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Approximation by linear means of Fourier series in Orlicz spaces

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Abstract. *In this work the approximation of functions by linear means of Fourier series in reflexive Orlicz spaces was studied. This result was applied to the approximation of the functions by linear means of Faber series in Smirnov-Orlicz classes defined on simply connected domain of the complex plane*

Keywords. Young function, Orlicz space, Boyd indices, modulus of smoothness, best approximation.

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1 Introduction and main results

Let $M(u)$ be a continuous increasing convex function on $[0,\infty)$ such that $M(u)/u \to 0$ if $u \to 0$, and $M(u)/u \to \infty$ if $u \to \infty$. We denote by N *the complementary of* M *in Young's sense,* i.e. $N(u) = \max\{uv - M(v) : v \geq 0\}$ if $u \geq 0$. We will say that M satisfies the Δ_2-condition if $M(2u) \leq cM(u)$ for any $u \geq u_0$ with some constant c independent of u.

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$.

For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\int_{\mathbb{T}} M(|f(x)|)\, dx < \infty.$$

Let N be the complementary Young function of M. It is well-known [23, p. 69], [30, pp. 52-68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup\left\{\int_{\mathbb{T}} |f(x)g(x)|\, dx : g \in \widetilde{L}_N(\mathbb{T}), \int_{\mathbb{T}} N(|g(x)|)\, dx \leq 1\right\},$$

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or with the *Luxembourg norm*

$$\|f\|^*_{L_M(\mathbb{T})} := \inf\left\{k>0 : \int\limits_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right)dx \le 1\right\}$$

becomes a Banach space. The space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [23, p. 26]. The Luxembourg norm is equivalent to the Orlicz norm as

$$\|f\|^*_{L_M(\mathbb{T})} \le \|f\|_{L_M(\mathbb{T})} \le 2\,\|f\|_{L_M(\mathbb{T})}\,, \quad f\in L_M(\mathbb{T})$$

holds true [23, p. 80].

If we choose $M(u)=u^p/p \ \ (1<p<\infty)$ then the complementary function is $N(u)=u^q/q$ with $1/p+1/q=1$ and we have the relation

$$p^{-1/p}\,|u|_{L_p(\mathbb{T})} = \|u\|^*_{L_M(\mathbb{T})} \le \|u\|_{L_M(\mathbb{T})} \le q^{1/q}\,\|u\|_{L_p(\mathbb{T})}\,,$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int\limits_{\mathbb{T}} |u(x)|^p\,dx\right)^{1/p}$ denotes the usual norm of the $L_p(\mathbb{T})-$space.

If N is complementary to M in Young's sense and $f\in L_M(\mathbb{T})$, $g\in L_N(\mathbb{T})$ then the so-called strong Hölder inequalities [23, p.80]

$$\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx \le \|f\|_{L_M(\mathbb{T})}\,\|g\|^*_{L_N(\mathbb{T})}\,,$$

$$\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx \le \|f\|^*_{L_M(\mathbb{T})}\,\|g\|_{L_N(\mathbb{T})}\,.$$

are satisfied.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the $\varDelta_2-$condition [30, p. 113].

Let $M^{-1}:[0,\infty)\to[0,\infty)$ be the inverse function of the $N-$function M. The *lower* and *upper indices* α_M,β_M [5, p. 350]

$$\alpha_M := \lim_{t\to+\infty} -\frac{\log h(t)}{\log t},\ \beta_M := \lim_{t\to o^+} -\frac{\log h(t)}{\log t}$$

of the function

$$h:(0,\infty)\to(0,\infty],\quad h(t):=\lim_{y\to\infty}\sup\frac{M^{-1}(y)}{M^{-1}(ty)},\ \ t>0$$

first considered by Matuszewska and Orlicz [27], are called the *Boyd indices* of the Orlicz spaces $L_M(\mathbb{T})$.

It is known that the indices α_M and β_M satisfy $0\le\alpha_M\le\beta_M\le1, \alpha_N+\beta_M=1$, $\alpha_M+\beta_N=1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0<\alpha_M\le\beta_M<1$. The detailed information about the Boyd indices can be found in [4], [5], [6], [20] and [26].

