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Asymptotic formulas for eigenvalues and eigenfunctions of some ordinary differential operators of fourth order

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Abstract. *In this paper we consider a spectral problem for a fourth order ordinary differential equations with a spectral parameter fractionally linearly entering in the boundary condition. This problem, in particular, describes the bending vibrations of a homogeneous rod, in cross-sections of which a longitudinal force acts, the left end of which is fixed and a mass is concentrated at the right end. We obtain refined asymptotic formulas for the natural frequencies and harmonics of this problem.*

Keywords. spectral parameter, eigenvalue, eigenfunction, asymptotic formula

Mathematics Subject Classification (2010): 34B05, 34B08, 34B24, 34L10, 34L20

1 Introduction

We consider the following spectral problem

$$y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ 0 < x < 1, \tag{1.1}$$

$$U_1(y) \equiv y(0) = 0,\, U_2(y) \equiv y'(0) = 0, \tag{1.2}$$

$$U_3(y) \equiv y''(1) = 0, \tag{1.3}$$

$$U_4(y) \equiv (a\lambda + b)y(1) - (c\lambda + d)Ty(1) = 0, \tag{1.4}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv y''' - qy'$, $q(x)$ is a positive absolutely continuous function on the interval $[0, 1]$, a, b, c and d are real constants such that $\sigma = bc - ad \neq 0$ and $c \neq 0$.

Problem (1.1)-(1.4) in a more general case was considered in [2, 11, 12]. In these papers were investigated the location of eigenvalues in the complex plane (on the real axis), the structure of the root subspaces, obtained the asymptotic formulas for the eigenvalues and eigenfunctions, studied the basis property of subsystems of root functions of this problem. In particular, were established a necessary and sufficient condition for the subsystems of root functions of problem (1.1)-(1.4) (in a more general case) to form a basis in the space $L_p(0, 1),\ 1 < p < \infty$.

The uniform convergence of the expansions for the functions in terms of root functions of the Sturm-Liouville operators with a spectral parameter in the boundary conditions were investigate in [7-10, 13-15]. The uniform convergence of Fourier series expansions in the

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system of root functions of problem (1.1)-(1.4) with $c = 0$ were investigated in recent works [1-3].

Note that in order to study the uniform convergence of spectral expansions in terms of the root functions of differential operators, we need to obtain more refined asymptotic formulas for eigenvalues and eigenfunctions of problem (1.1)-(1.4) and and the so-called "asymptotic" problem (1.1)-(1.3) and $ay(1) - cTy(1) = 0$. The aim of our paper is obtaining of refined asymptotic formulas for the eigenvalues and eigenfunctions of problem (1.1)-(1.4) and "asymptotic" problem.

2 Some auxiliary facts and statements

We introduce the boundary condition

$$\tilde{U}_4(y) \equiv ay(1) - cTy(1) = 0. \tag{2.1}$$

Alon with problem (1.1)-(1.4) consider the spectral problem (1.1)-(1.3), (2.1).

Theorem 2.1 (see [4, 5]) *The eigenvalues of the problem* (1.1)-(1.3), (2.1) *are real, simple and form an infinitely increasing sequence* $\{\mu_k\}_{k=1}^{\infty}$ *such that* $\mu_k > 0$ *for* $k \geq 2$. *Moreover, the eigenfunction* $v_k(x)$, $k \geq 2$, *corresponding to the eigenvalue* μ_k *has precisely* $k - 1$ *simple zeros in* $(0, 1)$.

Theorem 2.2 (see [2, Theorem 4.1], [11, Theorem 5.1] and [12, Theorem 2.2]) *If* $\sigma > 0$, *then the eigenvalues of the problem* (1.1)-(1.4) *are real, simple and form an infinitely increasing sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *such that* $\lambda_k > 0$ *for* $k \geq 3$. *If* $\sigma > 0$, *then the eigenvalues of problem* (1.1)-(1.4) *form an infinitely sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *without finite limit point and only the following cases are possible: (a) all the eigenvalues are real and simple; (b) all the eigenvalues are real and all, except one double, are simple; (c) all the eigenvalues are real and all, except one triple, are simple; (d) all the eigenvalues are simple and all, except a conjugate pair of non-real, are real (then we will assume that* $\lambda_1 \in \mathbb{C}\backslash\mathbb{R}$ *and* $\lambda_2 = \bar{\lambda}_1$*). In this case also* $\lambda_k > 0$ *for* $k \geq 3$.

