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Degenerate convolution equations in vector-valued weighted Besov spaces

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Abstract. *In a series of recent publications, Fourier multiplier theorems with operator-valued multiplier functions have found many applications. From this point of view Besov spaces form one class of function spaces which are of special interest. In the present paper $B-$separability properties of degenerate convolution differential-operator equations with unbounded operator coefficients in vector-valued weighted Besov spaces are investigated. These results are applied to obtain the maximal $B-$regularity properties for anisotropic integro-differential equations and many convolution equations.*

Keywords. weighted Besov spaces, degenerate convolution equations, coercive uniform estimate, $B-$separability.

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1 Introduction

In recent years, operator valued Fourier multiplier theorems on diverse vector-valued function spaces have been studied (see, [2, 5, 8, 9, 13, 15]). They are needed to establish existence and uniqueness as well as regularity for differential equations in Banach spaces, and thus also for convolution differential operator equations (CDOEs).

Besov spaces form one class of function spaces which are of special interest. Fourier multiplier theorems with operator valued multiplier functions have found many applications in the theory of convolution equations, in particular, in connection with: maximal regularity of parabolic convolution equations, elliptic CDOEs, infinite system of integro-differential equations, and pseudo differential operators. Used is a general Fourier multiplier theorem for operator-valued multiplier functions on vector-valued Besov spaces.

The maximal regularity properties of differential operator equations has been studied extensively, e.g., in [1-3, 6, 7, 10, 18]. Convolution-differential equations (CDEs) have been treated, e.g., in [12, 14] and the references therein. Convolution operators in Banach-valued function spaces studied in [8, 11, 15]. Note that, the CDOEs are relatively less investigated subject, in particular, degenerate CDOEs. The main aim of present paper is to study the $B-$separability property of the anisotropic degenerate CDOEs

$$\sum_{|\alpha|\leq l} a_\alpha * D^{[\alpha]}u + A * u + \lambda u = f(x) \tag{1.1}$$

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in E-valued weighted Besov spaces, where E is an arbitrary Banach space, l is a natural number, $a_\alpha = a_\alpha(x)$ are complex-valued functions, $\alpha = (\alpha_1, \alpha_2, ...\alpha_n)$, α_k are nonnegative integers, λ is a complex parameter and $A = A(x)$ is a linear operator in E. Here, the convolutions $a_\alpha * D^{[\alpha]}u$, $A * u$ are defined in the distribution sense (see e.g. [2]) and

$$D^{[\alpha]} = D_{x_1}^{[\alpha_1]} D_{x_2}^{[\alpha_2]} ... D_{x_n}^{[\alpha_n]}, \quad D_{x_i}^{[\alpha_i]} = \left(\gamma(x)\frac{\partial}{\partial x_i}\right)^{\alpha_i},$$

where $\gamma(x)$ is a positive measurable function on $\Omega \subset R^n$, $i = \overline{1, n}$.

In this paper, we establish the $B-$separability properties of the equation (1.1), in particular the maximal regularity of Cauchy problem for the corresponding parabolic degenerate CDOE.

The main tools of this work is the theory of the operator-valued Fourier multipliers. This facts is derived by using the representation formula for the solution of nondegenerate case equation (1.1) and operator valued multipliers in weighted Besov space.

The paper is organized as follows. The first section of the paper contains an introduction. Section 2 collects definitions and basic properties of vector-valued function spaces, in particular, Besov spaces. Section 3 contains basic estimates for the Fourier transform and multiplier theorems in Banach valued weighted Besov spaces $B_{p,q,\gamma}^s(R^n; E)$. Applying this results gives a uniformly coercive estimate holds. Namely, we prove that for all $f \in B_{p,q}^s(R^n; E)$ there is a unique solution $u \in B_{p,q}^{[l],s}(R^n; E(A), E)$ of the problem (1.1) and the following uniformly coercive estimate holds:

$$\sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\| a_\alpha * D^{[\alpha]}u \right\|_{B_{p,q}^s(R^n;E)} + \|A * u\|_{B_{p,q}^s(R^n;E)}$$

$$+ |\lambda| \, \|u\|_{B_{p,q}^s(R^n;E)} \leq C \|f\|_{B_{p,q}^s(R^n;E)}.$$

In the particular case, by putting concrete vector spaces instead of E and concrete linear differential operators instead of A, the regularity properties of different class of the convolution equations and Cauchy problem for parabolic CDOEs are obtained in vector valued weighted Besov spaces.

