Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
38 (2018), no. 4, Mathematics, 63-69.

Application of finite integral transformation method to the solution of a mixed problem with integral condition of vibration process

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Received: 17.03.2017 / Revised: 28.06.2018 / Accepted: 04.10.2018

Abstract. *In the present paper, using the finite integral transformation method, we obtain analytic representation of classic solution of mixed problems with integral condition of vibrations process, described by a wave equation and running its course for period of time* $0 \leq t \leq T$*. It is assumed that points with coordinates* $x = 0$ *and* $x = 1$ *are the ends of the vibrating string*

Keywords. finite integral transformation method, mixed problem with integral condition, analytic representation of classic solution

Mathematics Subject Classification (2010): 35L02, 35L15, 3520

1 Introduction

One of the methods for solving mixed problems for linear differential partial equations is operational calculus and method of integral transformations that were successfully used in the classic works of Cauchy, Heaviside, Laplace and others and then by Ephros A.M. and Danilevsckiy D.M.[6], A.N. Tikhonov [10], M.L.Rasulov [9] and others M.L.Rasulov showed that the Laplace integral transformation (consequently, the Heaviside transformation) is a weak tool in solving dynamic problems under nonzero initial conditions.

In the present paper, we suggest finite integral tharsformation method [10] that is used for solving dynamics problem under nonzero initial conditions.

In [7] V.A. Il'in and E.I. Moiseev considered a problem of boundary control of the process described by telegraph (hyperbolic) equation. At initial moment $t = 0$ when displacement and velocity of the points of string are known, V.A. Il'in in [8] studied boundary value problem of control when of vibration process at finite moment $t = T$ the displacement and velocity of the points of the string are known.

In the present paper at initial moment $t = 0$ when displacements and velocities of the points of the string are known, we study boundary control of vibrations process when time average (integral) displacements of the problem itself and the technique for solving it differs from the problems considered in references.

To find the classical solution of vibrations process described by the hyperbolic equation [10]

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$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + F(x),\ \ 0 < x < 1,\ \ 0 < t < T, \tag{1.1}$$

satisfying the initial conditions (at initial moment $t = 0$ the displacement and velocity of the string's points are unknown)

$$u(x,0) = \varphi_0(x),\ \ \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \varphi_1(x),\ \ 0 < x < 1, \tag{1.2}$$

and the boundary condition (generated on one end of the string $x = 0$ by elastic force)

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \mu(t),\ \ 0 < t < T, \tag{1.3}$$

and the integral condition (time averagle displacement of the string)

$$\frac{1}{T}\int_0^T u(x,\tau)d\tau = \varphi(x),\ \ 0 \le x \le 1, \tag{1.4}$$

where a, T are positive numbers, $u \equiv u(x,t)$ is the sought- for classic solution, the remaining's are known functions.

2 Solution of the mixed problem

We a priory assume that problem (1.1)-(1.4) has a classic solution, $u = u(x,t)$ is continuous in a closed set $0 \le x \le 1,\ \ 0 \le t \le T.$

Assume

$$\left.u(x,t)\right|_{x=1} = \nu(t),\ \ 0 < t < T, \tag{2.1}$$

where the function $\nu(t)$ should be determined.

For solving problem (1.1)-(1.4) it suffices to solve the following problem with a boundary control:to find classic solution $u \equiv u(x,t)$ of problem (1.1)-(1.3),(2.1), where the unknown control $\nu(t)$ is determined from integral condition (1.4).

Applying the finite integral transformation [1]

$$\widetilde{\varphi}(\lambda,t) \equiv \int_0^t \exp(-\lambda\tau)\varphi(\tau)d\tau, \tag{2.2}$$

(λ is a complex parameter) to mixed problem (1.1)-(1.3),(2.1), by the methods used in [3-5] we prove the following theorem.

