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Fractional integral associated to Schrödinger operator on the Heisenberg groups in vanishing generalized Morrey spaces

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Abstract. *Let $L = -\Delta_{\mathbb{H}_n} + V$ be a Schrödinger operator on the Heisenberg groups $\mathbb{H}_n$, where the non-negative potential V belongs to the reverse Hölder class $RH_{Q/2}$ and Q is the homogeneous dimension of $\mathbb{H}_n$. Let b belong to a new $BMO_\theta(\mathbb{H}_n, \rho)$ space, and let $\mathcal{I}_\beta^L$ be the fractional integral operator associated with L. In this paper, we study the boundedness of the operator $\mathcal{I}_\beta^L$ and its commutators $[b, \mathcal{I}_\beta^L]$ with $b \in BMO_\theta(\mathbb{H}_n, \rho)$ on vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ associated with Schrödinger operator. We find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operator $\mathcal{I}_\beta^L$ from $VM_{p,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $LM_{q,\varphi_2}^{\alpha,V}(\mathbb{H}_n)$, $1/p - 1/q = \beta/Q$. When b belongs to $BMO_\theta(\mathbb{H}_n, \rho)$ and (φ_1, φ_2) satisfies some conditions, we also show that the commutator operator $[b, \mathcal{I}_\beta^L]$ are bounded from $VM_{p,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $VM_{q,\varphi_2}^{\alpha,V}(\mathbb{H}_n)$, $1/p - 1/q = \beta/Q$.*

Keywords. Schrödinger operator; Heisenberg group; vanishing generalized Morrey space; fractional integral; commutator; BMO.

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1 Introduction and main results

Heisenberg groups, in discrete and continuous versions, appear in many parts of mathematics, including Fourier analysis, several complex variables, geometry, and topology. We state some basic results about Heisenberg group. More detailed information can be found in [5, 7, 8] and the references therein.

Let us consider the Schrödinger operator on Heisenberg group $\mathbb{H}_n$

$$L = -\Delta_{\mathbb{H}_n} + V \ \text{ on } \ \mathbb{H}_n, \ n \geq 3,$$

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where V is a non-negative, $V \neq 0$, and belongs to the reverse Hölder class RH_q for some $q \geq Q/2$, i.e., there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(g,r)|}\int_{B(g,r)} V^q(h)dh\right)^{1/q} \leq \frac{C}{|B(g,r)|}\int_{B(g,r)} V(h)dh \tag{1.1}$$

holds for every $g \in \mathbb{H}_n$ and $0 < r < \infty$, where $B(g,r)$ denotes the ball centered at g with radius r. In particular, if V is a nonnegative polynomial, then $V \in RH_\infty$.

Obviously, $RH_{q_2} \subset RH_{q_1}$, if $q_2 > q_1$. The reverse Hölder class RH_q have property, that is, if $V \in RH_q$, then $V \in RH_{q+\epsilon}$ for some $\epsilon > 0$.

We define the auxiliary function $0 < \rho(g) < \infty$ for a given potential $V \in RH_q$ with $q \geq Q/2$,

$$\rho(g) := \frac{1}{m_V(g)} = \sup_{r>0}\left\{r : \frac{1}{r^{Q-2}}\int_{B(g,r)} V(h)dh \leq 1\right\}$$

for any $g \in \mathbb{H}_n$ (for example, see [20]).

The BMO space $BMO_\theta(\mathbb{H}_n, \rho)$ associated with Schrödinger operator with $\theta \geq 0$ is defined as a set of all locally integrable functions b such that

$$\frac{1}{|B(g,r)|}\int_{B(g,r)} |b(h) - b_B|dh \leq C\Big(1 + \frac{r}{\rho(g)}\Big)^\theta$$

for all $g \in \mathbb{H}_n$ and $r > 0$, where $b_B = \frac{1}{|B|}\int_B b(h)dh$ (see [3]). A norm for $b \in BMO_\theta(\mathbb{H}_n, \rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Clearly, $BMO(\mathbb{H}_n) \subset BMO_\theta(\mathbb{H}_n, \rho)$.