In [20-22] the following asymptotic formulas were found for the eigenvalues and root functions of problems (1.1)-(1.3), (2.1) and (1.1)-(1.4) up to the order $\frac{1}{k}$.

Theorem 2.3 (see [20, Theorem 5.1], [11, Theorem 6.1] and [12, Theorem 3.1]) *The following asymptotic formulas hold:*

$$\sqrt[4]{\eta_k} = \left(k - \frac{1}{2}\right)\pi + O\left(\frac{1}{k}\right), \tag{2.2}$$

$$\sqrt[4]{\lambda_k} = \left(k - \frac{3}{2}\right)\pi + O\left(\frac{1}{k}\right), \tag{2.3}$$

$$\begin{aligned} v_k(x) = \sin\left(k - \tfrac{1}{2}\right)\pi x - \cos\left(k - \tfrac{1}{2}\right)\pi x + e^{-\left(k-\frac{1}{2}\right)\pi x} \\ +(-1)^{k+1} e^{\left(k-\frac{1}{2}\right)\pi(x-1)} + O\left(\tfrac{1}{k}\right), \end{aligned} \tag{2.4}$$

$$\begin{aligned} y_k(x) = \sin\left(k - \tfrac{3}{2}\right)\pi x - \cos\left(k - \tfrac{3}{2}\right)\pi x + e^{-\left(k-\frac{3}{2}\right)\pi x} \\ +(-1)^{k} e^{\left(k-\frac{3}{2}\right)\pi(x-1)} + O\left(\tfrac{1}{k}\right), \end{aligned} \tag{2.5}$$

where relations (2.4)-(2.5) *hold uniformly for* $x \in [0, 1]$.

In the following sections, we will refine the asymptotic formulas (2.2) and (2.5) with $q \equiv 0$ to order $\frac{1}{k^2}$.

3 Asymptotic formulas for eigenvalues and eigenfunctions of problem (1.1)-(1.3), (2.1) with $q(x) \equiv 0$

In this section we consider the spectral problem (1.1)-(1.3), (2.1) with $q(x) \equiv 0$.

Theorem 3.1 *For the eigenvalues μ_k and eigenfunctions $v_k(x)$ of problem* (1.1)-(1.3), (2.1) *with $q(x) \equiv 0$ one has the asymptotic formulas*

$$\sqrt[4]{\mu_k} = \left(k - \frac{1}{2}\right)\pi + O\left(\frac{1}{k^3}\right), \tag{3.1}$$

$$\begin{aligned} v_k(x) &= \sin\left(k - \tfrac{1}{2}\right)\pi x - \cos\left(k - \tfrac{1}{2}\right)\pi x + e^{-\left(k-\frac{1}{2}\right)\pi x} \\ &+ (-1)^{k+1} e^{\left(k-\frac{1}{2}\right)\pi(x-1)} + O\left(\tfrac{1}{k^3}\right), \end{aligned} \tag{3.2}$$

where relation (3.2) *holds uniformly for $x \in [0,1]$.*

Proof. Let $\rho = \sqrt[4]{\lambda}$ and $\omega_k,\ k = 1,\ 2,\ 3,\ 4$, are distinct fourth roots of unity and we let

$$\omega_1 = -\omega_4 = -1,\ \omega_2 = -\omega_3 = -i. \tag{3.3}$$

In the case $q \equiv 0$ equation (1.1) has four linearly independent solutions

$$z_k(x,\rho) = e^{\rho\,\omega_k x},\ k = 1,\ 2,\ 3,\ 4\,.$$

Hence we have

$$z_k^{(s)}(x,\rho) = e^{\rho\,\omega_k x}(\rho\omega_k)^s,\ k = 1,\ 2,\ 3,\ 4,\ s = 0,\ 1,\ 2,\ 3\,. \tag{3.4}$$

Then in view of (3.4), by (1.2), (1.3) and (2.1) we obtain

$$\begin{aligned} &U_1(z_k) \equiv z_k(0,\rho) = 1,\ U_2(z_k) \equiv z_k'(0,\rho) = \rho\omega_k, \\ &U_3(z_k) \equiv z_k''(1) = \rho^2\omega_k^2 e^{\rho\omega_k},\ \tilde{U}_4(z_k) \equiv z_k(1) = \rho^3\omega_k^3 e^{\rho\omega_k}(1 + O\left(\tfrac{1}{k^3}\right)). \end{aligned} \tag{3.5}$$