Let $L_M(T)$ be an Orlicz space. Suppose that x,h are real, and let us take into account the following series

$$\varDelta^\alpha_h f(x) = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k} f\left(x+(\alpha-k)h\right),\quad \alpha>0,\quad f\in L_M(T).$$

Then, by [24, Theorem 11, p.135] the last series converges absolutely almost everywhere (a. e). on T. Hence the operator Δ_h^{α} by [24, Theorem 10, p.134] is bounded in the space $L_M(T)$. Namely,

$$\Delta_h^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f\left(x + (\alpha - k)h\right) = \sum_{k=0}^{\alpha} (-1)^{\alpha-k} \binom{\alpha}{k} f\left(x + kh\right).$$

for $\alpha \in Z^+$. Let $L_M(\mathbb{T}, \omega)$ be a weighted Orlicz space.

The function

$$\omega_{\alpha}(f,\delta)_M := \sup_{|h| \le \delta} \|\Delta_h^{\alpha} f\|_{L_M(\mathbb{T})}, \alpha \in Z^+$$

is called *α-th modulus of smoothness* of $f \in L_M(\mathbb{T})$

It can easily be shown that $\omega_{\alpha}(f,\delta)_M$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta \to 0} \omega_{\alpha}(f,\delta)_M = 0, \ \omega_{\alpha}(f+g,\delta)_M \le \omega_{\alpha}(f,\delta)_M + \omega(g,\delta)_M$$

for $f, g \in L_M(\mathbb{T})$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k(f) \cos kx + b_k(f) \sin kx\right) \tag{1.1}$$

be the Fourier series of the function $f \in L_1(\mathbb{T})$, where $\alpha_k(f)$ are $b_k(f)$ the Fourier coefficients of the function f.

Let (1.1) be the Fourier series of the function f . For $f \in L_M(\mathbb{T})$ we define the summability method by the tringular matrix $\Lambda = \{\lambda_{ij}\}_{i,j=0}^{j,\infty}$ by the linear means

$$U_n(x,f) = \lambda_{0n} \frac{a_0}{2} + \sum_{i=1}^{n} \lambda_{in}(a_i(f) \cos ix + b_i(f) \sin ix)$$

If the Fourier series of f is given by (1.1), then *Zygmund-Riesz means of order k* is defined as

$$Z_n^k(x,\ f) = \frac{a_0}{2} + \sum_{i=1}^{n} \left(1 - \frac{i^k}{(n+1)^k}\right) \left(a_i(f) \cos ix + b_i(f) \sin ix\right).$$

We denote by $E_n(f)_M$ the best approximation of $f \in L_M(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_M = \inf\left\{\|f - T_n\|_{L_M(\mathbb{T})} : T_n \in \Pi_n\right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

Let $T_n \in \Pi_n$

$$T_n = \frac{c_0}{2} + \sum_{i=1}^{n} (c_i \cos ix + d_i \sin ix).$$

The conjugate polynomial $\widetilde{T_n}$ is defined by

$$\widetilde{T_n} = \sum_{i=1}^{n} (c_i \sin ix - d_i \cos ix).$$

We will say that the method of summability by the matrix Λ satisfies condition $b_{k,M}$ (respectively $b^*_{k,M}$) if for $T_n \in \Pi_n$ the inequality

$$\|T_n - U_n(T_n)\|_{L_M(\mathbb{T})} \le c(n+1)^{-k} \left\| T_n^{(k)} \right\|_{L_M(\mathbb{T})}$$

$$\left(\|T_n - U_n(T_n)\|_{L_M(\mathbb{T})} \le c(n+1)^{-k} \left\| \widetilde{T_n}^{(k)} \right\|_{L_M(\mathbb{T})} \right)$$

holds and the norms

$$\|\Lambda\|_1 := \int_0^{2\pi} \left| \frac{\lambda_{0n}}{2} + \sum_{i=1}^{n} \lambda_{in} \cos it \right| dt$$

are bounded.

We use the constants $c, c_1, c_2, ...$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest.

The problems of approximation theory in the weighted and non-weighted Orlicz spaces have been investigated by several authors (see, for example, [1]-[3], [11], [12], [15]-[19], [21] and [29]).

In the present paper necessary and sufficient condition about the relationship between the approximation of functions by linear means of Fourier series and by Zygmund-Riesz means of order k was investigated in reflexive Orlicz spaces. Also, we investigate the approximation of functions by linear means of Fourier series in terms of the modulus of smoothness of these functions in Orlicz spaces. This result was applied to the approximation of the functions by linear means of Faber series in Smirnov-Orlicz classes defined on simply connected domain of the complex plane. The similar problems in different spaces were investigated in [7] ,[8], [13], [14], [22], [31], and [33]-[35].

