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Asymptotic behavior of eigenvalues of a boundary value problem for Sturm-Liouville operator equation with a spectral parameter in one of the boundary conditions

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Abstract. *In separable Hilbert space H we consider asymptotic behavior of eigenvalues of a boundary value problem for Sturm-Liouville operator equation on a finite segment in the case when one and the same spectral parameter linearly particientering the equation and one of the boundary condition. Asymptotic formula of eigenvalues of the considered boundary value problem is obtained.*

Keywords. Sturm-Liouville operator, equation, spectral parameter, asymptotical behavior of eigenvalues.

Mathematics Subject Classification (2010): 35G25, 35P20, 47A75

1 Introduction

In the paper, in separable Hilbert space H we consider asymptotic behavior of eigenvalues of the following boundary value problems for the Strum-Lioville operator equation

$$-u''(x) + Au(x) = \lambda u(x),\ x \in (0,1), \tag{1.1}$$

subject boundary conditions

$$u'(1) = 0, \quad u(0) - d\lambda u'(0) = 0, \tag{1.2}$$

where $d > 0$ is some positive number; A is a linear, unbounded, self-adjoint, positive-definite operator in H and A^{-1} is completely continuous in H .

It is proved that eigenvalues of boundary value problem (1.1), (1.2) are real. Further, it is shown that problem (1.1), (1.2) has a set of eigenvalues that behaves as $\mu_k + n^2\pi^2$, where $\mu_k = \mu_k(A)$ are eigenvalues of the operator A.

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Asymptotic behavior of eigenvalues of boundary value problems for equation (1.1) in the case of boundary conditions of the form

$$u(1) = 0,\ u'(0) + \lambda u(0) = 0, \tag{1.3}$$

was studied in the papers [4], [6], in the case of boundary conditions of the from

$$u'(1) - \lambda u(1) = 0,\ u'(0) + \lambda u(0) = 0, \tag{1.4}$$

was studied in the paper [1]. It is proved that spectres of boundary value problems (1.1), (1.3) and (1.1), (1.4) are discrete and have two sets of eigenvalues $\lambda_k \sim \sqrt{\mu_k}$, $\lambda_{k,n} \sim \mu_k + n^2\pi^2$, where $\mu_k = \mu_k(A)$ are eigenvalues of the operator A.

In the papers [2], [3] asymptotic behavior of eigenvalues of boundary value problems are studied for Sturm-Liouville operator equation in the case when one and the same spectral parameter enters quadratically into the equations and linearly into the boundary conditions.

2 Main results

Lemma 2.1 *Eigenvalues of boundary value problem* (1.1), (1.2) *are real.*

Proof. We denote eigenelements of the operator A, corresponding to eigenvalues μ_k, by $\varphi_k,\ k = 1, 2, 3, ...$ It is known that $\{\varphi_k\}_{k=1}^{\infty}$ forms a complete orthonormed basis in H. Then from the expansion $u(x) = \sum_{k=1}^{\infty} (u(x), \varphi_k)_H \varphi_k$, for Fourier coefficients $u_k(x) = (u(x), \varphi_k)_H$, we get the following spectral problem:

$$-u_k''(x) + \mu_k u_k(x) = \lambda u_k(x),\ \ x \in (0, 1), \tag{2.1}$$

$$u_k'(1) = 0, \quad u_k(0) - d\lambda u_k'(0) = 0. \tag{2.2}$$

Thus, study of eigenvalues of boundary value problem (1.1), (1.2) is reduced to the study of eigenvalues of boundary value problem (2.1), (2.2), for different natural k. Eigenvalues of boundary problem (1.1), (1.2) consist of those λ, under which problem (2.1), (2.2) has nontrivial solution $u_k(x)$, even if for one k. The number $\lambda = \mu_k$ may not be an eigenvalue of problem (2.1), (2.2), as in this case this problem has only a trivial solution.

