_p\| = 1$.

The following lemma holds.

Lemma 2.1. *Let a compact set Q have closed paths. Then*

$$\sup_{p\subset Q}|G_p(f)|\leq E\left(f\right), \tag{2.1}$$

where the sup is taken over all closed paths. Moreover, inequality (2.1) is sharp, i.e. there exist functions for which (2.1) turns into equality.

Proof. Let p be a closed path of Q and r be any function from $\mathcal{D}$. Then by the linearity of G_p and properties (a) and (b),

$$|G_p(f)|=|G_p(f-r)|\leq\|f-r\|\,. \tag{2.2}$$

Since the left-hand and the right-hand sides of (2.2) do not depend on r and p respectively, it follows from (2.2) that

$$\sup_{p\subset Q}|G_p(f)|\leq\inf_{r\in\mathcal{D}}\|f-r\|\,. \tag{2.3}$$

Now we prove the sharpness of (2.1). By assumption Q has closed paths. Then Q has closed paths $p'=(\mathbf{p}'_1,...,\mathbf{p}'_{2m})$ such that all points $\mathbf{p}_1,...,\mathbf{p}_{2m}$ are distinct. On the other hand there exist continuous functions $g=g(\mathbf{x})$ on Q such that $g(\mathbf{p}'_i)=1$, $i=1,3,...,2m-1$, $g(\mathbf{p}'_i)=-1$, $i=2,4,...,2m$ and $-1<g(\mathbf{x})<1$ elsewhere. For such functions we have

$$G_{p'}(g)=\|g\|=1 \tag{2.4}$$

and

$$E(g)\leq\|g\|, \tag{2.5}$$

where the last inequality follows from the fact that $0\in\mathcal{D}$. From (2.3)-(2.5) it follows that

$$\sup_{p\subset Q}|G_p(g)|=E\left(g\right).$$

We have proved the sharpness of (2.1) and hence the lemma.

The images of the distance functions $\|\mathbf{x}-\mathbf{c}_1\|$ and $\|\mathbf{x}-\mathbf{c}_2\|$ on Q denote by X_1 and X_2, respectively. For any function $h\in C(Q)$, consider the real functions

$$s_1(a)=\max_{\substack{\mathbf{x}\in Q\\ \|\mathbf{x}-\mathbf{c}_1\|=a}}h(x),\ s_2(a)=\min_{\substack{\mathbf{x}\in Q\\ \|\mathbf{x}-\mathbf{c}_1\|=a}}h(x),\ a\in X_1,$$
$$g_1(b)=\max_{\substack{\mathbf{x}\in Q\\ \|\mathbf{x}-\mathbf{c}_2\|=b}}h(x),\ g_2(b)=\min_{\substack{\mathbf{x}\in Q\\ \|\mathbf{x}-\mathbf{c}_2\|=b}}h(x),\ b\in X_2.$$

When are these functions continuous on the appropriate sets X_1 and X_2. The following lemma, which we use in the proof of our main result, Theorem 2.1, answers this question.

Lemma 2.2. (see [3]). *Let $Q\subset\mathbb{R}^d$ be a compact set. Then the functions s_1 and s_2 are continuous on X_1 (g_1 and g_2 are continuous on X_2) for any $h\in C(Q)$ if the following condition, which we call the condition A, holds:*

(A) for any two points $\mathbf{x}$ and $\mathbf{y}$ in Q with $\|\mathbf{x}-\mathbf{c}_1\|=\|\mathbf{y}-\mathbf{c}_1\|$ ($\|\mathbf{x}-\mathbf{c}_2\|=\|\mathbf{y}-\mathbf{c}_2\|$) and any sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ tending to $\mathbf{x}$, there exists a sequence $\{\mathbf{y}_n\}_{n=1}^{\infty}$ tending to $\mathbf{y}$ such that $\|\mathbf{x}_n-\mathbf{c}_1\|=\|\mathbf{y}_n-\mathbf{c}_1\|$ ($\|\mathbf{x}_n-\mathbf{c}_2\|=\|\mathbf{y}_n-\mathbf{c}_2\|$) for all $n=1,2,...$

The following theorem is valid. In the proof we use the method exploited in [13] (see also [4]).

