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Analytic properties of Euler-Zagier multiple Lucas-balancing zeta functions

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Abstract. *In the article we study the analytic continuation of Euler-Zagier multiple Lucas-balancing zeta function which is a multi-variable generalization of Lucas-balancing zeta function. We compute a complete list of poles and their residues. We also examine the Euler-Zagier multiple Lucas-balancing zeta values at negative integers.*

Keywords. Analytic continuation, Lucas-balancing sequence, Lucas-balancing zeta function, Euler-Zagier multiple Lucas-balancing zeta function
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1 Introduction

The Euler-Zagier's multiple zeta function $\zeta_{EZ,k}$ is defined by

$$\zeta_{EZ,k}(s_1, s_2, \ldots, s_k) = \sum_{0<m_1<m_2\cdots<m_k<\infty} \frac{1}{{m_1}^{s_1}{m_2}^{s_2}\cdots{m_k}^{s_k}}, \tag{1.1}$$

where $s_1, s_2, \ldots, s_k$ are complex variables [13]. This series is absolutely convergent in the region

$$\{(s_1, s_2, \ldots, s_k) \in C^k | Re(s_{k-r+1} + s_{k-r+2} + \cdots + s_k) > r,\ r = 1, 2, \ldots, k\}.$$

In [3], Atkinson proved the analytic continuation of $\zeta_{EZ,2}(s_1, s_2)$ with application to the study of the asymptotic behaviour of the mean values of the zeta functions using Poisson summation formula. Arakawa and Kaneko [2] demonstrated the analytic continuation of (1.1) as a function of single variable s_k, where $s_1, s_2, \ldots, s_{k-1}$ are positive integers. Akiyama et al. [1] proved the analytic continuation of (1.1) by applying Euler-Maclaurin summation formula. Zhao [14] obtained the same result considering (1.1) as a function of $s_i (i = 1, 2, \ldots, k)$ variables using Gelfand and Shilov's generalized functions. Mehta et al.

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[8] studied the meromorphic continuation of multiple zeta functions with their poles and residues, by means of an elementary and simple translation formula for specified multiple zeta functions.

In [9], Navas introduced Fibonacci Dirichlet series $\zeta_F(s) = \sum\limits_{n=1}^{\infty} F_n^{-s},\ Re(s) > 0$ for $s \in \mathbb{C}$, where F_n denotes the n-th Fibonacci number and proved that $\zeta_F(s)$ is analytically continued to a meromorphic function on the complex plane $\mathbb{C}$. In [6], Kamano considered the Lucas zeta function $\Phi^{(P,\,Q)}(s) = \sum\limits_{n=1}^{\infty} \frac{1}{U_n^s},\ Re(s) > 0,\ s \in \mathbb{C}$, which is a generalization of Fibonacci zeta function and derived its analytic continuation, where U_n is the n-th Lucas number of first kind. Rout and Meher [11] defined the multiple Fibonacci zeta function

$$\zeta_F(s_1, s_2, \ldots, s_d) = \sum_{0<n_1<n_2\cdots<n_d} \frac{1}{F_{n_1}^{s_1} F_{n_2}^{s_2} \cdots F_{n_d}^{s_d}},$$

where d and $s_1 + s_2 + \cdots + s_d$ are the depth and weight of $\zeta_F(s_1, s_2, \ldots, s_d)$ respectively. They studied the analytic continuation of $\zeta_F(s_1, s_2, \ldots, s_d)$ of depth $d = 2$ and found a complete list of poles and their corresponding residues. In [11], they also examined the arithmetic nature of $\zeta_F(s_1, s_2, \ldots, s_d)$ at negative integer arguments. Recently, Meher and Rout [7] proved the meromorphic continuation of multiple Lucas zeta functions of depth d :

$$\zeta_U(s_1, \ldots, s_d) = \sum_{0<n_1<n_2\cdots<n_d} \frac{1}{U_{n_1}^{s_1} \cdots U_{n_d}^{s_d}},$$

where U_n is the n-th Lucas number of first kind. They calculated a complete list of poles and their residues and proved that the multiple Lucas zeta values are rational at negative integers.

