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# Approximation of functions by linear positive operators in variable Lebesgue spaces

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**Abstract.** *In this paper analog of Korovkin type approximation theorem is proved for trigonometric polynomials in variable Lebesgue spaces.*

**Keywords.** linear positive operator · approximation theorem · variable Lebesgue spaces

**Mathematics Subject Classification (2010):** 41A17, 41A35

## 1 Introduction

Approximation of functions by positive linear operators is a classical topic in the field of approximation theory. It was motivated by the Weierstrass approximation theorem verifying the denseness of polynomials in the space $C[0,1]$ of continuous functions on the interval [0,1] and started with the investigation of approximation of continuous functions by the classical Bernstein operators defined in [10] as

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), \;\; 0 < x < 1, \;\; f \in C[0,1] \tag{1.1}$$

where $\{b_{n,k}\}_{k=0}^{n}$ is the Bernstein basis given by

$$b_{n,k} = C_n^k x^k (1-x)^{n-k}.$$

To approximate discontinuous functions, one often replaces the point evaluation functionals in (1.1) by some integrals and considers the corresponding Bernstein type positive linear operators on $L_p[0,1]$ spaces with $1 \le p < \infty$, where $L_p[0,1]$ is the Banach space consisting of all integrable functions $f$ on [0,1] with the $L_p$-norm

$$\|f\|_{L_p[0,1]} := \left(\int_0^1 |f(x)|^p \, dx\right)^{\frac{1}{p}} \tag{1.2}$$

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finite. Examples of such positive linear operators on $L_p[0,1]$ include the Kantorovich operators [8] defined by

$$K_n(f,x) = \sum_{k=0}^{n}(n+1)\int\limits_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dtb_{n,k}(x), \;\; x \in [0,1] \tag{1.3}$$

and the Durrmeyer operators [4] by

$$D_n(f,x) = \sum_{k=0}^{n}(n+1)\int_0^1 b_{n,k}(t)f(t)dtb_{n,k}(x), \;\; x \in [0,1] \tag{1.4}$$

Quantitative behaviors of the approximation by the above mentioned positive linear operators have been well understood due to a large literature.

In this paper we study the approximation of functions by positive linear operators on variable $L_p$ spaces. Note that (1.3) and (1.4) may be regarded as operators on the space $L_p[0,1]$. So the functions for approximation considered in this paper are defined on a connected open subset $\Omega$ of $R$ such as $\Omega = (0,1), (0,\infty)$ and $(-\infty,\infty)$.

The variable $L_{p(x)}(\Omega)$ space, $L_{p(x)}(\Omega)$, is associated with a measurable function $p : \Omega \to [1,\infty)$ called the exponent function. The space $L_{p(x)}(\Omega)$ consists of all measurable function $f$ on $\Omega$ such that $\int\limits_{\Omega} \left(\frac{|f(x)|}{\lambda_0}\right)^{p(x)} dx < \infty$ for some $\lambda_0 > 0$. Its norm cannot be defined through replacing the constant $p$ in (1.2) by the exponent function $p(x)$. It is defined by scaling as

$$\|f\|_{L_{p(x)}(\Omega)} := \|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int\limits_{\Omega}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\} \tag{1.5}$$

With this definition, $L_{p(x)}(\Omega)$ becomes a Banach space [9].

The idea of variable spaces was introduced by Orlicz [13] who considered necessary and sufficient conditions on a sequence $\{y_k\}$ for the series $\sum\limits_k x_k y_k$ to converge, given that $\{p_k\}$, $p_k \geq 1$ and $\{x_k\}$ are real sequences with the series$\sum\limits_k x_k^{p_k}$ convergent. Mathematical analysis [11, 12, 15] of variable spaces $L_{p(x)}(\Omega)$ was motivated by connections between these function spaces and variational integrals with non-standard growth related to modeling of electrorheological fluids, which can be found in [14, 16] and references therein. Important analysis topics include boundedness of various operators, continuity of translates, and denseness of smooth functions [2, 3]. Recently, simultaneous approximation in Lebesgue spaces with variable exponent was proved in [7]. Also, some approximation theorem for Bernstein-Chlodowsky polynomials was investigated in [1].

The purpose of this paper is to raise the issue of approximation on the variable spaces $L_{p(x)}(\Omega)$ by positive linear operators.