Main results in the present work are the following theorems:

Theorem 1.1 *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space. In order that for $f \in L_M(\mathbb{T})$*

$$\|f - U_n(\cdot,\ f)\|_{L_M(\mathbb{T})} \le c_1 \left\| f - Z_n^k(\cdot,\ f) \right\|_{L_M(\mathbb{T})} \tag{1.2}$$

it is sufficient and necessary that for $f \in L_M(\mathbb{T})$

$$\|T_n - U_n(\cdot,\ T_n)\|_{L_M(\mathbb{T})} \le c_2 \left\| T_n - Z_n^k(\cdot,\ T_n) \right\|_{L_M(\mathbb{T})}. \tag{1.3}$$

Theorem 1.2 *Let $L_M(T)$ be a reflexive Orlicz space. In order that for every $f \in L_M(\mathbb{T})$*

$$\|f - U_n(\cdot,\ f)\|_{L_M(\mathbb{T})} \le c_1 \left\| f - Z_n^k(\cdot,\ f) \right\|_{L_M(\mathbb{T})} \tag{1.4}$$

it is sufficient and necessary that

(i) $\|U_n(\cdot,\ f)\|_{L_M(\mathbb{T})} = O(1)$;

(ii) if k is even, $U_n(\cdot, f)$ satisfies the condition $(b_{k,M})$; if k is odd $U_n(\cdot, f)$ satisfies the condition $\left(b^*_{k,M}\right)$.

Theorem 1.3 *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space . If the summability method with the matrix Λ satisfies the condition $(b_{k,M})$ or $\left(b^*_{k,M}\right)$, then for $f \in L_M(\mathbb{T})$ the inequality*

$$\|f - U_n(\cdot,\ f)\|_{L_M(\mathbb{T})} \leq c_3 \omega_\alpha \left(f, \frac{1}{n}\right)_M \tag{1.5}$$

holds with a constant $c_3 > 0$ independent of n.

Theorem 1.4 *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space . If the summability method with the matrix Λ satisfies the condition $(b_{k,M})$ or $\left(b^*_{k,M}\right)$, then for $f \in L_M(\mathbb{T})$ the estimate*

$$\omega_\alpha \left(U_n(\cdot,\ f), \delta\right)_M \leq c_4 \omega_\alpha \left(f, \delta\right)_M , \tag{1.6}$$

holds with a constant $c_4 > 0$ not depend on n, f and δ.

Corollary 1.1 *This results obtained in Theorems 1.1 and 1.3 are valid for the Zugmund-Riesz means of order k.*

Note that Theorem 1.3 in weighted Orlicz spaces for different modulus of continuity and Zygmund-Riesz means of order 2 was proved in [12].

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ, and let $G^- := ext\Gamma$. Further let

$$\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}, \mathbb{D} := int\, \mathbb{T} \text{ and } \mathbb{D}^- := ext\, \mathbb{T}.$$

Let $w = \phi(z)$ be the conformal mapping of G^- onto D^- normalized by

$$\phi(\infty) = \infty, \quad \lim_{z \to \infty} \frac{\phi(z)}{z} > 0,$$

and let ψ denote the inverse of ϕ.

Let $w = \phi_1(z)$ denote a function that maps the domain G conformally onto the disk $|w| < 1$. The inverse mapping of ϕ_1 will be denoted by ψ_1. Let Γ_r denote circular images in the domain G, that is, curves in G corresponding to circle $|\phi_1(z)| = r$ under the mapping $z = \psi_1(w)$.

Let us denote by E_p, where $p > 0$, the class of all functions $f(z) \neq 0$ which are analytic in G and have the property that the integral

$$\int_{\Gamma_r} |f(z)|^p \, |dz|$$

is bounded for $0 < r < 1$. We shall call the E_p-class the *Smirnov class*. If the function $f(z)$ belongs to E_p, then $f(x)$ has definite limiting values $f(z')$ almost every where on Γ, over all nontangential paths; $|f(z')|$ is summable on Γ; and

$$\lim_{r \to 1} \int_{\Gamma_r} |f(z)|^p \, |dz| = \int_{\Gamma} \left|f(z')\right|^p |dz| .$$

It is known that $\varphi' = E_1(G^-)$ and $\psi' \in E_1(D^-)$. Note that the general information about Smirnov classes can be found in the books [9, pp. 168-185] and [10, pp. 483-453].