Obviously, the eigenvalues of problem (1.1)-(1.3), (2.1) are the roots of the characteristic determinant

$$\Delta(\lambda) = \begin{vmatrix} U_1(z_1) & U_1(z_2) & U_1(z_3) & U_1(z_4) \\ U_2(z_1) & U_2(z_2) & U_2(z_3) & U_2(z_4) \\ U_3(z_1) & U_3(z_2) & U_3(z_3) & U_3(z_4) \\ \tilde{U}_4(z_1) & \tilde{U}_4(z_2) & \tilde{U}_4(z_3) & \tilde{U}_4(z_4) \end{vmatrix}. \tag{3.6}$$

By (3.3) and (3.5) it follows from (3.6) that

$$\Delta(\lambda) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -\rho & -\,i\rho & i\rho & \rho \\ \rho^2 e^{-\rho}\left(1 - \frac{q_0}{4\rho}\right) & -\,\rho^2 e^{-i\rho}\left(1 - \frac{q_0}{4\rho i}\right) & -\,\rho^2 e^{i\rho}\left(1 + \frac{q_0}{4\rho i}\right) & \rho^2 e^{\rho}\left(1 + \frac{q_0}{4\rho}\right) \\ -\rho^3 e^{-\rho}\left(1 - \frac{q_0}{4\rho}\right) & -\,i\rho^3 e^{-i\rho}\left(1 - \frac{q_0}{4\rho i}\right) & -\,i\rho^3 e^{i\rho}\left(1 + \frac{q_0}{4\rho i}\right) & \rho^3 e^{\rho}\left(1 + \frac{q_0}{4\rho}\right) \end{vmatrix}$$

$$= \rho^6 e^{\rho}\left\{\begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & -\,i & i & 0 \\ 0 & -\,e^{-i\rho} & -\,e^{i\rho} & 1 \\ 0 & ie^{-i\rho} & -\,ie^{i\rho} & 1 \end{vmatrix} + O\left(\frac{1}{\rho^3}\right)\right\}$$

$$= 2i\rho^6 e^{\rho}\left\{e^{i\rho} + e^{-i\rho} + O\left(\frac{1}{\rho^3}\right)\right\}.$$

Thus, the roots of $\Delta(\lambda)$ are the roots of the equation

$$e^{i\rho} + e^{-i\rho} + O\left(\frac{1}{\rho^3}\right) = 0. \tag{3.7}$$

Note that equation (3.7) is equivalent to the equation

$$e^{2i\rho} = -1 + O\left(\frac{1}{\rho^3}\right). \tag{3.8}$$

By the asymptotic formula (2.2) we have

$$\rho_k = \sqrt[4]{\mu_k} = \left(k - \frac{1}{2}\right)\pi + \varepsilon_k, \tag{3.9}$$

where $\varepsilon_k = o(1)$.

Using (3.9) from (3.8) we find

$$e^{2i\varepsilon_k} = 1 + O\left(\frac{1}{k^3}\right),$$

which implies that

$$\varepsilon_k = O\left(\frac{1}{k^3}\right). \tag{3.10}$$

Then the asymptotic formula (3.1) follows from (3.8) and (3.9).

By (3.1) we have

$$\begin{aligned} e^{i\rho_k} &= i\,(-1)^k + O\left(\tfrac{1}{k^3}\right), \\ e^{-i\rho_k} &= -i\,(-1)^k + O\left(\tfrac{1}{k^3}\right). \end{aligned} \tag{3.11}$$

Now we prove the validity of the asymptotic formula (3.2). Obviously, the eigenfunction $v(x,\rho)$, corresponding to the eigenvalue $\lambda = \rho^4$, can be represented in the following form

$$v(x,\rho) = A_\rho \begin{vmatrix} z_1(x,\rho) & z_2(x,\rho) & z_3(x,\rho) & z_4(x,\rho) \\ U_1(z_1) & U_1(z_2) & U_1(y_3) & U_1(z_4) \\ U_2(z_1) & U_2(z_2) & U_2(z_3) & U_2(z_4) \\ \tilde{U}_4(z_1) & \tilde{U}_4(z_2) & \tilde{U}_4(z_3) & \tilde{U}_4(z_4) \end{vmatrix}, \tag{3.12}$$

where A_ρ is a constant which depends on ρ.