Theorem 2.1. *Let $Q \subset \mathbb{R}^d$ be a compact set and $f \in C(Q)$. Assume the following conditions hold.*

1) Q satisfies the condition A;

2) there exists an extremal element $r_0 \in \mathcal{D}$ for the function f;

3) for any path $q = (\mathbf{q}_1, ..., \mathbf{q}_n) \subset Q$ there exist points $\mathbf{q}_{n+1}, \mathbf{q}_{n+2}, ..., \mathbf{q}_{n+s} \in Q$ such that $(\mathbf{q}_1, ..., \mathbf{q}_n, \mathbf{q}_{n+1}, ..., \mathbf{q}_{n+s})$ is a closed path and s is not more than some positive integer n_0 independent of q.

Then the approximation error can be computed by the formula

$$E(f) = \sup_{p \subset Q} |G_p(f)|,$$

where the $\sup$ *is taken over all closed paths.*

Proof. For brevity of the exposition, in the sequel, we use the concept of "an extremal path". A finite or infinite path $(\mathbf{p}_1, \mathbf{p}_2, ...)$ is said to be extremal for a function $u \in C(Q)$ if $u(\mathbf{p}_i) = (-1)^i \|u\|, i = 1, 2, ...$ or $u(\mathbf{p}_i) = (-1)^{i+1} \|u\|, i = 1, 2, ...$ (see [13]). Regarding extremal paths for the function $f_1 = f - r_0$, there are only two possible options. The first option is when there exists a closed path $p_0 = (\mathbf{p}_1, ..., \mathbf{p}_{2n})$ extremal for the function f_1. In this case, it is easy to see that

$$|G_{p_0}(f)| = |G_{p_0}(f - r_0)| = \|f - r_0\| = E(f).$$

Considering this, the assertion of the theorem follows from (2.1). The second option is when there does not exist a closed path extremal for the function f_1. Let us prove that in this case, there exists an infinite path extremal for f_1. Suppose the contrary. Suppose that there exists a positive integer N such that the length of each path extremal for f_1 is not more than N. Here by length of a path we mean its number of points. Define the following functions:

$$f_n = f_{n-1} - r_{1,n-1} - r_{2,n-1}, \ \ n = 2, 3, ...,$$

where

$$r_{1,n-1} = r_{1,n-1}(\|\mathbf{x} - \mathbf{c}_1\|) = \frac{1}{2}\left(\max_{\substack{\mathbf{y} \in Q \\ \|\mathbf{y}-\mathbf{c}_1\| = \|\mathbf{x}-\mathbf{c}_1\|}} f_{n-1}(\mathbf{y}) + \min_{\substack{\mathbf{y} \in Q \\ \|\mathbf{y}-\mathbf{c}_1\| = \|\mathbf{x}-\mathbf{c}_1\|}} f_{n-1}(\mathbf{y}) \right)$$

$$r_{2,n-1} = r_{2,n-1}(\|\mathbf{x} - \mathbf{c}_2\|) = \frac{1}{2}\left(\max_{\substack{\mathbf{y} \in Q \\ \|\mathbf{y}-\mathbf{c}_2\| = \|\mathbf{x}-\mathbf{c}_2\|}} (f_{n-1}(\mathbf{y}) - r_{1,n-1}(\|\mathbf{y} - \mathbf{c}_1\|)) \right.$$

$$\left. + \min_{\substack{\mathbf{y} \in Q \\ \|\mathbf{y}-\mathbf{c}_2\| = \|\mathbf{x}-\mathbf{c}_2\|}} (f_{n-1}(\mathbf{y}) - r_{1,n-1}(\|\mathbf{y} - \mathbf{c}_1\|)) \right).$$