Theorem 2.1 *Let $a(a > 0)$ be a real number, and $F(x), \varphi_0(x), \varphi_1(x) \in C([0,1])$. Then for $\mu(t),\ \nu(t) \in C([0,T])$ if classic problem (1.1)-(1.3),(2.1) has the classic solution $u \equiv u(x,t)$, then*

i) this solution is unique,

ii) thus solution is represented by the formula

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t),\ \ 0 < x < 1,\ \ 0 < t < T, \tag{2.3}$$

where

$$u_0(x,t) = \frac{1}{a}\sum_{k=0}^{\infty}\frac{1}{\lambda_k}(\varphi_1)_k \sin a\lambda_k t \cos\lambda_k x + \sum_{k=0}^{\infty}(\varphi_0)_k \cos a\lambda_k t \cos\lambda_k x$$

$$+\frac{1}{a^2}\sum_{k=0}^{\infty}\frac{1}{\lambda_k^2}(F)_k(1-\cos a\lambda_k t)\cos\lambda_k x;$$

$$u_1(x,t)=-2a\sum_{k=0}^{\infty}\frac{1}{\lambda_k}\cos\lambda_k x\int_0^t \mu(\tau)\sin a\lambda_k(t-\tau)d\tau;$$

$$u_2(x,t)=2a\sum_{k=0}^{\infty}(-1)^k\cos\lambda_k x\int_0^t \nu(\tau)\sin a\lambda_k(t-\tau)d\tau$$

$$(\psi)_k\equiv 2\int_0^t \psi(x)\cos\lambda_k x dx;$$

$$\lambda_k=\frac{\pi}{2}+k\pi. \tag{2.4}$$

1^0. *Let* $aT=1$.
2^0. *Let* $F(x)\in C^5([0,1])$, $\varphi_0(x)\in C^6([0,1])$, $\varphi_1(x)\in C^5([0,1])$, $\varphi_2(x)\in C^7([0,1])$,
$\varphi_2(x)\equiv T\varphi(x)$.
3^0.*Let* $\nu(0)=\varphi_0(1)$.
4^0. *Let* $\mu(t)\in C^5([0,T])$.
5^0. *Let*

$$\varphi_2'(0)-\int_0^1\mu(\tau)d\tau=0;$$
$$a^3\varphi_2''(1)+a\varphi_1(1)-a^2\varphi_0'(0)+a^2\mu(0)+F(1)=0;$$

$$a^3\varphi_2'''(0)+a\varphi_1'(0)+a^2\varphi_0''(1)+F'(0)+F(1)-a\mu'(1)=0;$$
$$a^3\varphi_2^{(4)}(1)+a\varphi_1''(1)+F''(1)+\mu''(0)-a^2\varphi_0'''(0)-F'(0)=0;$$
$$a^4\varphi_2^{(5)}(0)+a^2\varphi_1'''(0)+a^3\varphi_0^{(4)}(1)+aF'''(0)+aF''(1)-\mu'''(1)=0.$$

It holds

Theorem 2.2 *Let constraints* 1^0-5^0 *be fulfilled. Then problem (1.1)-(1.4) has a unique classic solution* $u\equiv u(x,t)$*, and this solutions determined by formula (2.3), where*

$$\nu(t)=\varphi_0(1)+\frac{1}{a}\int_0^{at}\nu'(\xi)\big|_{\xi=\frac{\tau}{a}}d\tau,\ \ 0\le t\le T; \tag{2.5}$$

$$\nu'(\xi)\Big|_{\xi=\frac{\tau}{a}}=\sum_{k=0}^{\infty}\nu_k\cos\lambda_k\tau,\ \ 0<\tau<aT=1, \tag{2.6}$$

$$\nu_k \equiv 2\int_0^1 \nu'(\xi)\big|_{\xi=\frac{\tau}{a}} \cos\lambda_k\tau d\tau = -\frac{2(-1)^k}{a\lambda_k^5}F'''(0) + \frac{2a^2}{\lambda_k^4}\int_0^1 \varphi_2^{(6)}(x)\times$$

$$\times\cos\lambda_k x dx + \frac{2}{\lambda_k^4}\int_0^1 \varphi_1^{(4)}(x)\cos\lambda_k x dx - \frac{2a(-1)^k}{\lambda_k^4}\int_0^1 \varphi_0^{(5)}(x)\sin\lambda_k x dx$$

$$+\frac{2}{a\lambda_k^4}\int_0^1 F^{(4)}(x)\cos\lambda_k x dx - \frac{2(-1)^k}{a\lambda_k^5}\int_0^1 F^{(4)}(x)\cos\lambda_k x dx$$

$$+\frac{2(-1)^k}{a^2\lambda_k^4}\int_0^T \mu^{(4)}(\tau)\sin a\lambda_k\tau d\tau. \tag{2.7}$$