We give the definition of central (local) and global generalized Morrey spaces (including weak version) associated with Schrödinger operator, which introduced by Guliyev in [14] on the Euclidean setting (see also [1–3,22]) and in [11] on the Heisenberg setting.

Definition 1.1 *[11] Let $\varphi(r)$ be a positive measurable function on $(0,\infty)$, $1 \leq p < \infty$, $\alpha \geq 0$, and $V \in RH_q$, $q \geq 1$. We denote by $M^{\alpha,V}_{p,\varphi} = M^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ the generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in L^p_{loc}(\mathbb{H}_n)$ with finite quasinorm*

$$\|f\|_{M^{\alpha,V}_{p,\varphi}} = \sup_{g\in\mathbb{H}_n, r>0}\Big(1 + \frac{r}{\rho(g)}\Big)^\alpha \varphi(r)^{-1} r^{-Q/p}\|f\|_{L_p(B(g,r))}.$$

Also $WM^{\alpha,V}_{p,\varphi} = WM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ we denote the weak generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in WL^p_{loc}(\mathbb{H}_n)$ with

$$\|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{g\in\mathbb{H}_n, r>0}\Big(1 + \frac{r}{\rho(g)}\Big)^\alpha \varphi(r)^{-1} r^{-Q/p}\|f\|_{WL_p(B(g,r))} < \infty.$$

Remark 1.1 (i) When $\alpha = 0$, and $\varphi(r) = r^{(\lambda-Q)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the classical Morrey space $M_{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey in [16] on the Euclidean setting;

(ii) When $\varphi(r) = r^{(\lambda-Q)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the Morrey space associated with Schrödinger operator $M^{\alpha,V}_{p,\lambda}(\mathbb{R}^n)$ studied by Tang and Dong in [22] on the Euclidean setting;

(iii) When $\alpha = 0$, $M^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ is the generalized Morrey space $M_{p,\varphi}(\mathbb{H}_n)$ studied by Guliyev et al. in [13].

(iv) $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the generalized Morrey space associated with Schrödinger operator, studied by Guliyev in [14] on the Euclidean setting, see also [1,2] and [11].

The classical Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ was introduced by Morrey in [16] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [6,16,17]. The generalized Morrey spaces are defined with r^λ replaced by a general non-negative function $\varphi(r)$ satisfying some assumptions (see, for example, [9,10,12,13] and etc).

For brevity, in the sequel we use the notations

$$\mathfrak{A}_{p,\varphi}^{\alpha,V}(f;g,r) := \Big(1+\frac{r}{\rho(g)}\Big)^{\alpha} r^{-Q/p}\,\varphi(r)^{-1}\|f\|_{L_p(B(g,r))}$$

and

$$\mathfrak{A}_{\Phi,\varphi}^{W,\alpha,V}(f;g,r) := \Big(1+\frac{r}{\rho(g)}\Big)^{\alpha} r^{-Q/p}\,\varphi(r)^{-1}\|f\|_{WL_p(B(g,r))}.$$

Definition 1.2 *The vanishing generalized Morrey space associated with Schrödinger operator $VM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ is defined as the spaces of functions $f \in M_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ such that*

$$\lim_{r\to 0}\sup_{g\in\mathbb{H}_n}\mathfrak{A}_{p,\varphi}^{\alpha,V}(f;g,r) = 0. \tag{1.2}$$

The vanishing weak generalized Morrey space associated with Schrödinger operator $VWM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ is defined as the spaces of functions $f \in WM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ such that

$$\lim_{r\to 0}\sup_{g\in\mathbb{H}_n}\mathfrak{A}_{p,\varphi}^{W,\alpha,V}(f;g,r) = 0.$$