Note that by Lemma 2.2, all the above functions $f_n(\mathbf{x})$, $n = 2, 3, ...$, are continuous on Q. Since r_0 is an extremal element for f, the equality $\|f_1\| = E(f)$ holds. Let us show that $\|f_2\| = E(f)$. Indeed, for any $\mathbf{x} \in Q$

$$f_1(\mathbf{x}) - r_{1,1}(\|\mathbf{x} - \mathbf{c}_1\|)$$

$$\leq \frac{1}{2}\left(\max_{\substack{\mathbf{y} \in Q \\ \|\mathbf{y}-\mathbf{c}_1\| = \|\mathbf{x}-\mathbf{c}_1\|}} f_1(\mathbf{y}) - \min_{\substack{\mathbf{y} \in Q \\ \|\mathbf{y}-\mathbf{c}_1\| = \|\mathbf{x}-\mathbf{c}_1\|}} f_1(\mathbf{y}) \right) \leq E(f) \qquad (2.6)$$

and

$$f_1(\mathbf{x}) - r_{1,1}(\|\mathbf{x}-\mathbf{c}_1\|)$$

$$\geq \frac{1}{2}\left(\min_{\substack{\mathbf{y}\in Q\\ \|\mathbf{y}-\mathbf{c}_1\|=\|\mathbf{x}-\mathbf{c}_1\|}} f_1(\mathbf{y}) - \max_{\substack{\mathbf{y}\in Q\\ \|\mathbf{y}-\mathbf{c}_1\|=\|\mathbf{x}-\mathbf{c}_1\|}} f_1(\mathbf{y})\right) \geq -E(f). \tag{2.7}$$

Considering the definition of $r_{2,1}(\|\mathbf{x}-\mathbf{c}_2\|)$, for any $\mathbf{x}\in Q$ we can write

$$f_1(\mathbf{x}) - r_{1,1}(\|\mathbf{x}-\mathbf{c}_1\|) - r_{2,1}(\|\mathbf{x}-\mathbf{c}_2\|)$$

$$\leq \frac{1}{2}\left(\max_{\substack{\mathbf{y}\in Q\\ \|\mathbf{y}-\mathbf{c}_2\|=\|\mathbf{x}-\mathbf{c}_2\|}} (f_1(\mathbf{y}) - r_{1,1}(\|\mathbf{y}-\mathbf{c}_1\|)) - \min_{\substack{\mathbf{y}\in Q\\ \|\mathbf{y}-\mathbf{c}_2\|=\|\mathbf{x}-\mathbf{c}_2\|}} (f_1(\mathbf{y}) - r_{1,1}(\|\mathbf{y}-\mathbf{c}_1\|))\right)$$

and

$$f_1(\mathbf{x}) - r_{1,1}(\|\mathbf{x}-\mathbf{c}_1\|) - r_{2,1}(\|\mathbf{x}-\mathbf{c}_2\|)$$

$$\leq \frac{1}{2}\left(\min_{\substack{\mathbf{y}\in Q\\ \|\mathbf{y}-\mathbf{c}_2\|=\|\mathbf{x}-\mathbf{c}_2\|}} (f_1(\mathbf{y}) - r_{1,1}(\|\mathbf{y}-\mathbf{c}_1\|)) - \max_{\substack{\mathbf{y}\in Q\\ \|\mathbf{y}-\mathbf{c}_2\|=\|\mathbf{x}-\mathbf{c}_2\|}} (f_1(\mathbf{y}) - r_{1,1}(\|\mathbf{y}-\mathbf{c}_1\|))\right).$$

Using (2.6) and (2.7) in the last two inequalities, we obtain that for any $\mathbf{x}\in Q$

$$-E(f) \leq f_2(\mathbf{x}) = f_1(\mathbf{x}) - r_{1,1}(\|\mathbf{x}-\mathbf{c}_1\|) - r_{2,1}(\|\mathbf{x}-\mathbf{c}_2\|) \leq E(f).$$

Thus,

$$\|f_2\| \leq E(f). \tag{2.8}$$

Since $f_2 - f \in \mathcal{D}$, it follows from (2.8) that $\|f_2\| = E(f)$.

Similarly, one can show that $\|f_3\| = E(f)$, $\|f_4\| = E(f)$, and so on. Thus, $\|f_n\| = E(f)$ for all $n = 1, 2, ...$

Let us now prove the following implications

$$f_1(\mathbf{p}_0) < E(f) \Rightarrow f_2(\mathbf{p}_0) < E(f) \tag{2.9}$$

and

$$f_1(\mathbf{p}_0) > -E(f) \Rightarrow f_2(\mathbf{p}_0) > -E(f), \tag{2.10}$$

where $\mathbf{p}_0 \in Q$. First, we are going to prove the implication

$$f_1(\mathbf{p}_0) < E(f) \Rightarrow f_1(\mathbf{p}_0) - r_{1,1}(\|\mathbf{p}_0-\mathbf{c}_1\|) < E(f). \tag{2.11}$$

There are two possible cases.