Proof. Substituting (2.3) in (1.4), we have

$$\varphi_2(x) = \sum_{k=0}^{\infty}\frac{(\varphi_1)_k}{(a\lambda_k)^2}(1-\cos a\lambda_k T)\cos\lambda_k x + \sum_{k=0}^{\infty}(\varphi_0)_k\frac{\sin a\lambda_k T}{a\lambda_k}\cos\lambda_k x$$

$$+\frac{1}{a^2}\sum_{k=0}^{\infty}\frac{(F)_k}{\lambda_k^2}\left(T-\frac{\sin a\lambda_k T}{a\lambda_k}\right)\cos\lambda_k x - 2a\sum_{k=0}^{\infty}\frac{1}{\lambda_k}\cos\lambda_k x\times$$

$$\times\left[\frac{1}{a\lambda_k}\int_0^T \mu(\tau)d\tau - \frac{1}{(a\lambda_k)^3}\left(\mu'(T)-\mu'(0)\right) - \frac{\mu(0)}{(a\lambda_k)^2}\sin a\lambda_k T\right.$$

$$\left.-\frac{\mu'(0)}{(a\lambda_k)^3}(1-\cos a\lambda_k T) + \frac{\mu''(0)}{(a\lambda_k)^4}\sin a\lambda_k T + \frac{\mu'''(0)}{(a\lambda_k)^5}(1-\cos a\lambda_k T)\right]$$

$$-2\sum_{k=0}^{\infty}\frac{1}{a^4\lambda_k^6}\cos\lambda_k x\left[\mu'''(T)-\mu'''(0)-\int_0^T \mu^{(4)}(\tau)\cos a\lambda_k(T-\tau)d\tau\right]$$

$$+2\sum_{k=0}^{\infty}\frac{(-1)^k}{\lambda_k}\cos\lambda_k x\left[\int_0^T \nu(\tau)d\tau - \frac{\nu(0)}{a\lambda_k}\sin a\lambda_k T\right]$$

$$-2\sum_{k=0}^{\infty}\frac{(-1)^k}{a\lambda_k^2}\cos\lambda_k x\int_0^T \nu'(\tau)\sin a\lambda_k(T-\tau)d\tau. \tag{2.8}$$

By uniqueness of the expansions formula [4]

$$\varphi_2(x) = \sum_{k=0}^{\infty}(\varphi_2)_k\cos\lambda_k x,\;\; 0<x<1, \tag{2.9}$$

from (2.8) we have

$$\nu_k \equiv 2\int_0^T \nu'(\tau)\sin a\lambda_k(T-\tau)d\tau = (-1)^{k+1}\, a\lambda_k^2\,(\varphi_2)_k$$
$$+\frac{(-1)^k}{a}\,(\varphi_1)_k\,(1-\cos a\lambda_k T) + (-1)^k\,\lambda_k\,(\varphi_0)_k \sin a\lambda_k T$$
$$+\frac{(-1)^k (F)_k}{a}\left(T-\frac{\sin a\lambda_k T}{a\lambda_k}\right) - (-1)^k\, 2a\left[\int_0^T \mu(\tau)d\tau - \frac{1}{(a\lambda_k)^2}\mu^{'}(T)\right.$$
$$\left.-\frac{\mu(0)}{a\lambda_k}\sin a\lambda_k T + \frac{1}{(a\lambda_k)^2}\cos a\lambda_k T + \frac{\mu''(0)}{(a\lambda_k)^3}\sin a\lambda_k T + \frac{\mu^{'''}(0)}{(a\lambda_k)^4}\,(1-\cos a\lambda_k T)\right]$$
$$-\frac{(-1)^k 2}{a^3\lambda_k^4}\left[\mu^{'''}(T) - \mu^{'''}(0) - \int_0^T \mu^{(4)}(\tau)(\cos a\lambda_k(T-\tau)d\tau\right]$$
$$+2a\lambda_k\left[\varphi_2(1) - \frac{\varphi_0(1)}{a\lambda_k}\sin a\lambda_k T\right], \tag{2.10}$$