Using formulas (3.1), (3.3)-(3.5) and (3.11) from (3.12) we obtain

$$v_k(x) = v(x,\rho_k)$$

$$= A_{\rho_k}\rho_k^4 e^{\rho_k}\left\{\begin{vmatrix} e^{-\rho_k x} & e^{-i\,\rho_k x} & e^{i\,\rho_k x} & e^{\rho_k(x-1)} \\ 1 & 1 & 1 & 0 \\ -1 & -\,i & i & 0 \\ 0 & ie^{-i\rho_k} & -\,ie^{i\rho_k} & 1 \end{vmatrix} + O\left(\frac{1}{\rho_k^3}\right)\right\}$$

$$= A_{\rho_k}\rho_k^4 e^{\rho_k}\left\{\begin{vmatrix} e^{-\rho_k x} & e^{-i\,\rho_k x} & e^{i\,\rho_k x} & e^{\rho_k(x-1)} \\ 1 & 1 & 1 & 0 \\ 0 & 1-i & 1+i & 0 \\ 0 & ie^{-i\rho_k} & -\,ie^{i\rho_k} & 1 \end{vmatrix} + O\left(\frac{1}{\rho_k^3}\right)\right\}$$

$$= 2i\, A_{\rho_k} \rho_k^4 e^{\rho_k} \left\{ \sin\left(k - \frac{1}{2}\right)\pi x - \cos\left(k - \frac{1}{2}\right)\pi x + e^{-\left(k-\frac{1}{2}\right)\pi x} \right.$$

$$\left. + (-1)^{k+1} e^{\left(k-\frac{1}{2}\right)\pi(x-1)} + O\left(\frac{1}{k^3}\right) \right\}.$$

By virtue of (2.4) we can choose the constant A_{ρ_k} as follows: $A_{\rho_k} = \frac{e^{-\rho_k}}{2i\rho_k^4}$. Then it follows from last relation that the asymptotic formula (3.2) is true. The proof of this theorem is complete.

4 Asymptotic formulas for eigenvalues and eigenfunctions of problem (1.1)-(1.4)

In this section we refine the asymptotic formulas for eigenvalues and eigenfunctions of the boundary value problem (1.1)-(1.4).

Theorem 4.1 *For the eigenvalues λ_k and corresponding eigenfunctions $y_k(x)$ of the spectral problem* (1.1)-(1.4) *one has the asymptotic formulas*

$$\sqrt[4]{\lambda_k} = \left(k - \frac{3}{2}\right)\pi + \frac{q(0)}{4k\pi} + O\left(\frac{1}{k^2}\right), \tag{4.1}$$

$$\begin{aligned} y_k(x) &= \sin\left(k - \tfrac{3}{2}\right)\pi x - \cos\left(k - \tfrac{3}{2}\right)\pi x + e^{-\left(k-\frac{3}{2}\right)\pi x} \\ &+ \tfrac{q_0 x - q_0(x)}{4k\pi}\left\{\sin\left(k - \tfrac{3}{4}\right)\pi x + \cos\left(k - \tfrac{3}{4}\right)\pi x\right\} - \tfrac{q_0 x + q_0(x)}{4k\pi} e^{-\left(k-\frac{3}{2}\right)\pi x} \\ &+ (-1)^k \tfrac{q_0 x - q_1(x) - q_0}{4k\pi} e^{\left(k-\frac{3}{2}\right)\pi(x-1)} + O\left(\tfrac{1}{k^2}\right), \end{aligned} \tag{4.2}$$

where $q_0(x) = \int\limits_0^x q(t)dt$, $q_0 = \int\limits_0^1 q(x)dx$, $q_1(x) = \int\limits_x^1 q(t)dt$ *and the relation* (4.2) *holds uniformly for* $x \in [0, 1]$.