1) $\max\limits_{\substack{\mathbf{y}\in Q\\ \|\mathbf{y}-\mathbf{c}_1\|=\|\mathbf{p}_0-\mathbf{c}_1\|}} f_1(\mathbf{y}) = E(f)$ and $\min\limits_{\substack{\mathbf{y}\in Q\\ \|\mathbf{y}-\mathbf{c}_1\|=\|\mathbf{p}_0-\mathbf{c}_1\|}} f_1(\mathbf{y}) = -E(f)$.

In this case, $r_{1,1}(\|\mathbf{p}_0-\mathbf{c}_1\|) = 0$. Therefore,

$$f_1(\mathbf{p}_0) - r_{1,1}(\|\mathbf{p}_0-\mathbf{c}_1\|) < E(f).$$

2) $\max\limits_{\substack{\mathbf{y}\in Q\\ \|\mathbf{y}-\mathbf{c}_1\|=\|\mathbf{p}_0-\mathbf{c}_1\|}} f_1(\mathbf{y}) = E(f) - \varepsilon_1$ and $\min\limits_{\substack{\mathbf{y}\in Q\\ \|\mathbf{y}-\mathbf{c}_1\|=\|\mathbf{p}_0-\mathbf{c}_1\|}} f_1(\mathbf{y}) = -E(f) + \varepsilon_2$,

where $\varepsilon_1, \varepsilon_2 \geq 0$ and $\varepsilon_1 + \varepsilon_2 \neq 0$.

In this case,

$$f_1(\mathbf{p}_0) - r_{1,1}(\|\mathbf{p}_0-\mathbf{c}_1\|) \leq \max_{\substack{\mathbf{y}\in Q\\ \|\mathbf{y}-\mathbf{c}_1\|=\|\mathbf{p}_0-\mathbf{c}_1\|}} f_1(\mathbf{y}) - r_{1,1}(\|\mathbf{p}_0-\mathbf{c}_1\|)$$

$$= \frac{1}{2}\left(\max_{\substack{\mathbf{y}\in Q \\ \|\mathbf{y}-\mathbf{c}_1\|=\|\mathbf{p}_0-\mathbf{c}_1\|}} f_1(\mathbf{y}) - \min_{\substack{\mathbf{y}\in Q \\ \|\mathbf{y}-\mathbf{c}_1\|=\|\mathbf{p}_0-\mathbf{c}_1\|}} f_1(\mathbf{y}) \right)$$

$$= E(f) - \frac{\varepsilon_1 + \varepsilon_2}{2} < E(f).$$

Thus we have proved (2.11). Using the same method, we can also prove that

$$f_1(\mathbf{p}_0) - r_{1,1}(\|\mathbf{p}_0 - \mathbf{c}_1\|) < E(f)$$

$$\Rightarrow f_1(\mathbf{p}_0) - r_{1,1}(\|\mathbf{p}_0 - \mathbf{c}_1\|) - r_{2,1}(\|\mathbf{p}_0 - \mathbf{c}_2\|) < E(f). \tag{2.12}$$

Implications (2.11) and (2.12) yield (2.9). By the same way one can prove the validity of (2.10). From implications (2.9) and (2.10) it follows that if $f_2(\mathbf{p}_0) = E(f)$, then $f_1(\mathbf{p}_0) = E(f)$ and if $f_2(\mathbf{p}_0) = -E(f)$, then $f_1(\mathbf{p}_0) = -E(f)$. This simply means that each path extremal for f_2 is extremal for f_1.