Let $L_M(\mathbb{T})$ is a Orlicz space defined on Γ. We define also the *Smirnov-Orlicz class* $E_M(G)$ as

$$E_M(G) := \{f \in E_1(G) : f \in L_M(\Gamma)\} .$$

With every weight function ω on Γ, we associate another weight ω_0 on $\mathbb{T}$ defined by

$$\omega_0(t) := \omega(\psi(t)), \ t \in \mathbb{T}.$$

For $f \in L_M(\Gamma, \omega)$ we define the function

$$f_0(t) := f(\psi(t)), \ t \in \mathbb{T}.$$

Let h be a continuous function on $[0, 2\pi]$. Its modulus of continuity is defined by

$$\omega(t, h) := \sup\{|h(t_1) - h(t_2)| : t_1, t_2 \in [0, 2\pi], |t_1 - t_2| \leq t\}, t \geq 0.$$

The curve Γ is called *Dini-smooth* if it has a parameterization

$$\Gamma : \varphi_0(s), \ 0 \leq s \leq 2\pi$$

such that $\varphi_0'(s)$ is Dini-continuous, i.e.

$$\int_0^\pi \frac{\omega(t, \varphi_0')}{t} dt < \infty$$

and $\varphi_0'(s) \neq 0$ [28, p. 48].

If Γ is Dini-smooth curve, then there exist [36] the constants c_5 and c_6 such that

$$0 \leq c_5 \leq |\psi'(t)| \leq c_6 < \infty, \ |t| > 1. \tag{1.7}$$

Note that if Γ is a Dini-smooth curve, then by (1.6) we have $f_0 \in L_M(\mathbb{T})$ and $f \in L_M(\Gamma)$.

Let Γ be a rectifiable Jordan curve and $f \in L_1(\Gamma)$. Then the function f^+ defined by

$$f^+(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(s)ds}{s - z}, \ z \in G$$

is analytic in G. Note that if $0 < \alpha_M \leq \beta_M < 1$and $f \in L_M(\Gamma)$, then by [2] $f^+ \in E_M(G)$.

Let $\phi_k(z), \ k = 0, 1, 2, ...$ be the Faber polynomials for G. The Faber polynomials $\phi_k(z)$, associated with $G \cup \Gamma$, are defined through the expansion

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^\infty \frac{\phi_k(z)}{w^{k+1}}, \ z \in G, \ w \in D^- \tag{1.8}$$

and the equalities

$$\phi_k(z) = \frac{1}{2\pi i} \int_T \frac{t^k \psi'(t)}{\psi(t) - z} dt, \ z \in G, \tag{1.9}$$

$$\phi_k(z) = \phi^k(z) + \frac{1}{2\pi i} \int_T \frac{\phi^k(s)}{s - z} ds, \ z \in G^-$$

hold [32, p. 33-48].

Let $f \in E_M(G, \omega)$. Since $f \in E_1(G)$. we obtain

$$f(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(s)ds}{s - z} = \frac{1}{2\pi i} \int_\mathbb{T} \frac{f(\psi(t))\psi'(t)}{\psi(t) - z} dt,$$

for every $z \in G$. Considering this formula and expansion (1.8), we can associate with f the formal series

$$f(z) \sim \sum_{i=0}^{\infty} a_i(f)\phi_i(z),\ z \in G, \tag{1.10}$$

where

$$a_i(f) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\psi(t))}{t^{i+1}} dt.\ i = 0, 1,\ 2,...$$

This series is called the *Faber series* expansion of f, and the coefficients $a_i(f)$ are said to be the *Faber coefficients* of f.