2 Notations and background

Let $S = S(R^n; E)$ denote a Schwartz class, i.e., a space of E-valued rapidly decreasing smooth functions on R^n and $S'(R^n; E)$ denotes the space of E-valued tempered distributions. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, where α_i are integers. An E-valued generalized function $D^\alpha f$ is called a generalized derivative in the sense of Schwartz distributions of the function $f \in S'(R^n; E)$, if the equality

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle$$

holds for all $\varphi \in S$.

It is known that

$$F(D_x^\alpha f) = (i\xi_1)^{\alpha_1} ... (i\xi_n)^{\alpha_n} \widehat{f}, \; D_\xi^\alpha(F(f)) = F\left[(-ix_1)^{\alpha_1} ... (-ix_n)^{\alpha_n} f\right],$$

where $f \in S'(R^n; E)$, and F denote the Fourier transform.

The space $C^{(m)}(\Omega; E)$ will denote the space of $E-$valued uniformly bounded, m-times continuously differentiable functions on Ω.

Denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$.

Let E_1 and E_2 be two Banach spaces. $\mathcal{L}(E_1, E_2)$ denotes the space of all bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ this space will denoted by $\mathcal{L}(E)$.

Let $L_{p,\gamma}(\Omega; E)$ denotes the space of all strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm

$$\|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int_{\Omega} \|f(x)\|_E^p \, \gamma(x)\, dx \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)} = ess \sup_{x\in\Omega} [\|f(x)\|_E \, \gamma(x)],$$

where $x = (x_1, x_2, ..., x_n) \in \Omega \subset R^n$, $\gamma(x)$ be a measurable positive function in domain $\Omega \subset R^n$.

For $\gamma(x) \equiv 1$ the spaces $L_{p,\gamma}(\Omega; E)$ will denoted by $L_p(\Omega; E)$.

The weight function $\gamma = \gamma(x)$ satisfiy an A_p condition (i.e. $\gamma \in A_p$) if there is a positive constant C such that

$$\left(\frac{1}{|Q|} \int_Q \gamma(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q \gamma^{-\frac{1}{p-1}}(x)\, dx \right)^{p-1} \le C, \; p \in (1, \infty)$$

for all cubes $Q \subset R^n$.

By virtue of [16] the following weighted functions

$$\gamma(x) = |x|^{\alpha}, x \in \mathbb{R}, \; -1 < \alpha < p-1, \gamma(x) = \prod_{k=1}^{N} \left(1 + \sum_{j=1}^{n} |x_j|^{\alpha_{jk}} \right)^{\beta_k},$$

belong to A_p class, where

$x = (x_1, x_2, ..., x_n), \alpha_{jk} \ge 0, N \in \mathbb{N}, x_k, \beta_k \in \mathbb{R}$.

Suppose that

$$S_{\varphi} = \{\lambda : \lambda \in \mathbb{C}, \; |\arg \lambda| \le \varphi\} \cup \{0\}, \; 0 \le \varphi < \pi.$$

A linear operator $A = A(x)$ is said to be uniformly positive in a Banach space E, if $D(A(x))$ is dense in E and does not depend on x,

$$\left\| (A(x) + \lambda I)^{-1} \right\|_{\mathcal{L}(E)} \le C(1 + |\lambda|)^{-1},$$

with $C > 0$, $\lambda \in S_{\varphi}$, $\varphi \in [0, \pi)$, where I is the identity operator in E. Sometimes instead of $A + \lambda I$ we will write $A + \lambda$ and denote it by A_{λ}.