Proof. We set $\lambda = \eta^4$ in equation (1.1). As is known (see [16, Ch. II, § 4.5 Theorem 1 and § 4.6 formula (27)-(29)]), in each subdomain $\mathcal{T}$ of the complex η-plane equation (1.1) has four linearly independent solutions $\varphi_k(x, \eta)$, $k = 1, 2, 3, 4$, regular in η (for sufficiently large η) and satisfying the relations

$$\varphi_k^{(s)}(x, \eta) = (\eta\omega_k)^s e^{\eta\,\omega_k x}\left\{1 + \frac{q_0(x)}{4\eta\omega_k} + O\left(\frac{1}{\eta^2}\right)\right\},\ k = 1, 2, 3, 4,\ s = 0, 1, 2, 3. \tag{4.3}$$

By (4.3) it follows from (1.2)-(1.4) that

$$\begin{aligned} &U_1(\varphi_k) = 1 + O\left(\eta^{-2}\right),\ U_2(\varphi_k) = \eta\omega_k\left(1 + O\left(\eta^{-2}\right)\right), \\ &U_3(\varphi_k) = \eta^2\omega_k^2 e^{\rho\omega_k}\left(1 + \tfrac{q_0}{4\eta\omega_k} + O\left(\eta^{-2}\right)\right), \\ &U_4(\varphi_k) = \eta^3\omega_k^3 e^{\eta\omega_k}\left(1 + \tfrac{q_0}{4\eta\omega_k} + O\left(\eta^{-2}\right)\right). \end{aligned} \tag{4.4}$$

Recall that the eigenvalues of problem (1.1)-(1.4) are the roots of the characteristic determinant

$$\Delta(\lambda)=\begin{vmatrix} U_1(\varphi_1) & U_1(\varphi_2) & U_1(\varphi_3) & U_1(\varphi_4)\\ U_2(\varphi_1) & U_2(\varphi_2) & U_2(\varphi_3) & U_2(\varphi_4)\\ U_3(\varphi_1) & U_3(\varphi_2) & U_3(\varphi_3) & U_3(\varphi_4)\\ U_4(\varphi_1) & U_4(\varphi_2) & U_4(\varphi_3) & U_4(\varphi_4) \end{vmatrix}. \tag{4.5}$$

In view of (4.3) and (4.4) from (4.5) we get

$$\Delta(\lambda)=\begin{vmatrix} 1 & 1 & 1 & 1\\ -\eta & -i\eta & i\eta & \eta\\ \eta^2e^{-\eta}\left(1-\frac{q_0}{4\eta}\right) & -\eta^2e^{-i\eta}\left(1-\frac{q_0}{4\eta i}\right) & -\eta^2e^{i\eta}\left(1+\frac{q_0}{4\eta i}\right) & \eta^2e^{\eta}\left(1+\frac{q_0}{4\eta}\right)\\ -\eta^3e^{-\eta}\left(1-\frac{q_0}{4\eta}\right) & i\eta^3e^{-i\eta}\left(1-\frac{q_0}{4\eta i}\right) & -i\eta^3e^{i\eta}\left(1+\frac{q_0}{4\eta i}\right) & \eta^3e^{\eta}\left(1+\frac{q_0}{4\eta}\right) \end{vmatrix}+O\left(\eta^{-2}\right)$$

$$=\eta^6e^{\eta}\left\{\begin{vmatrix} 1 & 1 & 1 & 0\\ -1 & -i & i & 0\\ 0 & -e^{-i\eta}\left(1-\frac{q_0}{4\eta i}\right) & -e^{i\eta}\left(1+\frac{q_0}{4\eta i}\right) & 1+\frac{q_0}{4\eta}\\ 0 & ie^{-i\eta}\left(1-\frac{q_0}{4\eta i}\right) & -ie^{i\eta}\left(1+\frac{q_0}{4\eta i}\right) & 1+\frac{q_0}{4\eta} \end{vmatrix}+O\left(\eta^{-2}\right)\right\}$$

$$=\eta^6e^{\eta}(1+\frac{q_0}{4\eta})\left\{\begin{vmatrix} 1 & 1 & 1 & 0\\ 0 & 1-i & 1+i & 0\\ 0 & -e^{-i\eta}\left(1-\frac{q_0}{4\eta i}\right) & -e^{i\eta}\left(1+\frac{q_0}{4\eta i}\right) & 1\\ 0 & ie^{-i\eta}\left(1-\frac{q_0}{4\eta i}\right) & -ie^{i\eta}\left(1+\frac{q_0}{4\eta i}\right) & 1 \end{vmatrix}+O\left(\eta^{-2}\right)\right\}$$