Let (1.9) be the Faber series of the function $f \in E_M(G)$. For the function f we define the summability method by the tringular matrix $\Lambda = \{\lambda_{ij}\}_{i,j=0}^{j,\infty}$ by the linear means

$$U_n(z,\ f) = \sum_{i=0}^{n} \lambda_{in} a_i(f)\phi_i(z),$$

The n-the *partial sums* and *Zygmund means of order* k of the series (1.9) are defined, respectively, as

$$S_n(z,\ f) = \sum_{k=0}^{n} a_k(f)\phi_k(z),$$

$$Z_n^k(z,\ f) = \sum_{i=0}^{n} \left(1 - \frac{i^k}{(n+1)^k}\right) a_i(f)\phi_i(z).$$

Let Γ be a Dini-smooth curve. Using the nontangential boundary values of f_0^+ on $\mathbb{T}$ we define the $\alpha - th$ *modulus of smoothness*

of $f \in L_M(\Gamma)$ as

$$\omega_{\alpha,\Gamma}(f,\delta)_M := \omega_{\alpha,\Gamma}\left(f_0^+,\delta\right),\ \delta > 0,$$

for $\alpha = 1, 2, 3, ...$

The following theorem holds.

Theorem 1.5 *Let Γ be a Dini-smoth curve, and let $L_M(\Gamma)$ be a reflexive Orlicz space. If the summability method with the matrix Λ satisfies the condition $(b_{k,M})$ or $\left(b_{k,M}^*\right)$, then for $f \in E_M(G)$ the estimate*

$$\|f - U_n(\cdot, f)\|_{L_M(\Gamma)} \le c_7 \omega_{\alpha,\Gamma}\left(f, \frac{1}{n}\right)_M \tag{1.11}$$

holds with a constant $c_7 > 0$, independent of n.

Let $\mathcal{P}$ be the set of al algebraic polynomials (with no restriction on the degree), and let $\mathcal{P}(\mathbb{D})$ be the set of traces of members of $\mathcal{P}$ on $\mathbb{D}$. We define the operator

$$T : \mathcal{P}(\mathbb{D}) \longrightarrow E_M(G,\omega)$$

as

$$T(P)(z) := \frac{1}{2\pi i} \int_{T} \frac{P(w)\psi'(w)}{\psi(w) - z} dw,\ z \in G.$$

Then from (1.8) we have

$$T\left(\sum_{k=0}^{n}\beta_k w^k\right)=\sum_{k=0}^{n}\beta_k\phi_k(z).$$

The following result hold for the linear operator $\mathbb{T}$ [2].

Theorem 1.6 *Let Γ be a Dini- smooth curve and $L_M(\Gamma)$ be a reflexive Orlicz space. Then linear operator $T : P(D) \longrightarrow E_M(G)$ is bounded.*

Theorem 1.7 *If Γ is a Dini- smooth curve. Then the operator*

$$T : E_M(\mathbb{D}) \longrightarrow E_M(G)$$

is one-to-one and onto.

2 Proof of main results

Proof of Theorem1.1. Necessity. It is clear that the inequality (1.3) follows from the inequalitu (1.2).

Sufficiency. Let $f \in L_M(\mathbb{T})$ and let $T_n \in \Pi_n$ $(n = 0, 1, 2, ...)$ be the polynomial of best approximation to f . We obtain

$$\begin{aligned}
&\|f-U_n(\cdot,\ f)\|_{L_M(\mathbb{T})}\\
&\le \|f-T_n\|_{L_M(\mathbb{T})}-\|T_n-U(\cdot,T_n)\|_{L_M(\mathbb{T})}+\|U_n(\cdot,\ f-T_n\|_{L_M(\mathbb{T})}\\
&\le E_n(f)_M+c_2\left\|T_n-Z_n^k(\cdot,T_n)\right\|_{L_M(\mathbb{T})}+c_8E_n(f)_M\le c_9E_n(f)_M\\
&\quad +c_2\left(\|T_n-f\|_{L_M(\mathbb{T})}+\left\|f-Z_n^k(\cdot,f)\right\|_{L_M(\mathbb{T})}+\left\|Z_n^k(\cdot,f-T_n)\right\|_{L_M(\mathbb{T})}\right)\\
&\le c_9E_n(f)_M+c_2E_n(f)_M+c_2\left\|f-Z_n^k(\cdot,f)\right\|_{L_M(\mathbb{T})}+c_2c_{10}E_n(f)_M\\
&\le c_{11}E_n(f)_M+c_2\left\|f-Z_n^k(\cdot,f)\right\|_{L_M(\mathbb{T})}\le c_{12}\left\|f-Z_n^k(\cdot,f)\right\|_{L_M(\mathbb{T})}
\end{aligned}$$

and Theorem 1.1 is proved.