**Definition 1.1** *We say that a linear operator $A_n$ on $L_{p(x)}(\Omega)$ is positive if it maps $(L_{p(x)}(\Omega))_+$ into itself, where $(L_{p(x)}(\Omega))_+$ denotes the positive cone of $L_{p(x)}(\Omega)$ consisting of all functions $f$ in $L_{p(x)}(\Omega)$ such that $f(x) \geq 0$ almost everywhere.*

In this direction recently was the work [6].

## 2 Main result

Let $p = p(x)$ be a Lebesgue measurable $2\pi$-periodic function such that $1 \le \underline{p} \le \overline{p} < \infty$, where $\underline{p} = \operatorname{ess\,inf} p(x)$ and $\overline{p} = \operatorname{ess\,sup} p(x)$. Throughout the paper $L_{p(x)}$ denotes the space of all $2\pi$ periodic Lebesgue measurable functions $f$ equipped with the norm (1.5).

**Theorem 2.1** *Let $1 \le \underline{p} \le \overline{p} < \infty$ and let $A_n$ be a sequence of linear positive operators acting from $L_{p(x)}$ to $L_{p(x)}$. In order to a sequence of linear positive operators $A_n$ for any function $f \in L_{p(x)}$ converged in the $L_{p(x)}$ metrics to this function, it is necessary and sufficient that the following two conditions are satisfied:*
*1) There exists a $M > 0$ such that for any $f \in L_{p(x)}$ and $n \in \mathbb{N}$*

$$\|A_n f\|_{L_{p(\cdot)}} \le M\|f\|_{L_{p(\cdot)}}; \tag{2.1}$$

*2) For the sequence of operators $\{A_n(1;x)\}_{n=1}^{\infty}$ , $\{A_n(\cos\, t;x)\}_{n=1}^{\infty}$ and $\{A_n(\sin\, t;x)\}_{n=1}^{\infty}$ the following equalitys holds*

$$\lim_{n\to\infty} \|1 - A_n(1;x)\|_{L_{p(\cdot)}} = \lim_{n\to\infty} \|\cos\, x - A_n(\cos\, t;x)\|_{L_{p(\cdot)}}$$
$$= \lim_{n\to\infty} \|\sin\, x - A_n(\sin\, t;x)\|_{L_{p(\cdot)}} = 0. \tag{2.2}$$

**Necessity:** The completeness of variable Lebesgue spaces as providing (see [9]). The necessity of first condition immediately implies from Banach-Steinhaus theorem. Note that the necessity of the second condition is obvious.

**Suffiiciency**. We have

$$A_n \sin^2 \frac{x-t}{2} = A_n \left( \frac{1-\cos(x-t)}{2} \right) = \frac{1}{2}(A_n(1-\cos(x-t)))$$
$$= \frac{1}{2}\left(A_n 1 - \cos x A_n \cos t - \sin x A_n \sin t\right).$$

By triangle inequality in $L_{p(x)}$ , we have

$$\left\| A_n \sin^2 \frac{x-t}{2} \right\|_{L_{p(\cdot)}} = \frac{1}{2} \|A_n 1 - \cos x A_n \cos t - \sin x A_n \sin t\|_{L_{p(\cdot)}}$$
$$\le \frac{1}{2} \left( \|A_n 1 - 1\|_{L_{p(\cdot)}} + \|1 - \cos x A_n \cos t - \sin x A_n \sin t\|_{L_{p(\cdot)}} \right)$$
$$= \frac{1}{2} \left( \|A_n 1 - 1\|_{L_{p(\cdot)}} + \left\|\sin^2 x + \cos^2 x - \cos x A_n \cos t - \sin x A_n \sin t\right\|_{L_{p(\cdot)}} \right)$$
$$\le \frac{1}{2} \left( \|A_n 1 - 1\|_{L_{p(\cdot)}} + \|\sin x(\sin x - A_n \sin t)\|_{L_{p(\cdot)}} + \|\cos x(\cos x - A_n \cos t\|_{L_{p(\cdot)}} \right)$$
$$\le \frac{1}{2} \left( \|A_n 1 - 1\|_{L_{p(\cdot)}} + \|\sin x - A_n \sin t\|_{L_{p(\cdot)}} + \|\cos x - A_n \cos t\|_{L_{p(\cdot)}} \right) \to 0, n \to \infty \Rightarrow$$

$$\left\| A_n \sin^2 \frac{x-t}{2} \right\|_{L_{p(\cdot)}} \to 0,\ n \to \infty. \tag{2.3}$$