Before going to the study of analytic continuation of Euler-Zagier multiple Lucas balancing zeta functions, our foremost task is to discuss the theory of balancing numbers, Lucas-balancing numbers and their zeta functions. A natural number m is said to be a balancing number if it is the solution of a simple Diophantine equation $1+2+\cdots+(m-1) = (m+1) + (m+2) + \cdots + (m+r)$, where r is a balancer corresponding to m [5]. Let $\{B_m\}_{m\geq 0}$ be the balancing sequence and is recursively defined as $B_0 = 0,\ B_1 = 1$ and $B_m = 6B_{m-1} - B_{m-2}$ for $m \geq 2$. For any balancing number B_n, the positive square roots of $8B_n^2 + 1$ also generate a sequence known as Lucas-balancing sequence $\{C_n\}_{n\geq 0}$. Lucas-balancing sequence satisfies the same recurrence as that of balancing sequence but with different initials, that is, $C_n = 6C_{n-1} - C_{n-2}$ for $n \geq 2$ with $C_0 = 1$ and $C_1 = 3$ [10]. The closed form expressions for both these sequences are $B_m = \dfrac{\alpha^m - \beta^m}{\alpha - \beta}$, and $C_m = \dfrac{\alpha^m + \beta^m}{2}$, where $\alpha = 3 + 2\sqrt{2}$ and $\beta = \alpha^{-1} = 3 - 2\sqrt{2}$ are the roots of the balancing characteristic equation $x^2 - 6x + 1 = 0$ [10]. Rout and Panda [12] considered balancing zeta function $\zeta_B(s) = \sum\limits_{m=1}^{\infty} B_m^{-s},\ Re(s) > 0$ for $s \in \mathbb{C}$, where B_m denotes the m-th balancing number and derived that $\zeta_B(s)$ can be meromorphically continued to the whole complex plane. They also shown that $\zeta_B(-m)$ is an irrational number when m is an odd natural number. In a subsequent paper, Behera et al. [4] proved the analytical continuation of $\zeta_C(s) = \sum\limits_{n=1}^{\infty} C_n^{-s},\ Re(s) > 0$ for $s \in \mathbb{C}$, where C_n denotes the n-th Lucas-balancing number and $\zeta_C(-m)$ is a rational number for any odd natural number m.

In this note we investigate the analytic continuation of Euler-Zagier multiple Lucas-balancing zeta function which is a k-variable generalization of Lucas-balancing zeta function. We calculate a complete list of poles and their residues and examine the values of Euler-Zagier multiple Lucas-balancing zeta functions at negative integer arguments.

2 Analytic continuation of Euler-Zagier multiple Lucas-balancing zeta functions

In this section we define Euler-Zagier multiple Lucas-balancing zeta function and demonstrate its analytic continuation.

Definition 2.1 *The Euler-Zagier multiple Lucas-balancing zeta function ζ_{EZC} is defined by*

$$\begin{aligned}\zeta_{EZC}(S) &= \sum_{1\le m_1<m_2<\cdots<m_k<\infty} \frac{1}{C_{m_1}^{s_1} C_{m_2}^{s_2} \cdots C_{m_k}^{s_k}} \\ &= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_k=1}^{\infty} \frac{1}{C_{m_1}^{s_1} C_{m_1+m_2}^{s_2} \cdots C_{m_1+m_2+\cdots+m_k}^{s_k}},\end{aligned} \tag{2.1}$$

where $S=(s_1,s_2,\ldots,s_k)$ and $s_i\in\mathbb{C}, i=1,2,\ldots k$.

Proposition 2.1 *The sum $\zeta_{EZC}(S)$ converges absolutely in the domain*

$$D_k = \{(s_1,s_2,\ldots,s_k)\in\mathbb{C}^k \mid \sum_{i=d}^{k} Re(s_i) > 0, 1\le d\le k\}.$$