Let $E(A)$ denote the space $D(A)$ with the graph norm

$$\|u\|_{E(A)} = (\|u\|^p + \|Au\|^p)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

Let $A = A(x) \in S'(R^n; \mathcal{L}(D(A), E))$. Then the Fourier transformation of $A(x)$ in the sense of Schwartz distributions is defined as follows:

$$\left(\widehat{A} u, \varphi \right) = (Au, \widehat{\varphi}), \;\; u \in D(A) \text{ and } \varphi \in S(R^n).$$

$A(x)$ is differentiable if there is the limit

$$\left(\frac{\partial A}{\partial x_k}\right)u = \lim_{h\to 0}\frac{\triangle_k(h)\,A(x)\,u}{h}, \quad u\in D(A)$$

in the sense of $E-$norm, where $\triangle_k(h)\,f(x) = f(x+he_k) - f(x)$, $h\in\mathbb{R}$, and e_k, $k=\overline{1,n}$, be the standard unit vectors of R^n.

Let $A = A(x)$ be a uniformly positive operator in E and $u\in B^s_{p,q}(R^n;E(A))$. Then define:

$$(A*u)(t) = \int\limits_{R^n} A(t-y)u(y)dy.$$

Let $h\in\mathbb{R}, m\in\mathbb{N}$ and e_i, $i=1,2,...,n$, be the standard unit vectors of R^n. Let (see [4, sec.16])

$$\triangle_i(h)f(x) = f(x+he_i) - f(x),$$
$$\triangle_i^m(h)f(x) = \triangle_i(h)\left[\triangle_i^{m-1}(h)f(x)\right] = \sum_{k=0}^{m}(-1)^{m-k}C_m^k f(x+khe_i).$$

Let

$$\triangle_i^m(h,\Omega)f(x) = \begin{cases}\triangle_i^m(h)f(x) \text{ for } [x,\ x+mhe_i]\subset\Omega \\ 0 \qquad \text{for } [x,\ x+mhe_i]\overline{\subset}\Omega\end{cases}$$

Let m_i be positive integers,k_i nonnegative integers, s_i positive numbers, and $m_i > s_i - k_i > 0$, $i=1,2,...,n$, $s=(s_1,s_2,...,s_n)$, $1\le p\le\infty$, $1\le q\le\infty$.The Banach-valued weighted Besov space $B^s_{p,q,\gamma}(\Omega;E)$ is defined as

$$B^s_{p,q,\gamma}(\Omega;E) = \left\{\begin{array}{c} f: f\in L_{p,\gamma}(\Omega;E), \|f\|_{B^s_{p,q,\gamma}(\Omega;E)} = \|f\|_{L_{p,\gamma}(\Omega;E)} + \\ +\sum_{i=1}^n\left(\int\limits_0^{h_0} h^{-(s_i-k_i)q}\left\|\triangle_i^m(h,\Omega)\,D_i^{k_i}f\right\|^q_{L_{p,\gamma}(\Omega;E)}\frac{dh}{h}\right)^{1/q} < \infty\end{array}\right\},$$

$$\|f\|_{B^s_{p,q,\gamma}(\Omega;E)} = \|f\|_{L_{p,\gamma}(\Omega;E)} + \sum_{i=1}^n\sup_{0<h<h_0}\frac{\left\|\triangle_i^{m_i}(h,\Omega)\,D_i^{k_i}f\right\|^q_{L_{p,\gamma}(\Omega;E)}}{h^{s_i-k_i.}},\ \text{for } q=\infty.$$

For $E=R$ and $\gamma(x)\equiv 1$, we obtain a scalar-valued anisotropic Besov space $B^s_{p,q,\gamma}(\Omega)$ [4, sec. 18].