Let $f$ be an arbitrary $2\pi$-periodic function from $L_{p(x)}$. Using absolute continuity of Lebesgue integral and applying theorem on Luzin $C$-property of measurable function, for arbitrary $\varepsilon > 0$ we can find $2\pi$-periodic continuous function $f^*$ such that

$$\|f - f^*\|_{L_{p(\cdot)}} < \varepsilon \tag{2.4}$$

and exists $\delta > 0$ with $|x' - x''| < \delta$,

$$\left|f^*(x') - f^*(x'')\right| < \varepsilon. \tag{2.5}$$

We put $\|f^*\|_C = M_1$. Then for any $x$ and $t$ we have

$$\left|f^*(x') - f^*(t)\right| < \varepsilon + \frac{2M_1}{\sin^2 \frac{\delta}{2}} \sin^2 \frac{x-t}{2}. \tag{2.6}$$

Take into account (2.1)-(2.6) and positivity of linear operators $A_n$, for any sufficiency large $n \in N$

$$\|f - A_n(f)\|_{L_{p(\cdot)}} \leq \|f - f^*\|_{L_{p(\cdot)}} + \|f^* - f^* A_n(1;x)\|_{L_{p(\cdot)}}$$

$$+ \|f^* A_n(1;x) - A_n(f^*(t);x)\|_{L_{p(\cdot)}} + \|A_n(f^* - f;x)\|_{L_{p(\cdot)}}$$

$$\leq \varepsilon + M_1 \|1 - A_n(1;x)\|_{L_{p(\cdot)}} + \|A_n(|f^*(\cdot) - f^*(t)|;x)\|_{L_{p(\cdot)}} + M \|f - f^*\|_{L_{p(\cdot)}}$$

$$\leq \varepsilon + M_1 \|1 - A_n(1;x)\|_{L_{p(\cdot)}} + \|A_n(|f^*(\cdot) - f^*(t)|;x)\|_{L_{p(\cdot)}} + M\varepsilon$$

$$\leq \varepsilon(1 + M_1 + M) + \varepsilon \|A_n(1;x)\|_{L_{p(\cdot)}} + \frac{2M_1}{\sin^2 \frac{\delta}{2}} \left\|A_n\left(\sin^2 \frac{x-t}{2};x\right)\right\|_{L_{p(\cdot)}}$$

$$\leq \varepsilon(1 + M_1 + M) + \varepsilon M (2\pi)^{\frac{1}{p}} + \varepsilon.$$

Thus, $\lim\limits_{n\to\infty} \|f - A_n(f;x)\|_{L_{p(\cdot)}} = 0$.
The proof of Theorem 2.1 is complete .

**Remark 2.1** Note that in the constant exponent case theorem 2.1 was proved in [5].

Let a function $p = p(x)$ satisfies the Dini-Lipschitz condition

$$|p(x) - p(y)| \leq \frac{C}{\ln \frac{1}{|x-y|}}, \quad \forall\, x, y \in [-\pi, \pi]. \tag{2.7}$$

Let $f \in L_{p(x)}[-\pi, \pi]$ and let $K_\mu$- $2\pi$-periodic function, where $1 \leq \mu < \infty$. We consider the following convolution operator

$$\mathcal{L}_\mu(f) := \mathcal{L}_\mu(f)(x) = \int_{-\pi}^{\pi} f(t) K_\mu(t-x) dt.$$

**Corollary 2.1** *[15] Let $f \in L_{p(x)}[-\pi, \pi]$ and variable exponent function p satisfy condition (2.7). Suppose a kernel $K_\mu$ satisfy following conditions:*
*a)* $\int_{-\pi}^{\pi} |K_\mu(x)|\, dx \leq C_1$;
*b)* $\sup\limits_{x\in[-\pi,\pi]} |K_\mu(x)| \leq C_2 \mu^\vartheta$;
*c)* $|K_\mu(x)| \leq C_3,$ *for* $\mu^{-\gamma} \leq |x| \leq \pi$ *with some constants* $C_i$ $(i = 1, 2, 3)$ *and* $\vartheta, \gamma > 0$ *independent of* $\mu$.
*Then the family of operators* $\{\mathcal{L}(f)\}_{\mu \geq 1}$ *is uniformly bounded in* $L_{p(x)}[-\pi, \pi]$.

In particular, as an example of operators $\mathcal{L}_\mu(f)$ can be by the operators of Fejer, Poisson, Jackson, Steklov, and Cesaro.

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