Prof of Theorem 1.2. Necessity. According to [33] $Z_n^k(f)$ is bounded in $L_p(\mathbb{T})$ and $L_q(\mathbb{T})$. Then by [16, Theorem 7] (see also [4, p.153]) $\left\|Z_n^k(\cdot,f)\right\|_{L_M(\mathbb{T})}=O(1)$. Taking account of (1.4) we have $\|U_n(\cdot,f)\|_{L_M(\mathbb{T})}=O(1)$. Let $f\in L_M(\mathbb{T})$. Then the following inequality holds:

$$\begin{aligned}
&\|f-Z_{n,k}(\cdot,f)\|_{L_M(\mathbb{T})}\\
&\le \|f-S_n(\cdot,f)\|_{L_M(\mathbb{T})}+(n+1)^{-k}\left\|\sum_{\nu=1}^{n}\nu^kA_\nu(\cdot,f)\right\|_{L_M(\mathbb{T})}\\
&=U_1+U^{(k)}
\end{aligned}\tag{2.1}$$

It is well known from [33], [14] that

$$\begin{aligned}
U_1=\|f-S_n(\cdot,f)\|_{L_M(\mathbb{T})}&\le c_{17}(M)E_n(f)_M\\
&\le \frac{c}{(n+1)^k}\left\|f^{(k)}\right\|_{L_M(\mathbb{T})}
\end{aligned}\tag{2.2}$$

We note that if k is even

$$\sum_{\nu=1}^{n} \nu^k A_\nu(x,f) = (-1)^{k/2} S_n^{(k)}(x,f),$$

if k is odd

$$\sum_{\nu=1}^{n} \nu^k A_\nu(x,f) = (-1)^{(k+3)/2} \widetilde{S_n}^{(k)}(x,f),$$

where $\widetilde{g(x)}$ is the function that is trigonometricaly conjugate to $g(x)$.Then

$$U_2^{(k)} = \begin{cases} (n+1)^{-k} \left\| S_n^{(k)}(\cdot,f) \right\|_{L_M(\mathbb{T})}, & k - \text{even} \\ (n+1)^{-k} \left\| \widetilde{S_n}^{(k)}(\cdot,f) \right\|_{L_M(\mathbb{T})}, & k - \text{odd}. \end{cases} \tag{2.3}$$

Using (2.3) if k is even, we have

$$\begin{aligned} U_2^{(k)} &= (n+1)^{-k} \left\| S_n^{(k)}(\cdot,f) \right\|_{L_M(\mathbb{T})} \\ &\leq c_{20} (n+1)^{-k} \left\| f^{(k)} \right\|_{L_M(\mathbb{T})}, \end{aligned} \tag{2.4}$$

if k is odd, we find that

$$\begin{aligned} U_2^{(k)} &= (n+1)^{-k} \left\| \widetilde{S_n}^{(k)}(\cdot,f) \right\|_{L_M(\mathbb{T})} \\ &\leq c_{25} (n+1)^{-k} \left\| \widetilde{f}^{(k)} \right\|_{L_M(\mathbb{T})}. \end{aligned} \tag{2.5}$$

Taking into account the relations (2.1), (2.2), (2.4) and (2.5), if k is even for $f \in L_M(\mathbb{T})$ and $f^{(k)} \in L_M(\mathbb{T})$ we obtain the ineguality

$$\left\| f - Z_{n,k}(\cdot,f) \right\|_{L_M(\mathbb{T})} \leq \frac{c}{(n+1)^k} \left\| \widetilde{f}^{(k)} \right\|_{L_M(\mathbb{T})}$$

and if k is odd for $f \in L_M(\mathbb{T})$ and $\widetilde{f}^{(k)} \in L_M(\mathbb{T})$ we reach

$$\left\| f - Z_{n,k}(\cdot,f) \right\|_{L_M(\mathbb{T})} \leq \frac{c}{(n+1)^k} \left\| \widetilde{f}^{(k)} \right\|_{L_M(\mathbb{T})}.$$

Sufficiency. We note that for $T_n \in \Pi_n$ we get

$$\left\| T_n - Z_{n,k}(\cdot,T_n) \right\|_{L_M(\mathbb{T})} = \frac{\left\| T_n^{(k)} \right\|_{L_M(\mathbb{T})}}{(n+1)^k}, \text{ if } k \text{ is even,} \tag{2.6}$$

$$\left\| T_n - Z_{n,k}(\cdot,T_n) \right\|_{L_M(\mathbb{T})} = \frac{\left\| \widetilde{T_n}^{(k)} \right\|_{L_M(\mathbb{T})}}{(n+1)^k}, \text{ if } k \text{ is odd} \tag{2.7}$$