The vanishing spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ and $VWM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM_{p,\varphi}^{\alpha,V}} \equiv \|f\|_{M_{p,\varphi}^{\alpha,V}} = \sup_{g\in\mathbb{H}_n, r>0}\mathfrak{A}_{p,\varphi}^{\alpha,V}(f;g,r),$$

$$\|f\|_{VWM_{p,\varphi}^{\alpha,V}} \equiv \|f\|_{WM_{p,\varphi}^{\alpha,V}} = \sup_{g\in\mathbb{H}_n, r>0}\mathfrak{A}_{W,p,\varphi}^{\alpha,V}(f;g,r),$$

respectively.

In the case $\alpha = 0$, and $\varphi(r) = r^{(\lambda-Q)/p}$ $VM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ is the vanishing Morrey space $VM_{p,\lambda}$ introduced in [23] on the Euclidean setting, where applications to PDE were considered.

We refer to [2,4,18,19] for some properties of vanishing generalized Morrey spaces.

Definition 1.3 *Let $L = -\Delta_{\mathbb{H}_n} + V$ with $V \in RH_{Q/2}$. The fractional integral associated with L is defined by*

$$\mathcal{I}_\beta^L f(g) = L^{-\beta/2} f(g) = \int_0^\infty e^{-tL}(f)(g)\, t^{\beta/2-1}dt$$

for $0<\beta<Q$. The commutator of $\mathcal{I}_\beta^L$ is defined by

$$[b,\mathcal{I}_\beta^L]f(g) = b(g)\mathcal{I}_\beta^L f(g) - \mathcal{I}_\beta^L(bf)(g).$$

In this paper, we consider the boundedness of the operator $\mathcal{I}_\beta^L$ and its commutators $[b,\mathcal{I}_\beta^L]$ with $b \in BMO_\theta(\mathbb{H}_n,\rho)$ on vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ associated with Schrödinger operator.

Our main results are as follows.

Theorem 1.1 *Let $V \in RH_{Q/2}$, $\alpha \geq 0$, $1 \leq p < Q/\beta$, $1/q = 1/p - \beta/Q$ and $\varphi_1 \in \Omega^{\alpha,V}_{p,1}$, $\varphi_2 \in \Omega^{\alpha,V}_{q,1}$ satisfies the conditions*

$$c_\delta := \int_\delta^\infty \varphi_1(t)\frac{dt}{t} < \infty$$

for every $\delta > 0$, and

$$\int_r^\infty \varphi_1(t)\frac{dt}{t^{1-\beta}} \leq C_0\varphi_2(r), \tag{1.3}$$

where C_0 does not depend on $r > 0$. Then the operator $\mathcal{I}^L_\beta$ is bounded from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{q,\varphi_2}$ for $p > 1$ and from $VM^{\alpha,V}_{1,\varphi_1}$ to $VWM^{\alpha,V}_{\frac{Q}{Q-\beta},\varphi_2}$.

Theorem 1.2 *Let $V \in RH_{Q/2}$, $b \in BMO_\theta(\rho)$, $1 < p < Q/\beta$, $1/q = 1/p - \beta/Q$, and $\varphi_1 \in \Omega^{\alpha,V}_{p,1}$, $\varphi_2 \in \Omega^{\alpha,V}_{q,1}$ satisfies the conditions*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)\varphi_1(t)\frac{dt}{t^{1-\beta}} \leq c_0\varphi_2(r), \tag{1.4}$$

where c_0 does not depend on r,

$$\lim_{r\to 0}\frac{\ln\frac{1}{r}}{\varphi_2(r)} = 0 \tag{1.5}$$

and

$$c_\delta := \int_\delta^\infty \Big(1 + |\ln t|\Big)\varphi_1(t)\frac{dt}{t^{1-\beta}} < \infty \tag{1.6}$$

for every $\delta > 0$. Then the operator $[b, \mathcal{I}^L_\beta]$ is bounded from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{q,\varphi_2}$.