Proof. Lucas-balancing numbers satisfy the inequality $C_m > \alpha^{m-1}$ [10]. Let $s_j=\sigma_j+it_j$ and $\sigma_j = Re(s_j) > 0$ for $j=1,2,\ldots,k$.
Then we have the following estimates

$$\begin{aligned}&\left|\frac{1}{C_{m_1}^{s_1}}\right| = \frac{1}{C_{m_1}^{\sigma_1}} < \frac{1}{\alpha^{\sigma_1(m_1-1)}},\\ &\left|\frac{1}{C_{m_1+m_2}^{s_2}}\right| = \frac{1}{C_{m_1+m_2}^{\sigma_2}} < \frac{1}{\alpha^{\sigma_2(m_1+m_2-1)}},\\ &\quad\vdots\\ &\left|\frac{1}{C_{m_1+m_2\cdots+m_k}^{s_k}}\right| = \frac{1}{C_{m_1+m_2+\cdots+m_k}^{\sigma_k}} < \frac{1}{\alpha^{\sigma_k(m_1+m_2+\cdots+m_k-1)}}.\end{aligned}$$

Using the above identities, from Definition 2.1 we have

$$\begin{aligned}&\sum_{0<m_1<m_2<\cdots<m_k}\left|\frac{1}{C_{m_1}^{s_1} C_{m_2}^{s_2}\cdots C_{m_k}^{s_k}}\right| \\ &< \sum_{\substack{m_1,m_2,\\ \ldots,m_k=1}}^{\infty} \frac{1}{\alpha^{\sigma_1(m_1-1)}\alpha^{\sigma_2(m_1+m_2-1)}\cdots\alpha^{\sigma_k(m_1+m_2+\cdots+m_k-1)}}\end{aligned}$$

$$
\begin{aligned}
&= \alpha^{\sigma_1+\sigma_2+\cdots+\sigma_k} \sum_{m_1=1}^{\infty} \frac{1}{\alpha^{(\sigma_1+\sigma_2+\cdots+\sigma_k)m_1}} \sum_{m_2=1}^{\infty} \frac{1}{\alpha^{(\sigma_2+\sigma_3+\cdots+\sigma_k)m_2}} \cdots \sum_{m_k=1}^{\infty} \frac{1}{\alpha^{\sigma_k m_k}} \\
&= \alpha^{\sigma_1+\sigma_2+\cdots+\sigma_k} \frac{1}{(\alpha^{\sigma_1+\sigma_2+\cdots+\sigma_k}-1)} \frac{1}{(\alpha^{\sigma_2+\sigma_3+\cdots+\sigma_k}-1)} \cdots \frac{1}{(\alpha^{\sigma_k}-1)} \\
&= \left(1+\frac{1}{\alpha^{\sigma_1+\sigma_2+\cdots+\sigma_k}-1}\right) \frac{1}{(\alpha^{\sigma_2+\sigma_3+\cdots+\sigma_k}-1)} \cdots \frac{1}{(\alpha^{\sigma_k}-1)} \\
&< \infty,
\end{aligned}
$$

as $\alpha > 1$.
This completes the proof.

Theorem 2.1 *$\zeta_{EZC}(s_1, s_2, \ldots, s_k)$ can be analytically continued to a meromorphic function on $\mathbb{C}^k$. It has possible simple poles on the hyperplanes*

$$
s_d + s_{d+1} + \cdots + s_k = -2(r_d + r_{d+1} + \cdots + r_k) + \frac{2\pi i a_d}{\log \alpha},\ 1 \leq d \leq k, \tag{2.2}
$$

where $r_1, r_2, \ldots, r_k \in \mathbb{Z}_{\geq 0},$ and $a_1, \ldots, a_k \in \mathbb{Z}$.

Proof. For any complex number $s \in \mathbb{C}$, we have

$$
\begin{aligned}
C_m^s = \left(\frac{\alpha^m+\beta^m}{2}\right)^s &= 2^{-s}\alpha^{ms}\left(1+\left(\frac{\beta}{\alpha}\right)^m\right)^s \\
&= 2^{-s}\alpha^{ms}\left(1+\frac{1}{\alpha^{2m}}\right)^s \\
&= 2^{-s}\sum_{r=0}^{\infty}(-1)^r\binom{s}{r}\alpha^{m(s-2r)}.
\end{aligned}
\tag{2.3}
$$