Let $l=(l_1,l_2,...,l_n)$ where l_k are positive integers. Let $B^{l,s}_{p,q,\gamma}(\Omega;E)$ denote a $E-$valued weighted Besov space of functions $u\in B^s_{p,q,\gamma}(\Omega;E)$ that have generalized derivatives $D_k^{l_k}u = \left(\partial^{l_k}/\partial x_k^{l_k}\right)u\in B^s_{p,q,\gamma}(\Omega;E)$, $k=1,2,...,n$, with the norm

$$\|u\|_{B^{l,s}_{p,q,\gamma}(\Omega;E)} = \|u\|_{B^s_{p,q,\gamma}(\Omega;E)} + \sum_{k=1}^n\left\|D_k^{l_k}u\right\|_{B^s_{p,q,\gamma}(\Omega;E)} < \infty.$$

Suppose E_0 and E be two Banach spaces with E_0 continuously and densely embedded into E. Let us introduce Besov-lions type spaces $B^{l,s}_{p,q,\gamma}(\Omega;E_0,E)$ which are collections of functions $u\in B^s_{p,q,\gamma}(\Omega;E_0)$ having the generalized derivatives $D_k^{l_k}u\in B^s_{p,q,\gamma}(\Omega;E)$, with the norm

$$\|u\|_{B^{l,s}_{p,q,\gamma}(\Omega;E_0,E)} = \|u\|_{B^s_{p,q,\gamma}(\Omega;E_0)} + \sum_{k=1}^n\left\|D_k^{l_k}u\right\|_{B^s_{p,q,\gamma}(\Omega;E)} < \infty.$$

Consider the E-valued weighted Besov space $B^{[l],s}_{p,q,\gamma}(\Omega;E_0,E)$ defined as

$$B^{[l],s}_{p,q,\gamma}(\Omega;E_0,E)=\left\{u:u\in B^{s}_{p,q,\gamma}(\Omega;E_0),\ \ D^{[l_k]}_{k}u\in B^{s}_{p,q,\gamma}(\Omega;E),\right.$$

$$\left.\|u\|_{B^{[l],s}_{p,q,\gamma}(\Omega;E_0,E)}=\|u\|_{B^{s}_{p,q,\gamma}(\Omega;E_0)}+\left\|D^{[l_k]}_{k}u\right\|_{B^{s}_{p,q,\gamma}(\Omega;E)}<\infty\right\}.$$

The function $\Psi\in L_\infty(R^n;\mathcal{L}(E_1,E_2))$ is called a multiplier from $B^{s}_{p,q,\gamma}(R^n;E_1)$ to $B^{s}_{p,q,\gamma}(R^n;E_2)$ for $p,q\in(1,\infty)$ if the map

$$u\rightarrow Ku=F^{-1}\Psi(\xi)Fu,u\in S(R^n;E_1)$$

is well defined and extends to a bounded linear operator

$$K:B^{s}_{p,q,\gamma}(R^n;E_1)\rightarrow B^{s}_{p,q,\gamma}(R^n;E_2).$$

For a corresponding nondegenerate case of equation (1.1) we have the following definition.

The CDOE (1.1) is said to be uniform weighted $B-$separable if for all $f\in B^{l,s}_{p,q,\gamma}(R^n;E)$ it has a unique solution $u\in B^{l,s}_{p,q,\gamma}(R^n;E(A),E)$ and

$$\sum_{|\alpha|\le l}\|a_\alpha*D^\alpha u\|_{B^{s}_{p,q,\gamma}(R^n;E)}+\|A*u\|_{B^{s}_{p,q,\gamma}(R^n;E)}\le C\|f\|_{B^{s}_{p,q,\gamma}(R^n;E)}.$$

Let $p,q\in[1,\infty)$, $\gamma\in A_p$.The Banach space E is said to be satisfying B-weighted multiplier condition if for any $M\in C^{(l)}\left(R^n,\mathcal{L}(E)\right)$ the condition

$$\sup_{x}\left\|(1+|x|)^{|\alpha|}D^\alpha M(x)\right\|_{\mathcal{L}(E)}\le C$$

for each multi-index with $|\alpha|\le l$ implies that M is a Fourier multiplier in $B^{s}_{p,q,\gamma}(R^n;E)$.

It is well known that any Hilbert space satisfies the $B-$multiplier condition. There are, however, Banach spaces which are not Hilbert spaces but satisfy the $B-$multiplier condition (for example, UMD spaces).