Remark 1.2 Note that, on the Euclidean setting the analogous of the Theorems 1.1 and 1.2 was proved in [2, Theorems 3 and 4]. The statements of the Theorems 1.1 and 1.2 in the case of $V \equiv 0$ also news.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Some Preliminaries

Let $\mathbb{H}_n$ be a Heisenberg group of dimension $2n + 1$, that is, a nilpotent Lie group with underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$. The group structure is given by

$$(x,\ t)(y,\ s)\ = (x + y,\ t + s\ + 2\sum_{j=1}^n (x_{n+j}y_j - x_j y_{n+j})).$$

The Lie algebra of left-invariant vector fields on $\mathbb{H}_n$ is spanned by

$$X_{2n+1} = \frac{\partial}{\partial t},\ X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial t},\ X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial t},\ j = 1,\ \cdots,\ n.$$

The non-trivial commutation relations are given by $[X_j,\ X_{n+j}] = -4X_{2n+1},\ j = 1, \cdots, n.$ The sub-Laplacian $\triangle_{\mathbb{H}_n}$ is defined by $\Delta_{\mathbb{H}_n} = \sum_{j=1}^{2n} X_j^2$. The Haar measure on $\mathbb{H}_n$ is simply the Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}$. The measure of any measurable set $E \subset \mathbb{H}_n$ is denoted by $|E|$. The homogeneous norm on $\mathbb{H}_n$ is defined by

$$|g| = (|x|^4 + |t|^2)^{\frac{1}{4}},\ g = (x,\ t) \in \mathbb{H}_n,$$

which leads to a left-invariant distance $d(g,\ h) = |g^{-1}h|$ on $\mathbb{H}_n$. The dilations on $\mathbb{H}_n$ have the form $\delta_r(x,\ t) = (rx,\ r^2t),\ r > 0$. The Haar measure on this group coincides with the Lebesgue measure $dx = dx_1 \ldots dx_{2n}dt$. The identity element in $\mathbb{H}_n$ is $e = 0 \in \mathbb{R}^{2n+1}$, while the element g^{-1} inverse to $g = (x,t)$ is $(-x,-t)$.

The ball of radius r and centered at g is $B(g,\ r) = \{h \in \mathbb{H}_n : |g^{-1}h| < r\}$ and $|B(g,r)| = r^Q|B(0,1)|$, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}_n$. If $B = B(g,\ r)$ then λB denote $B(g,\ \lambda r)$ for $\lambda > 0$. Clearly, we have $|\lambda B| = \lambda^Q|B|$.

For background on the analysis on the Heisenberg groups we refer the reader to [7,21].

We would like to recall the important properties concerning the critical function.

Lemma 2.1 *[15] Let $V \in RH_{Q/2}$. For the associated function ρ there exist C and $k_0 \geq 1$ such that*

$$C^{-1}\rho(g)\Big(1 + \frac{|h^{-1}g|}{\rho(g)}\Big)^{-k_0} \leq \rho(h) \leq C\rho(g)\Big(1 + \frac{|h^{-1}g|}{\rho(g)}\Big)^{\frac{k_0}{1+k_0}} \tag{2.1}$$

for all $g, h \in \mathbb{H}_n$.

Lemma 2.2 *[1], [9] Let φ be a positive measurable function on $(0,\infty)$, $1 \leq p < \infty$, $\alpha \geq 0$, and $V \in RH_q$, $q \geq 1$.*

(i) If

$$\sup_{t<r<\infty} \Big(1 + \frac{r}{\rho(g)}\Big)^{\alpha} \frac{r^{-\frac{Q}{p}}}{\varphi(r)} = \infty \quad \textit{for some } t > 0 \ \textit{ and for all } g \in \mathbb{H}_n, \tag{2.2}$$

then $M^{\alpha,V}_{p,\varphi}(\mathbb{H}_n) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \Big(1 + \frac{r}{\rho(g)}\Big)^{\alpha} \varphi(r)^{-1} = \infty \quad \textit{for some } \tau > 0 \ \textit{ and for all } g \in \mathbb{H}_n, \tag{2.3}$$

then $M^{\alpha,V}_{p,\varphi}(\mathbb{H}_n) = \Theta$.