The above series converges since $\alpha > 1$. Using (2.3) in (2.1), we get

$$
\begin{aligned}
\zeta_{EZC}(s_1, s_2, \ldots, s_k) &= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\frac{1}{C_{m_1}^{s_1}C_{m_1+m_2}^{s_2}\cdots C_{m_1+m_2+\cdots+m_k}^{s_k}} \\
&= \sum_{m_1,m_2,\ldots,m_k=1}^{\infty}\left(2^{s_1}\sum_{r_1=0}^{\infty}(-1)^{r_1}\binom{-s_1}{r_1}\alpha^{-m_1(s_1+2r_1)}\right) \\
&\quad \times \left(2^{s_2}\sum_{r_2=0}^{\infty}(-1)^{r_2}\binom{-s_2}{r_2}\alpha^{-(m_1+m_2)(s_2+2r_2)}\right)\cdots \\
&\quad \cdots\left(2^{s_k}\sum_{r_k=0}^{\infty}(-1)^{r_k}\binom{-s_k}{r_k}\alpha^{-(m_1+m_2+\cdots+m_k)(s_k+2r_k)}\right)
\end{aligned}
\tag{2.4}
$$

Using $\left|\binom{-s_i}{r_i}\right| \leq (-1)^{r_i}\binom{-|s_i|}{r_i}$, for $i = 1, 2, \ldots, k$, we have

$$\begin{aligned}
&\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\left|\frac{1}{C_{m_1}^{s_1}}\frac{1}{C_{m_1+m_2}^{s_2}}\cdots\frac{1}{C_{m_1+m_2+\cdots+m_k}^{s_k}}\right| \\
&= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\left|\left(2^{s_1}\sum_{r_1=0}^{\infty}(-1)^{r_1}\binom{-s_1}{r_1}\alpha^{-m_1(s_1+2r_1)}\right)\right. \\
&\quad\times\left(2^{s_2}\sum_{r_2=0}^{\infty}(-1)^{r_2}\binom{-s_2}{r_2}\alpha^{-(m_1+m_2)(s_2+2r_2)}\right)\cdots \\
&\quad\left.\cdots\left(2^{s_k}\sum_{r_k=0}^{\infty}(-1)^{r_k}\binom{-s_k}{r_k}\alpha^{-(m_1+m_2+\cdots+m_k)(s_k+2r_k)}\right)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{m_1,m_2,\ldots,m_k=1}^{\infty} 2^{\sigma_1+\sigma_2+\cdots+\sigma_k}\alpha^{-m_1\sigma_1}\sum_{r_1=0}^{\infty}\binom{-|s_1|}{r_1}(-1)^{r_1}\alpha^{-2m_1r_1} \\
&\quad\times\alpha^{-(m_1+m_2)\sigma_2}\sum_{r_2=0}^{\infty}\binom{-|s_2|}{r_2}(-1)^{r_2}\alpha^{-2(m_1+m_2)r_2}\cdots \\
&\quad\cdots\alpha^{-(m_1+m_2+\cdots+m_k)\sigma_k}\sum_{r_k=0}^{\infty}\binom{-|s_k|}{r_k}(-1)^{r_k}\alpha^{-2(m_1+m_2+\cdots+m_k)r_k} \\
&= \sum_{m_1,m_2,\ldots,m_k=1}^{\infty} 2^{\sigma_1+\sigma_2+\cdots+\sigma_k}\alpha^{-m_1\sigma_1}\left(1-\alpha^{-2m_1}\right)^{-|s_1|} \\
&\quad\times\alpha^{-(m_1+m_2)\sigma_2}\left(1-\alpha^{-2(m_1+m_2)}\right)^{-|s_2|}\cdots \\
&\quad\cdots\alpha^{-(m_1+m_2+\cdots+m_k)\sigma_k}\left(1-\alpha^{-2(m_1+m_2+\cdots+m_k)}\right)^{-|s_k|} \\
&\leq \sum_{m_1,m_2,\ldots,m_k=1}^{\infty} 2^{\sigma_1+\sigma_2+\cdots+\sigma_k}\alpha^{-m_1\sigma_1}\left(1-\alpha^{-2}\right)^{-|s_1|} \\
&\quad\times\alpha^{-(m_1+m_2)\sigma_2}\left(1-\alpha^{-4}\right)^{-|s_2|}\cdots\alpha^{-(m_1+m_2+\cdots+m_k)\sigma_k}\left(1-\alpha^{-2k}\right)^{-|s_k|}
\end{aligned}$$