$$=\eta^6e^{\eta}(1+\frac{q_0}{4\eta})\begin{vmatrix} 1-i & 1+i & 0\\ -e^{-i\eta}\left(1-\frac{q_0}{4\eta i}\right) & -e^{i\eta}\left(1+\frac{q_0}{4\eta i}\right) & 1\\ ie^{-i\eta}\left(1-\frac{q_0}{4\eta i}\right) & -ie^{i\eta}\left(1+\frac{q_0}{4\eta i}\right) & 1 \end{vmatrix}$$

$$=2\eta^6e^{\eta}(1+\frac{q_0}{4\eta})\left\{e^{i\eta}\left\{1+\frac{q_0}{4\eta i}\right\}+e^{-i\eta}\left\{1-\frac{q_0}{4\eta i}\right\}+O\left(\eta^{-2}\right)\right\}.$$

Hence it follows from last relation that the roots of the characteristic determinant $\Delta(\lambda)$ coincide with the roots of the equation

$$e^{i\eta}\left\{1+\frac{q_0}{4\eta i}\right\}+e^{-i\eta}\left\{1-\frac{q_0}{4\eta i}(\right\}+O\left(\eta^{-2}\right)=0. \tag{4.6}$$

The equation (4.6) is equivalent to the following equation

$$e^{2\eta i}=-1+\frac{q_0}{2\eta i}+O\left(\eta^{-2}\right). \tag{4.7}$$

In view of formula (2.3) we have

$$\eta_k=\sqrt[4]{\lambda_k}=\left(k-\frac{3}{2}\right)\pi+\varepsilon_k, \tag{4.8}$$

where $\varepsilon_k=o(1)$. By virtue of (4.8) from (4.7) we get

$$e^{2i\varepsilon_k}=1-\frac{q_0-2/a}{2k\pi i}+O\left(\frac{1}{k^2}\right).$$

Hence we have

$$\varepsilon_k=\frac{q_0}{4k\pi}+O\left(\frac{1}{k^2}\right). \tag{4.9}$$

Using (4.9) from (4.8) we obtain (4.1).

By virtue of (4.1) we have the following relations

$$
\begin{aligned}
e^{i\eta_k}\left(1+\tfrac{q_0}{4k\pi i}\right) &= i\,(-1)^k\left(1-\tfrac{q_0}{4_k\pi i}\right)\left(1+\tfrac{q_0}{4k\pi i}\right)+O\left(\tfrac{1}{k^2}\right)\\
&= i\,(-1)^k+O\left(\tfrac{1}{k^2}\right),\\
e^{-i\eta_k}\left(1-\tfrac{q_0}{4k\pi i}\right) &= -i\,(-1)^k\left(1+\tfrac{q_0}{4_k\pi i}\right)\left(1-\tfrac{q_0}{4k\pi i}\right)+O\left(\tfrac{1}{k^2}\right)\\
&= -i\,(-1)^k+O\left(\tfrac{1}{k^2}\right),
\end{aligned}
\tag{4.10}
$$

Now we prove that the asymptotic formula (4.2) is valid. Clearly, the eigenfunction $y(x,\mu)$ corresponding to the eigenvalue $\lambda=\eta^4$ has the form

$$
y(x,\eta)=B_\eta\begin{vmatrix}\varphi_1(x,\eta) & \varphi_2(x,\eta) & \varphi_3(x,\eta) & \varphi_4(x,\eta)\\ U_1(\varphi_1) & U_1(\varphi_2) & U_1(\varphi_3) & U_1(\varphi_4)\\ U_2(\varphi_1) & U_2(\varphi_2) & U_2(\varphi_3) & U_2(\varphi_4)\\ U_4(\varphi_1) & U_4(\varphi_2) & U_4(\varphi_3) & U_4(\varphi_4)\end{vmatrix}. \tag{4.11}
$$

where B_η is a constant which depends of η.