Let λ be an eigenvalue of boundary value problem (2.1) , (2.2) and $u_k(x, \lambda)$be an appropriate eigenfunction. Multiplying the both hand sides of equality (2.1) by the function $\overline{u_k(x, \lambda)}$ and integrating the obtained identity with respect to x from 0 to 1 , we get

$$-\int_0^1 u_k''(x,\lambda)\,\overline{u_k(x,\lambda)}dx + \mu_k \int_0^1 |u_k(x,\lambda)|^2 dx = \lambda \int_0^1 |u_k(x,\lambda)|^2 dx. \tag{2.3}$$

Using the formula of integration by parts and taking into account boundary conditions (2.2), we get

$$\int_0^1 u_k''(x,\lambda)\,\overline{u_k(x,\lambda)}dx = \overline{u_k(x,\lambda)}u_k'(x,\lambda)\Big|_0^1 - \int_0^1 u_k'(x,\lambda)\,\overline{u_k'(x,\lambda)}dx$$

$$= \overline{u_k(1,\lambda)}u_k'(1,\lambda) - \overline{u_k(0,\lambda)}\ u_k'(0,\lambda) - \int_0^1 |u_k'(x,\lambda)|^2 dx$$

$$= -\frac{1}{d\lambda}\left|u_k(0,\lambda)\right|^2 - \int\limits_0^1 \left|u_k'(x,\lambda)\right|^2 dx.$$

Hence from (2.3) we obtain

$$\lambda^2 \int\limits_0^1 \left|u_k(x,\lambda)\right|^2 dx - \lambda\left(\int_0^1 \left|u_k'(x,\lambda)\right|^2 dx + \mu_k \int\limits_0^1 \left|u_k(x,\lambda)\right|^2 dx\right)$$

$$-\frac{1}{d}\left|u_k(0,\lambda)\right|^2 = 0. \tag{2.4}$$

Denote

$$a_k(\lambda) = \int_0^1 \left|u_k(x,\lambda)\right|^2 dx, \quad b_k(\lambda) = -\left(\int_0^1 \left|u_k'(x,\lambda)\right|^2 + \mu_k \int_0^1 \left|u_k(x,\lambda)\right|^2 dx\right),$$

$$c_k(\lambda) = -\frac{1}{d}\left|u_k(0,\lambda)\right|^2.$$

Then we can rewrite equation (2.4) in the form

$$a_k(\lambda)\lambda^2 + b_k(\lambda)\lambda + c_k(\lambda) = 0. \tag{2.5}$$

Since $k, a_k(\lambda) > 0$, $c_k(\lambda) < 0$, for any $k \in N$, we have $b_k^2(\lambda) - 4a_k(\lambda)c_k(\lambda) > 0$. Therefore, for each k, equation (2.5) has only real roots. The lemma is proved

Theorem 2.1 *Let A be a self-adjoint, positive-definite operator in separable Hilbert space H and A^{-1} be completely continuous in H; $d > 0$ be some number.*

Then boundary value problem (1.1), (1.2) *has one set of eigenvalues*

$$\lambda_{n,k} = \mu_k + \gamma_n,$$

where μ_k, $k \in N$, are eigenvalues of the operator A such that $\mu_k \to +\infty$ as $k \to \infty$ and $\gamma_n \sim n^2\pi^2$ as $n \to \infty$.

Proof. The general solution of ordinary differential equation (2.1) has the following form:

$$u_k(x,\lambda) = c_1 e^{-x\sqrt{\mu_k-\lambda}} + c_2 e^{-(1-x)\sqrt{\mu_k-\lambda}}, \tag{2.6}$$

where c_i, $i = 1,2$ are arbitrary constants. Using substituted (2.6) from (2.2) we get a system with respect c_i , $i = 1,2$, whose determinant has the form

$$D(\lambda) = -\sqrt{\mu_k-\lambda}\left[\left(1 - d\lambda\sqrt{\mu_k-\lambda}\right)\ e^{-2\sqrt{\mu_k-\lambda}} + \left(1 + d\lambda\sqrt{\mu_k-\lambda}\right)\right].$$

Thus, eigenvalues of boundary value problem (2.1), (2.2) and by the same token, boundary value problem (1.1), (1.2), are the zeros of the following equation (with respect to λ, $\lambda \neq \mu_k$), even if for one k

$$\left(1 - d\lambda\sqrt{\mu_k-\lambda}\right)\ e^{-2\sqrt{\mu_k-\lambda}} + \left(1 + d\lambda\sqrt{\mu_k-\lambda}\right) = 0. \tag{2.7}$$

Rewrite equation (2.7) in the form

$$ch\sqrt{\mu_k-\lambda} + d\lambda\sqrt{\mu_k-\lambda}sh\sqrt{\mu_k-\lambda} = 0. \tag{2.8}$$