Use of (2.6), (2.7) and condition (ii) gives us

$$\left\| T_n - U(\cdot,T_n) \right\|_{L_M(\mathbb{T})} \leq c \left\| T_n - Z_{n,k}(\cdot,T_n) \right\|_{L_M(\mathbb{T})}.$$

The last inequality and Theorem 1.1 imply that (1.4) .Theorem 1.2 is completely proved.

Proof of Theorem 1.3. We suppose that the condition $b_{k,M}^{*}$ is satisfed. Let $f \in L_M(\mathbb{T})$ and $T_n \in \Pi_n$ be the polynomial of best approximation to f. Note that $U_n(f) = \Lambda_n * f$. The operator $U_n(f)$ is bounded in $L_p(\mathbb{T})$ and $L_q(\mathbb{T})$ [26]). Using the method of proof of Lemma 1 in [16] we can show that the operator $U_n(f)$ is bounded in $L_M(\mathbb{T})$, i.e. $\|U_n(\cdot, f)\|_{L_M(\mathbb{T})} \le c_5 \|f\|_{L_M(\mathbb{T})}$. Then we get

$$\begin{aligned}
&\|f - U_n(\cdot,\ f)\|_{L_M(\mathbb{T})} \\
&\le \|f - T_n\|_{L_M(\mathbb{T})} + \|T_n - U(\cdot, T_n)\|_{L_M(\mathbb{T})} + \|U(\cdot, T_n) - U(\cdot, f)\|_{L_M(\mathbb{T})} \\
&\le c_{13} E_n(f)_{M,\omega} + c_7 E_n(f)_M + c_{14} n^{-k} \left\|\widetilde{T_n}^{(k)}\right\|_{L_M(\mathbb{T})} \\
&\le c_{15} E_n(f)_{M,\omega} + c_{16} n^{-k} \left\|\widetilde{T_n}^{(k)}\right\|_{L_M(\mathbb{T})}.
\end{aligned} \tag{2.8}$$

Taking the Bernstein inequality and the boundedness of the linear operator $f \to \widetilde{f}$ in $L_M(\mathbb{T})$ into account [16, Lemma 3 and (15)] and [3] we have

$$\begin{aligned}
n^{-k} \left\|\widetilde{T_n}^{(k)}\right\|_{L_M(\mathbb{T})} &\le c_{17} n^{-k} \left\|T_n^{(k)}\right\|_{L_M(\mathbb{T})} \\
&\le c_{18} \omega_\alpha(f, \frac{1}{n})_M
\end{aligned} \tag{2.9}$$

where $\widetilde{f}$ is the conjugate function of $f \in L_M(\mathbb{T})$.

Note that according to the direct theorem of approximation in $L_M(\mathbb{T})$ given in [2] following inequality holds:

$$E_n(f)_M \le c_{19} \omega_\alpha \left(f, \frac{1}{n}\right)_M. \tag{2.10}$$

Taking into account the relations (2.8), (2.9), and (2.10) we have

$$\|f - U_n(\cdot, f)\|_{L_M(\mathbb{T})} \le c_{20} \omega_\alpha \left(f, \frac{1}{n}\right)_M.$$

If the summability method with the matrix Λ satisfies condition $(b_{k,M}^{*})$, the proof is made anologously to the above.

The proof of Theorem 1.3 is completed.

Proof of Theorem 1.4. By [3] the following inequality holds:

$$\omega_\alpha\left(U_n(f) - f, \delta\right)_M \le c_{21} \left\|U_n(\cdot, f) - f\right\|_{L_M(\mathbb{T})}. \tag{2.11}$$

Let $\delta \ge (n+1)^{-1}$. Using Theorem 1.3 and (2.11) we have

$$\begin{aligned}
\omega_\alpha\left(U_n(f), \delta\right)_M &\le \omega_\alpha\left(f,\delta\right)_M + \omega_\alpha\left(U_n(\cdot, f) - f, \delta\right)_M \\
&\le \omega_\alpha\left(f,\delta\right)_M + c_{21} \left\|U_n(\cdot, f) - f\right\|_{L_M(\mathbb{T},\omega)} \\
&\le \omega_\alpha\left(f,\delta\right)_M + c_{18} \omega_\alpha(f, \frac{1}{n})_M \\
&\le c_{22} \omega_\alpha\left(f,\delta\right)_M.
\end{aligned} \tag{2.12}$$