We supposed above that any path extremal for f_1 has the length not more than N. Let us show that in his case, any path extremal for f_2 has the length not more than $N-1$. Suppose the contrary. Suppose that there is a path extremal for f_2 with the length equal to N. Denote this path by $q = (\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N)$. Without loss of generality we may assume that $\mathbf{b}\cdot\mathbf{q}_{N-1} = \mathbf{b}\cdot\mathbf{q}_N$. As we have shown above, the path q is extremal for f_1. Assume $f_1(\mathbf{q}_N) = E(f)$. Then there is not a point $\mathbf{q}_0 \in Q$ such that $\mathbf{q}_0 \neq \mathbf{q}_N$, $\mathbf{a}\cdot\mathbf{q}_0 = \mathbf{a}\cdot\mathbf{q}_N$ and $f_1(\mathbf{q}_0) = -E(f)$. Indeed, if there was such $\mathbf{q}_0$ and $\mathbf{q}_0 \notin q$, then the path $(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N, \mathbf{q}_0)$ would be extremal for f_1. But this would contradict our assumption that any path extremal for f_1 has the length not more than N. On the other hand, if there was such $\mathbf{q}_0$ and $\mathbf{q}_0 \in q$, then from points of q we could form a closed extremal path for f_1, which would contradict our assumption that there does not exist a closed extremal path for f_1. Hence we conclude that

$$\max_{\substack{\mathbf{y}\in Q \\ \|\mathbf{y}-\mathbf{c}_1\|=\|\mathbf{q}_N-\mathbf{c}_1\|}} f_1(\mathbf{y}) = E(f), \qquad \min_{\substack{\mathbf{y}\in Q \\ \|\mathbf{y}-\mathbf{c}_1\|=\|\mathbf{q}_N-\mathbf{c}_1\|}} f_1(\mathbf{y}) > -E(f).$$

Therefore,

$$|f_1(\mathbf{q}_N) - r_{1,1}(\|\mathbf{q}_N - \mathbf{c}_1\|)| < E(f).$$

From the last inequality, by the similar way as above, one can obtain that

$$|f_2(\mathbf{q}_N)| < E(f).$$

This means that the path $(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N)$ can not be extremal for f_2. Thus any path extremal for f_2 has the length not more than $N-1$.

By the same way, it can be shown that any path extremal for f_3 has the length not more than $N-2$, any path extremal for f_4 has the length not more than $N-3$ and so on. Finally, we obtain that there is not a path extremal for f_{N+1}. Then there is not a point $\mathbf{p}_0 \in Q$ such that $|f_{N+1}(\mathbf{p}_0)| = \|f_{N+1}\|$. But the norm $\|f_{N+1}\|$ must be attained, since by Lemma 2.2, all the functions $f_2, f_3, ..., f_{N+1}$ are continuous on the compact set Q. The obtained contradiction means that there exists an infinite path extremal for f_1.

Let a path $p = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n, ...)$ be infinite and extremal for f_1. Note that all the points $\mathbf{p}_i$ must be distinct, otherwise we could form a closed extremal path, contrary to our assumption. Consider the sequence $p_n = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n)$, $n = 1, 2, ...$, of finite paths. By condition (3) of the theorem, for each path p_n there exists a closed path $p_n^{m_n} = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n, \mathbf{q}_{n+1}, ..., \mathbf{q}_{n+m_n})$, where $m_n \leq n_0$. The functional $G_{p_n^{m_n}}$ obeys the inequalities

$$\left|G_{p_n^{m_n}}(f)\right| = \left|G_{p_n^{m_n}}(f - r_0)\right| \leq \frac{n\,\|f - r_0\| + m_n\,\|f - r_0\|}{n + m_n} = \|f - r_0\| \tag{2.13}$$

and

$$\left|G_{p_n^{m_n}}(f)\right| \geq \frac{n\left\|f-r_0\right\|-m_n\left\|f-r_0\right\|}{n+m_n}=\frac{n-m_n}{n+m_n}\left\|f-r_0\right\|. \tag{2.14}$$

We obtain from (2.13) and (2.14) that

$$\sup_{p_n^{m_n}}\left|G_{p_n^{m_n}}(f)\right|=\left\|f-r_0\right\|. \tag{2.15}$$

Since r_0 is an extremal element, it follows from (2.15) and Lemma 2.1 that

$$E\left(f\right)=\sup_{p\subset Q}\left|G_p(f)\right|,$$

where the $\sup$ is taken over all closed paths of Q. The theorem has been proved.

Remark. Theorem 2.1 generalizes the result of Diliberto and Straus (see [6, Theorem 1]) from the sum of univariate functions to the sum of radial basis functions.

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