Find the eigenvalues of λ, for which $\lambda < \mu_k$. Put into the equation (2.8) $\sqrt{\mu_k - \lambda} = y$ $(0 < y \leq \sqrt{\mu_k})$. Then equation (2.8) will take the form

$$chy + dy\left(\mu_k - y^2\right) shy = 0,\ 0 < y \leq \sqrt{\mu_k}. \tag{2.9}$$

Equation (2.9) is equivalent to the equation

$$cthy + d\left(\mu_k - y^2\right) y = 0,\ \ 0 < y \leq \sqrt{\mu_k}.$$

Let us consider the function $f_k(y) = cthy + d\left(\mu_k - y^2\right) y,\ \ y \in \left(0, \sqrt{\mu_k}\,\right]$. It is obvious that for each k, and for any $y \in \left(0, \sqrt{\mu_k}\,\right]$, $f_k(y) > 0$. Therefore, problem (2.1), (2.2) and by the same token, problem (1.1), (1.2) have no eigenvalues for $\lambda < \mu_k$.

Now find the eigenvalues of λ, for which $\lambda > \mu_k$. Put into the equation (2.8) $\sqrt{\lambda - \mu_k} = z\ \ (0 < z < \infty)$. Then equation (2.8) takes the form

$$\cos z - d(z^2 + \mu_k) z \sin z = 0,\ \ \ z \in (0, \infty)\,. \tag{2.10}$$

Let $z \neq n\pi,\ \ n = 1, 2, ...$. Then equation (2.10) is equivalent to the equation

$$ctgz - d(z^2 + \mu_k)z = 0,\ \ \ z \in (0, +\infty)\,,\ z \neq n\pi,\ \ n = 1, 2, ... \tag{2.11}$$

Let us consider the function

$$\varphi_k(z) = ctgz - d(z^2 + \mu_k)z,\ \ \ z \in (0, \infty)\,, z \neq n\pi,\ \ n = 1, 2, ...$$

As at each interval $(n\pi,\ \ (n+1)\pi)$, $n = 1, 2, ...$ the function $\varphi_k(z)$ runs from $-\infty$ to $+\infty$, and its derivative

$$\varphi_k'(z) = -\frac{1}{\sin^2 z} - d(3z^2 + \mu_k) < 0,$$

then for each k, the function $\varphi_k(z)$ has only one zero $z_{n,k}$: $n\pi < z_{n,k} < (n+1)\pi$, $n = 1, 2, ...$. Let us find asymptotic formulas for $z_{n,k}$, for each k, as $n \to \infty$. From (2.11) we have

$$ctgz = d(z^2 + \mu_k)z,\ \ \ z \in (0, +\infty)\,,\ z \neq n\pi,\ \ n = 1, 2, ...$$

Denote, $q_k(z) = d(z^2 + \mu_k)z,\ \ z \in (0, +\infty)$. Obviously, for each k, $q_k(z) > 0$, $q_k'(z) = d(3z^2 + \mu_k) > 0,\ \ q_k''(z) = 6dz > 0$. So, for each k $q_k(z)$ a is a positive, increasing, strictly downwards convex function, $\lim\limits_{z \to 0^+} q_k(z) = 0$, and $\lim\limits_{z \to +\infty} q_k(z) = +\infty$. Obviously, the point $z_{n,k}$ are the abscissas of the points of intersection of the function, $q_k(z)$ and the branches of the function $ctgz$. When increasing n the points $z_{n,k}$ will approach the points $n\pi$, i.e., $z_{n,k} \sim n\pi$. Hence and from the equality $\sqrt{\lambda - \mu_k} = z$ for eigenvalues of boundary value problems (1.1),(1.2) satisfying the condition $\lambda > \mu_k$, we get the following asymptotic formula: $\lambda_{n,k} \sim \mu_k + n^2\pi^2$. The theorem is proved.

Remark 2.1 If for Hilbert space H we take $R = (-\infty, +\infty)$ and in equation (1.1) take $A = 0$, we get the following spectral problem for second order ordinary differential equation

$$u''(x) + \lambda u(x) = 0,\ \ x \in (0, 1)\,, \tag{2.12}$$

$$u'(1) = 0,\ \ \ \ u(0) - d\lambda u'(0) = 0,\ \ d > 0, \tag{2.13}$$

that was studied in the paper [5]. From the paper [5] it follows that eigenvalues of boundary value problems (2.12), (2.13) behave asymptotically as $n^2\pi^2$.

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