Now we suppose that $\delta < (n+1)^{-1}$. Then by virtue of [3, Lemma 3.2] and [16, Lemma 3] we obtain

$$\begin{aligned}
&\omega_\alpha\left(U_n(f),\delta\right)_M \\
&\leq c_{23}\delta^\alpha \left\|U_n^{(\alpha)}(\cdot,f)\right\|_{L_M(\mathbb{T})} \\
&\leq c_{24}\delta^\alpha n^\alpha \left\|U_n(\cdot,f)\right\|_{L_M(\mathbb{T})} \leq c_{25}\delta^\alpha n^\alpha \omega_\alpha(U_n(f),\frac{1}{n})_M \\
&\leq c_{26}\delta^\alpha (n+1)^\alpha \omega_\alpha(U_n(f),\frac{1}{n})_M \leq c_{27}\omega_\alpha\left(f,\delta\right)_M . \qquad (2.13)
\end{aligned}$$

Now combining (2.12) and (2.13) we obtain the inequality (1.6) of Theorem 1.3.

Proof of Theorem 1.5. Let $f \in L_M(G)$. Then by virtue of Theoerm 1.7 the operator $T : E_M(\mathbb{D},) \longrightarrow E_M(G)$ is bounded one-to-one and onto and $T(f_0^+) = f$.The function f has the following Faber series

$$f(z) \backsim \sum_{m=0}^{\infty} a_m(f)\phi_m(z).$$

Using Lemma 1 in [15, p.760] we conclude that $f_0^+ \in E_M(\mathbb{D})$. For the function f_0^+ the following Taylor series holds:

$$f_0^+(w) = \sum_{m=0}^{\infty} a_m(f)w^m.$$

Note that $f_0^+ \in E_1(\mathbb{D})$ and boundary function $f_0^+ \in L_M(\mathbb{T})$. Then by [9, Theorem, 3.4] for the function f_0^+ we have the following Fourier expansion:

$$f_0^+(w) \backsim \sum_{m=0}^{\infty} a_m(f)e^{imt}.$$

Hence, if we consider boundedness of the operator $T : E_M(\mathbb{D}) \longrightarrow E_M(G)$ and Theorem 1.3, we obtain

$$\begin{aligned}
&\left\|f - U_n(.,f)\right\|_{L_M(\Gamma)} \\
&= \left\|T(f_0^+) - T(U_n(.,f_0^+))\right\|_{L_M(\Gamma)} \leq c_{35}\left\|f_0^+ - U_n(.,f_0^+)\right\|_{L_M(\mathbb{T})} \\
&\leq c_{36}\omega_\alpha\left(f_0^+,\frac{1}{n}\right) = c_{37}\omega_{\alpha,\Gamma}(f,\frac{1}{n})_M.
\end{aligned}$$

and (1.11) is proved.

Remark 2.1. Let $L_M(\mathbb{T})$ be the Orlicz space Then by virtue of Theorem 2 in [2] for $f \in L_M(\mathbb{T})$ the inequality

$$\omega_\alpha\left(f,\frac{1}{n+1}\right)_M \leq \frac{c_{38}}{(n+1)^\alpha}\sum_{m=0}^{n}(m+1)^{\alpha-1}E_m(f)_M, \qquad (2.14)$$

holds with a constant c independent of n. If the summability method with the matrix Λ satisfy the condition $(b_{k,M})$ or $\left(b^*_{k,M}\right)$ then for $f \in L_M(\mathbb{T})$ relation (1.5) and inequality (2.14) immediately yield

$$\|f - U_n(.,f)\|_{L_M(\mathbb{T})} \leq \frac{c_{39}}{(n+1)^\alpha} \sum_{m=0}^{n} (m+1)^{\alpha-1} E_m(f)_M.$$

The inequality (2.15) holds for Zygmund-Riesz means of order k. Note that in the Lebesgue spaces $L_p(\mathbb{T})$, $1 < p \leq \infty$ the inequality (2.15) was proved in [33].

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