Remark 2.1 We denote by $\Omega^{\alpha,V}_p$ (see [1,9,11]) the sets of all positive measurable functions φ on $(0,\infty)$ such that for all $t > 0$,

$$\sup_{g\in\mathbb{H}_n} \Big\|\Big(1+\frac{r}{\rho(g)}\Big)^{\alpha} \frac{r^{-\frac{Q}{p}}}{\varphi(r)}\Big\|_{L_\infty(t,\infty)} < \infty, \quad \text{and} \quad \sup_{g\in\mathbb{H}_n} \Big\|\Big(1+\frac{r}{\rho(g)}\Big)^{\alpha} \varphi(r)^{-1}\Big\|_{L_\infty(0,t)} < \infty,$$

respectively.

Remark 2.2 We denote by $\Omega^{\alpha,V}_{p,1}$ the sets of all positive measurable functions φ on $(0,\infty)$ such that

$$\inf_{g\in\mathbb{H}_n} \inf_{r>\delta} \Big(1 + \frac{r}{\rho(g)}\Big)^{-\alpha} \varphi(r) > 0, \text{ for some } \delta > 0, \tag{2.4}$$

and

$$\lim_{r\to 0} \Big(1 + \frac{r}{\rho(g)}\Big)^{\alpha} \frac{r^{Q/p}}{\varphi(r)} = 0.$$

For the non-triviality of the spaces $M^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$, $VM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ we always assume that $\varphi \in \Omega^{\alpha,V}_p$, $\varphi \in \Omega^{\alpha,V}_{p,1}$, respectively.

The following theorems was proved in [11].

Theorem 2.1 *[11] Let $V \in RH_{Q/2}$, $\alpha \geq 0$, $1 < p < Q/\beta$, $1/q = 1/p - \beta/Q$ and $\varphi_1 \in \Omega^{\alpha,V}_p$, $\varphi_2 \in \Omega^{\alpha,V}_q$ satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(s) s^{\frac{Q}{p}}}{t^{\frac{Q}{q}}} \frac{dt}{t} \leq c_0 \varphi_2(r), \tag{2.5}$$

where c_0 does not depend on r. Then the operator $\mathcal{I}^L_\beta$ is bounded on $M^{\alpha,V}_{p,\varphi_1}(\mathbb{H}_n)$ to $M^{\alpha,V}_{q,\varphi_2}(\mathbb{H}_n)$ for $p > 1$ and from $M^{\alpha,V}_{1,\varphi_1}(\mathbb{H}_n)$ to $WM^{\alpha,V}_{\frac{Q}{Q-\beta},\varphi_2}(\mathbb{H}_n)$.

Theorem 2.2 *[11] Let $V \in RH_{Q/2}$, $\alpha \geq 0$, $1 < p < Q/\beta$, $1/q = 1/p - \beta/Q$ and $\varphi_1 \in \Omega^{\alpha,V}_p$, $\varphi_2 \in \Omega^{\alpha,V}_q$ satisfies the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(s) s^{\frac{Q}{p}}}{t^{\frac{Q}{q}}} \frac{dt}{t} \leq c_0 \varphi_2(r), \tag{2.6}$$

where c_0 does not depend on r. If $b \in BMO_\theta(\mathbb{H}_n, \rho)$, then the operator $[b, \mathcal{I}^L_\beta]$ is bounded from $M^{\alpha,V}_{p,\varphi_1}(\mathbb{H}_n)$ to $M^{\alpha,V}_{q,\varphi_2}(\mathbb{H}_n)$.