3 Elliptic convolution differential-operator equations

In this section, we shall consider the following nondegenerate convolution equations

$$\sum_{|\alpha|\le l}a_\alpha*D^\alpha u+A_\lambda*u=f, \tag{3.1}$$

in $E-$valued weighted Besov spaces, where l is a natural number, $a_\alpha=a_\alpha(x)$ are complex-valued functions, $a=(a_1,a_2,...,a_k)$,a_k are nonnegative integers, λ is a complex parameter, $A=A(x)$ is a linear operator in a Banach space E for $x\in R^n$.

Let us first recal the simplest form of Corollary 4.11 [see, 8] which will be used in investigation of CDOEs in this section.

Corollary 3.1 *Let $p,q\in[1,\infty)$, E_1 and E_2 are arbitrary Banach space. If $\Psi\in C^{(l)}(R^n;\mathcal{L}(E_1,E_2))$ satisfies, for some constant C,*

$$\sup_{x\in R^n}\left\|(1+|x|)^{|\alpha|}D^\alpha\Psi(x)\right\|_{\mathcal{L}(E_1,E_2)}\le C$$

for each multi-index α with $|\alpha|\le l=n+1$, then Ψ is Fourier multiplier from $B^{s}_{p,q}(R^n;E_1)$ to $B^{s}_{p,q}(R^n;E_2)$,

Corollary 3.2 *Suppose the following are satisfied:*

(1) $L(\xi) = \sum\limits_{|\alpha|\leq l} \hat{a}_{\alpha}(\xi)(i\xi)^{\alpha} \in S_{\varphi_1}, \varphi_1 \in [0,\pi)\ |L(\xi)| \geq C \sum\limits_{k=1}^{n} \left|\hat{\alpha}_{\alpha(l,k)}\right| |\xi_k|^l,$
$\alpha(l,k) = (0,0,...,l,0,0,...,0)$, *i.e.* $\alpha_i = 0,\ i \neq k,\ \alpha_k = l;$

(2) $\hat{a}_{\alpha} \in C^{(n)}(R^n), |\xi|^{\beta} \left|D^{\beta}\hat{a}_{\alpha}(\xi)\right| \leq C_1, \beta_k \in \{0,1\}, 0 \leq |\beta| \leq n+1;$

(3) $\left[D^{\beta}\hat{A}(\xi)\right]\hat{A}^{-1}(\xi_0) \in C(R^n; \mathcal{L}(E)), |\xi|^{\beta}\left\|\left[D^{\beta}\hat{A}(\xi)\right]\hat{A}^{-1}(\xi_0)\right\|_{\mathcal{L}(E)} \leq C_2,$ *for* $0 \leq |\beta| \leq n+1, \xi, \xi_0 \in R^n \backslash \{0\}.$

By using a similar technique as in [2] and [12] we obtain the following.

Theorem 3.1 *Assume that Condition 3.1 holds, $\gamma \in A_p$. E is a Banach space satisfying the multiplier condition with respect to weighted function γ and $\hat{A}$ be a uniformly positive operator in E with $\varphi \in [0,\pi)$. Then for all $f \in B^s_{p,q,\gamma}(R^n;E)$, $\lambda \in S_{\varphi}$, $p,q \in [1,\infty)$ the equation (3.1) has a unique solution $u \in B^{l,s}_{p,q,\gamma}(R^n;E(A),E)$ and the following coercive uniform estimate holds*

$$\sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \|a_{\alpha} * D^{\alpha}u\|_{B^s_{p,q,\gamma}(R^n;E)} + \|A*u\|_{B^s_{p,q,\gamma}(R^n;E)}$$
$$+ |\lambda| \|u\|_{B^s_{p,q,\gamma}(R^n;E)} \leq C\|f\|_{B^s_{p,q,\gamma}(R^n;E)}. \quad (3.2)$$