By virtue of (4.1), (4.3) and (4.10) it follows from (4.11) that

$$
y(x,\eta_k)=\eta_k^4 e^{\eta_k}B_{\eta_k}\times
$$

$$
\times\begin{vmatrix} e^{-\eta_k x}\left(1-\frac{q_0(x)}{4k\pi}\right) & e^{-i\eta_k x}\left(1-\frac{q_0(x)}{4k\pi i}\right) & e^{i\eta_k x}\left(1+\frac{q_0(x)}{4k\pi i}\right) & e^{\eta_k(x-1)}\left(1+\frac{q_0(x)}{4k\pi}\right)\\ 1 & 1 & 1 & 0\\ -1 & -\,i & i & 0\\ 0 & (-1)^k & (-1)^k & 1+\frac{q_0}{4k\pi}\end{vmatrix}
$$

$$
+O\left(\frac{1}{k^2}\right)+e^{-\eta_k x}\left(1-\frac{q_0(x)}{4k\pi}\right)\begin{vmatrix}1 & 1 & 0\\ -i & i & 0\\ (-1)^k & (-1)^k & \left(1+\frac{q_0}{4k\pi}\right)\end{vmatrix}
$$

$$
-e^{-i\eta_k x}\left(1-\frac{q_0(x)}{4k\pi i}\right)\begin{vmatrix}1 & 1 & 0\\ -1 & i & 0\\ 0 & (-1)^k & \left(1+\frac{q_0}{4k\pi}\right)\end{vmatrix}
$$

$$
+e^{i\eta_k x}\left(1+\frac{q_0(x)}{4k\pi i}\right)\begin{vmatrix}1 & 1 & 0\\ -1 & -\,i & 0\\ 0 & (-1)^k & \left(1+\frac{q_0}{4k\pi}\right)\end{vmatrix}
$$

$$
-e^{\eta_k x(x-1)}\left(1+\frac{q_0(x)}{4k\pi}\right)\begin{vmatrix}1 & 1 & 1\\ -1 & -\,i & i\\ 0 & (-1)^k & (-1)^k\end{vmatrix}
$$

$$
=2i\,\eta_k^4 e^{\eta_k}\left(1+\frac{q_0}{4k\pi}\right)B_{\eta_k}
$$

$$
\times\left\{\left(1-\frac{q_0(x)}{4k\pi}\right)\sin\eta_k x-\left(1+\frac{q_0(x)}{4k\pi}\right)\cos\eta_k x+\left(1-\frac{q_0(x)}{4k\pi}\right)e^{-\eta_k x}\right.
$$

$$
\left.+\,(-1)^k e^{\eta_k(x-1)}\left(1-\frac{q_1(x)}{4k\pi}\right)+O\left(\frac{1}{k^2}\right)\right\}.
$$

In view of (2.5) we can choose B_{η_k} as follows:

$$B_{\eta_k} = \frac{\mu_k^{-4} e^{-\eta_k} \left(1 + \frac{q_0}{4k\pi}\right)^{-1}}{2i}.$$

Hence we have

$$y_k(x) = y(x, \eta_k)$$

$$= \left\{ \left(1 - \frac{q_0(x)}{4k\pi}\right) \sin \eta_k x - \left(1 + \frac{q_0(x)}{4k\pi}\right) \cos \eta_k x + \left(1 - \frac{q_0(x)}{4k\pi}\right) e^{-\eta_k x} \right. \tag{4.12}$$

$$\left. + (-1)^k e^{\eta_k (x-1)} \left(1 - \frac{q_1(x)}{4k\pi}\right) + O\left(\frac{1}{k^2}\right) \right\}.$$

By virtue of (4.1) we find the following relations

$$\sin \eta_k x = \sin \left(k - \frac{3}{2}\right) \pi x + \frac{q_0}{4k\pi} \cos \left(k - \frac{3}{2}\right) \pi x + O\left(\frac{1}{k^2}\right),$$

$$\cos \eta_k x = \cos \left(k - \frac{3}{2}\right) \pi x - \frac{q_0}{4k\pi} \sin \left(k - \frac{3}{2}\right) \pi x + O\left(\frac{1}{k^2}\right),$$

$$e^{-\eta_k x} = e^{-\left(k - \frac{3}{2}\right)\pi x} \left\{1 - \frac{q_0 x}{4k\pi}\right\} + O\left(\frac{1}{k^2}\right).$$

$$e^{\eta_k (x-1)} = e^{-\left(k - \frac{3}{2}\right)\pi x} \left\{1 + \frac{q_0(x-1)}{4k\pi}\right\} + O\left(\frac{1}{k^2}\right).$$

Then using these relations from (4.12) we obtain the asymptotic formula (4.2). The proof of this theorem is complete.

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