and in (2.10) we take into account the agreement conditions

$$\nu(0) = \varphi_0(1);\ \int_0^T \nu(\tau)d\tau = \varphi_2(1). \tag{2.11}$$

Under constraint 1^0 from (2.10) we have

$$2\int_0^T \nu'(\tau)\cos a\lambda_k\tau d\tau = -a\lambda_k^2(\varphi_2)_k + \frac{1}{a}(\varphi_1)_k + (-1)^k\lambda_k(\varphi_0)_k$$
$$+\frac{(F)_k}{a}\left(T-\frac{(-1)^k}{a\lambda_k}\right) - 2a\int_0^T \mu(\tau)d\tau + \frac{2}{a\lambda_k^2}\mu'(T) + \frac{(-1)^k 2}{\lambda_k}\mu(0)$$
$$-(-1)^k 2\frac{\mu^{''}(0)}{a^2\lambda_k^3} - \frac{2}{a^3\lambda_k^4}\mu^{'''}(T) + (-1)^k 2a\lambda_k\varphi_2(1)$$
$$-2\varphi_0(1) + \frac{2(-1)^k}{a^3\lambda_k^4}\int_0^T \mu^{(4)}(\tau)\sin a\lambda_k\tau d\tau. \tag{2.12}$$

Taking constraint 5^0 into account in (2.12) for ν_k we get formula (2.7). The function $\nu'(\xi)|_{\xi=r/a}$ is reconstructed by expansion formula (2.5) and by formula $\nu(t)$, and knowing this function, by formula (2.9) we construct the boundary control

From (2.7) we have

$$|\nu_k| \leq \frac{Const}{\lambda_k^5}. \tag{2.13}$$

Taking into account inequality (2.13) in (2.6) and (2.5), we have the inclusion

$$\nu(t) \in C^4([0,T]). \tag{2.14}$$

Taking inclusion (2.14) into account, we get

$$\int_0^t \nu(\tau)\sin a\lambda_k(t-\tau)d\tau = -\frac{\nu(0)}{a\lambda_k}\cos a\lambda_k t - \frac{1}{(a\lambda_k)^2}\nu^{'}(0)\sin a\lambda_k t +$$
$$+\frac{1}{(a\lambda_k)^3}\nu^{''}(0)\cos a\lambda_k t + \frac{1}{a\lambda_k}\nu(t) - \frac{1}{(a\lambda_k)^3}\nu^{''}(t) +$$
$$+\frac{1}{(a\lambda_k)^3}\int_0^t \nu^{'''}(\tau)\cos a\lambda_k(t-\tau)d\tau. \tag{2.15}$$

And from the inclusion $\mu(t) \in C^3([0,T])$ we have

$$\int_0^t \mu(\tau)\sin a\lambda_k(t-\tau)d\tau = -\frac{\mu(0)}{a\lambda_k}\cos a\lambda_k t - \frac{1}{(a\lambda_k)^2}\mu^{'}(0)\sin a\lambda_k t +$$
$$+\frac{1}{a\lambda_k}\mu(t) - \frac{1}{(a\lambda_k)^2}\int_0^t \mu^{''}(\tau)\sin a\lambda_k(t-\tau)d\tau. \tag{2.16}$$

Taking into account agreement condition 3^0 in (2.15) and substituting formula (2.16) in (2.4) and using constaint 2^0 (this constraint may be weakened) by direct verifaction we can see that (as [4] that for the function $u \equiv u(x,t)$, determined by formula (2.3), it holds the inclusion

$$u(x,t), \frac{\partial u(x,t)}{\partial x}, \frac{\partial u(x,t)}{\partial t} \in C([0,1]\times[0,T]),$$

$$\frac{\partial^2 u(x,t)}{\partial t^2}, \frac{\partial^2 u(x,t)}{\partial x^2} \in C((0,1)\times(0,T)),$$

And this function satisfies equation (1.1), initial conditions (1.2) and boundary condition (1.3).

Uniqueness of the classic solution of problem (1.1)-(1.4) is obtained from of the finite integral transformation method [1], [5].

The theorem is proved.

Remark 2.1 Using the known properties of trigonometric functions of sines and cosines, the cases $0 < aT < 1$ and $aT > 1$ are studied in the same way.

Remark 2.2 Using the results of the paper [11], [12] in the case with variable coefficients, the problem (1.1)-(1.4) is solved in the same way.

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