This theorem is proved in a similar way as in [15]. Let H_0 be the operator in $B^s_{p,q,\gamma}(R^n;E)$ generated by problem (3.1) for $\lambda = 0$, $i.e.$,

$$D(H_0) = B^{l,s}_{p,q,\gamma}(R^n;E(A),E), \qquad H_0u = \sum_{|\alpha|\leq l} a_{\alpha} * D^{\alpha}u + A*u.$$

From the Theorem 3.1 we have:

Result 3.1. Assume all conditions of the Theorem 3.1 hold. Then for all sufficiently large $\lambda \in S_{\varphi}$ there exist the resolvent of operator H_0 and the following estimate holds:

$$\sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\|a_{\alpha} * D^{\alpha}(H_0+\lambda)^{-1}\right\|_{\mathcal{L}\left(B^s_{p,q,\gamma}(R^n;E)\right)} +$$
$$\left\|A*(H_0+\lambda)^{-1}\right\|_{\mathcal{L}\left(B^s_{p,q,\gamma}(R^n;E)\right)} + |\lambda|\left\|(H_0+\lambda)^{-1}\right\|_{\mathcal{L}\left(B^s_{p,q,\gamma}(R^n;E)\right)} \leq C. \quad (3.3)$$

4 Linear degenerate elliptic convolution differential-operator equations in vector-valued weighted Besov space

As we mentioned before, from the differential operator equations point of view, Besov spaces form one class of function spaces which are of special interest.

Now we are ready to present our main results. Here used is a general Fourier multiplier theorem for operator-valued multiplier functions on vector-valued Besov spaces. Let us consider the following degenerate elliptic CDOE in vector-valued Besov space $B^s_{p,q}(R^n;E)$

$$\sum_{|\alpha|\leq l} a_{\alpha} * D^{[\alpha]}u + A*u + \lambda u = f. \quad (4.1)$$

We find sufficient conditions that guarantee the $B-$separability of the problem (4.1). For this purpose we need the following.

Recall that $\widetilde{\gamma}_k(x_k)$, $k=\overline{1,n}$ are positive measurable functions in $\mathbb{R}$ and

$$D^{[\alpha]}=D_{x_1}^{[a_1]}D_{x_2}^{[a_2]}...D_{x_n}^{[a_n]},\ D_{x_k}^{[\alpha_k]}=\left(\widetilde{\gamma}_k\left(x_k\right)\frac{\partial}{\partial x_k}\right)^{\alpha_k},$$
$$\widetilde{\gamma}=\left(\widetilde{\gamma}_1,\widetilde{\gamma}_2,...,\widetilde{\gamma}_n\right).$$

Consider the substitution

$$y_k=\int\limits_0^{x_k}\widetilde{\gamma}_k^{-1}(z)dz,\ k=\overline{1,n}. \tag{4.2}$$

Theorem 4.1 *Assume all conditions of the Theorem 3.1. and substitution (4.2) holds for* $a_\alpha\left(\widetilde{\gamma}\left(y\right)\right)$ *and* $A\left(\widetilde{\gamma}\left(y\right)\right)$. *Then equation (4.1) has a unique solution* $u(x)$ *that belongs to space* $B_{p,q}^{[l],s}(R^n;E(A),E)$ *and the coercive uniform estimate*

$$\sum_{|\alpha|\le l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha*D^{[\alpha]}u\right\|_{B_{p,q}^s(R^n;E)}+\|A*u\|_{B_{p,q}^s(R^n;E)}$$

$$+|\lambda|\,\|u\|_{B_{p,q}^s(R^n;E)}\le C\,\|f\|_{B_{p,q}^s(R^n;E)}$$

holds for all $f\in B_{p,q}^s(R^n;E)$ *and for sufficiently large* $\lambda\in S_\varphi$.