3 Proof of Theorem 1.1

The statement is derived from the estimate

$$\|\mathcal{I}^L_\beta(f)\|_{L_q(B(g,r))} \lesssim r^{\frac{Q}{q}} \int_{2r}^\infty \frac{\|f\|_{L_p(B(g,t))}}{t^{\frac{Q}{q}}} \frac{dt}{t} \tag{3.1}$$

holds for $1 \leq p < Q/\beta$, which proved in [11, Theorem 3.1]. The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Theorem 2.1. So we only have to prove that

$$\lim_{r\to 0} \sup_{g\in\mathbb{H}_n} \mathfrak{A}^{\alpha,V}_{p,\varphi_1}(f; g, r) = 0 \;\; \Rightarrow \;\; \lim_{r\to 0} \sup_{g\in\mathbb{H}_n} \mathfrak{A}^{\alpha,V}_{q,\varphi_2}(\mathcal{I}^L_\beta(f); g, r) = 0 \tag{3.2}$$

and

$$\lim_{r\to 0} \sup_{g\in\mathbb{H}_n} \mathfrak{A}^{\alpha,V}_{1,\varphi_1}(f; g, r) = 0 \;\; \Rightarrow \;\; \lim_{r\to 0} \sup_{g\in\mathbb{H}_n} \mathfrak{A}^{W,\alpha,V}_{Q/(Q-\beta),\varphi_2}(\mathcal{I}^L_\beta(f); g, r) = 0. \tag{3.3}$$

To show that $\sup_{g\in\mathbb{H}_n} \Big(1 + \frac{r}{\rho(g)}\Big)^\alpha \varphi_2(r)^{-1} r^{-Q/p} \|\mathcal{I}^L_\beta(f)\|_{L_q(B(g,r))} < \varepsilon$ for small r, we split the right-hand side of (3.1):

$$\Big(1 + \frac{r}{\rho(g)}\Big)^\alpha \varphi_2(r)^{-1} r^{-Q/p} \|\mathcal{I}^L_\beta(f)\|_{L_q(B(g,r))} \leq C[I_{\delta_0}(g, r) + J_{\delta_0}(g, r)], \tag{3.4}$$

where $\delta_0 > 0$ (we may take $\delta_0 > 1$), and

$$I_{\delta_0}(g,r) := \frac{\left(1+\frac{r}{\rho(g)}\right)^{\alpha}}{\varphi_2(r)} \int_r^{\delta_0} t^{-\frac{Q}{q}-1} \|f\|_{L_p(B(g,t))} dt$$

and

$$J_{\delta_0}(g,r) := \frac{\left(1+\frac{r}{\rho(g)}\right)^{\alpha}}{\varphi_2(r)} \int_{\delta_0}^{\infty} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(g,t))} dt$$

and it is supposed that $r < \delta_0$. We use the fact that $f \in VM_{p,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{H}_n} \left(1+\frac{t}{\rho(g)}\right)^{\alpha} \varphi_1(t)^{-1} t^{-Q/p} \|f\|_{L_p(B(g,t))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (1.3) and (3.4). This allows to estimate the first term uniformly in $r \in (0, \delta_0)$:

$$\sup_{g\in\mathbb{H}_n} CI_{\delta_0}(g,r) < \frac{\varepsilon}{2}, \ \ 0 < r < \delta_0.$$

The estimation of the second term now my be made already by the choice of r sufficiently small. Indeed, thanks to the condition (2.4) we have

$$J_{\delta_0}(g,r) \le c_{\sigma_0} \frac{\left(1+\frac{r}{\rho(g)}\right)^{\alpha}}{\varphi_1(r)} \|f\|_{VM_{p,\varphi_1}^{\alpha,V}},$$

where c_{σ_0} is the constant from (1.2). Then, by (2.4) it suffices to choose r small enough such that

$$\sup_{g\in\mathbb{H}_n} \frac{\left(1+\frac{r}{\rho(g)}\right)^{\alpha}}{\varphi_2(r)} \le \frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM_{p,\varphi_1}^{\alpha,V}}},$$

which completes the proof of (3.2).