Proof. It is clear that under the substitution (4.2) $D^{[\alpha]}u$ transform to $D^\alpha u$. Moreover the spaces $B_{p,q}^s(R^n;E)$ and $B_{p,q}^{[l],s}(R^n;E(A),E)$ are mapped isomorphically onto the weighted spaces $B_{p,q,\gamma}^s(R^n;E)$ and $B_{p,q,\gamma}^{l,s}(R^n;E(A),E)$ respectively, where

$$\gamma=\prod_{k=1}^n\widetilde{\gamma}_k\left(x_k\left(y_k\right)\right).$$

Furthermore, under substitution (4.2) the degenerate problem (4.1) considered in $B_{p,q}^s(R^n;E)$ is reduced into the nondegenerate problem (3.1) considered in the weighted space $B_{p,q,\gamma}^s(R^n;E)$, where $a_\alpha=a_\alpha\left(\widetilde{\gamma}\left(y\right)\right),\ u=u\left(\widetilde{\gamma}\left(y\right)\right),\ A=A\left(\widetilde{\gamma}\left(y\right)\right),\ f=f\left(\widetilde{\gamma}\left(y\right)\right)$. Then in view of Theorem 3.1 and estimate (3.2) we obtain the assertion.

Let H be an operator generated by the problem (4.1), i.e.,

$$D(H)=B_{p,q,\gamma}^{[l],s}(R^n;E(A),E),\ \ Hu=\sum_{|\alpha|\le l}a_\alpha*D^{[\alpha]}u+A*u.$$

From the Theorem 4.1 and Result 3.1 in a similar way we obtain the following result.

Result 4.1. Under the conditions of Theorem 4.1 for $\lambda\in S_\varphi$ there exist the resolvent of operator H and has the estimate

$$\sum_{|\alpha|\le l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha*D^{[\alpha]}\left(H+\lambda\right)^{-1}\right\|_{\mathcal{L}\left(B_{p,q}^s(R^n;E)\right)}+$$

$$+\left\|A*\left(H+\lambda\right)^{-1}\right\|_{\mathcal{L}\left(B_{p,q}^s(R^n;E)\right)}+|\lambda|\left\|\left(H+\lambda\right)^{-1}\right\|_{\mathcal{L}\left(B_{p,q}^s(R^n;E)\right)}\le C.$$

Applying this results, in the particular case, gives a operator $H+a, a>0$ is uniformly positive in $B_{p,q}^{s}(R^n;E)$. Then, by virtue of [17], it is clear to see that the operator $H+a$ is a generator of an analytic semigroup in $B_{p,q}^{s}(R^n;E)$.

As we mentioned before, Fourier multipliers theorems with operator valued multiplier functions have found many applications in the theory of convolution equations, in particular, in connection with: maximal regularity of parabolic convolution equations, elliptic CDOEs, infinite system of integro-differential equations, and pseudo differential operators.

Finally, for this purpose, as an application we consider the Cauchy problem for the degenerate parabolic CDOE

$$\frac{\partial u}{\partial t}+\sum_{|\alpha|\leq l} a_\alpha * D^{[\alpha]}u + A*u + au = f\left(t,x\right) \tag{4.3}$$

$$u(0,x)=0,\ t\in\mathbb{R}_+, x\in R^n,$$

where $a>0$, a_α are complex-valued functions defined in (1.1) and A is a linear operator in a Banach space E.

It is easy to see that, under the substitution (4.2) the degenerate Cauchy problem (4.3) considered in $B_{p,q}^{s}(R^n;E)$ is transformed into corresponding nondegenerate Cauchy problem considered in the weighted space $B_{p,q,\gamma}^{s}(R^n;E)$. By using a similar technique as in [11] and [15] we obtain the for all $f\in B_{p,q}^{s}(\mathbb{R}_+;X)$, where $X=B_{p,q,\gamma}(R^n;E)$, there is unique solution $u(t,x)$ of problem (4.3) satisfying the following coercive estimate:

$$\left\|\frac{\partial u}{\partial t}\right\|_{B_{p,q}^{s}(\mathbb{R}_+;X)}+\sum_{|\alpha|\leq l}\left\|a_\alpha * D^{[\alpha]}u\right\|_{B_{p,q}^{s}(\mathbb{R}_+;X)}+\left\|A*u\right\|_{B_{p,q}^{s}(\mathbb{R}_+;X)}\leq c\left\|f\right\|_{B_{p,q}^{s}(\mathbb{R}_+;X)}.$$

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