The proof of (3.3) is similar to the proof of (3.2).

4 Proof of Theorem 1.2

The norm inequality having already been provided by Theorem 2.2, we only have to prove the implication

$$\lim_{r\to 0} \sup_{g\in\mathbb{H}_n} \left(1+\frac{r}{\rho(g)}\right)^{\alpha} \varphi_1(r)^{-1} r^{-Q/p} \|f\|_{L_p(B(g,r))} = 0$$
$$\Longrightarrow \lim_{r\to 0} \sup_{g\in\mathbb{H}_n} \left(1+\frac{r}{\rho(g)}\right)^{\alpha} \varphi_2(r)^{-1} r^{-Q/p} \|[b,\mathcal{I}_\beta^L(f)]\|_{L_q(B(g,r))} = 0.$$

To check that

$$\sup_{g\in\mathbb{H}_n} \left(1+\frac{r}{\rho(g)}\right)^{\alpha} \varphi_2(r)^{-1} r^{-Q/p} \|[b,\mathcal{I}_\beta^L(f)]\|_{L_q(B(g,r))} < \varepsilon \ \text{ for small } r,$$

we use the following estimate, which proved in [11, Theorem 3.6].

$$\varphi_2(r)^{-1}r^{-Q/p}\|[b,\mathcal{I}^L_\beta(f)]\|_{L_q(B(g,r))}\lesssim \frac{[b]_\theta}{\varphi_2(r)}\int_r^\infty\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(g,t))}}{t^{\frac{Q}{q}}}\frac{dt}{t}.$$

We take $r<\delta_0$, where δ_0 will be chosen small enough and split the integration:

$$\Big(1+\frac{r}{\rho(g)}\Big)^\alpha\varphi_2(r)^{-1}r^{-Q/p}\|[b,\mathcal{I}^L_\beta(f)]\|_{L_q(B(g,r))}\le C[I_{\delta_0}(g,r)+J_{\delta_0}(g,r)], \tag{4.1}$$

where

$$I_{\delta_0}(g,r):=\frac{\big(1+\frac{r}{\rho(g)}\big)^\alpha}{\varphi_2(r)}\int_r^{\delta_0}\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(g,t))}}{t^{\frac{Q}{q}}}\frac{dt}{t}$$

and

$$J_{\delta_0}(g,r):=\frac{\big(1+\frac{r}{\rho(g)}\big)^\alpha}{\varphi_2(r)}\int_{\delta_0}^\infty\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(g,t))}}{t^{\frac{Q}{q}}}\frac{dt}{t}.$$

We choose a fixed $\delta_0>0$ such that

$$\sup_{g\in\mathbb{H}_n}\Big(1+\frac{r}{\rho(x)}\Big)^\alpha\varphi_1(r)^{-1}r^{-Q/p}\|f\|_{L_p(B(g,r))}<\frac{\varepsilon}{2CC_0},\quad r\le\delta_0,$$

where C and C_0 are constants from (4.1) and (1.4), which yields the estimate of the first term uniform in $r\in(0,\delta_0):\ \sup\limits_{g\in\mathbb{H}_n}CI_{\delta_0}(g,r)<\frac{\varepsilon}{2},\ \ 0<r<\delta_0$.

For the second term, writing $1+\ln\frac{t}{r}\le 1+|\ln t|+\ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(g,r)\le\frac{c_{\delta_0}+\widetilde{c_{\delta_0}}\ \ln\frac{1}{r}}{\varphi_2(r)}\|f\|_{M^{\alpha,V}_{p,\varphi_1}},$$

where c_{δ_0} is the constant from (1.6) with $\delta=\delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (1.5) we can choose small r such that $\sup_{g\in\mathbb{H}_n}J_{\delta_0}(g,r)<\frac{\varepsilon}{2}$, which completes the proof.

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