$$\begin{aligned}
&= 2^{\sigma_1+\sigma_2+\cdots+\sigma_k}\left(1-\alpha^{-2}\right)^{-|s_1|}\left(1-\alpha^{-4}\right)^{-|s_2|}\cdots\left(1-\alpha^{-2k}\right)^{-|s_k|} \\
&\quad\times\sum_{m_1=1}^{\infty}\alpha^{-(\sigma_1+\sigma_2+\cdots+\sigma_k)m_1}\sum_{m_2=1}^{\infty}\alpha^{-(\sigma_2+\sigma_3+\cdots+\sigma_k)m_2}\cdots\sum_{m_k=1}^{\infty}\alpha^{-\sigma_k m_k} \\
&< \infty.
\end{aligned}$$

Let

$$H = 2^{s_1+s_2+\cdots+s_k}\sum_{r_1=0}^{\infty}(-1)^{r_1}\binom{-s_1}{r_1}\sum_{r_2=0}^{\infty}(-1)^{r_2}\binom{-s_2}{r_2}\cdots\sum_{r_k=0}^{\infty}(-1)^{r_k}\binom{-s_k}{r_k}. \quad (2.5)$$

Then by interchanging the order of summation in (2.4) and using (2.5), we have

$$\begin{aligned}
&\zeta_{EZC}(s_1, s_2, \ldots, s_k) \qquad (2.6)\\
=\ & H \sum_{m_1=1}^{\infty} \left(\alpha^{-\left(s_1+s_2+\cdots+s_k+2(r_1+r_2+\cdots+r_k)\right)}\right)^{m_1}\\
&\times \sum_{m_2=1}^{\infty} \left(\alpha^{-\left(s_2+s_3+\cdots+s_k+2(r_2+r_3+\cdots+r_k)\right)}\right)^{m_2} \cdots \sum_{m_k=1}^{\infty} \left(\alpha^{-(s_k+2r_k)}\right)^{m_k}\\
&= H \frac{\alpha^{-\left(s_1+s_2+\cdots+s_k+2(r_1+r_2+\cdots+r_k)\right)}}{\left(1-\alpha^{-\left(s_1+s_2+\cdots+s_k+2(r_1+r_2+\cdots+r_k)\right)}\right)}\\
&\times \frac{\alpha^{-\left(s_2+s_3+\cdots+s_k+2(r_2+r_3+\cdots+r_k)\right)}}{\left(1-\alpha^{-\left(s_2+s_3+\cdots+s_k+2(r_2+r_3+\cdots+r_k)\right)}\right)} \cdots \frac{\alpha^{-(s_k+2r_k)}}{(1-\alpha^{-(s_k+2r_k)})}
\end{aligned}$$

$$\begin{aligned}
&= H \frac{1}{\left(\alpha^{s_1+s_2+\cdots+s_k+2(r_1+r_2+\cdots+r_k)}-1\right)} \frac{1}{\left(\alpha^{s_2+s_3+\cdots+s_k+2(r_2+r_3+\cdots+r_k)}-1\right)} \cdots\\
&\cdots \frac{1}{\left(\alpha^{s_k+2r_k}-1\right)}\\
&= 2^{s_1+s_2+\cdots+s_k} \sum_{r_1=0}^{\infty} (-1)^{r_1} \binom{-s_1}{r_1} \sum_{r_2=0}^{\infty} (-1)^{r_2} \binom{-s_2}{r_2} \cdots \sum_{r_k=0}^{\infty} (-1)^{r_k} \binom{-s_k}{r_k}\\
&\times \frac{1}{\left(\alpha^{s_1+s_2+\cdots+s_k+2(r_1+r_2+\cdots+r_k)}-1\right)} \frac{1}{\left(\alpha^{s_2+s_3+\cdots+s_k+2(r_2+r_3+\cdots+r_k)}-1\right)} \cdots\\
&\cdots \frac{1}{\left(\alpha^{s_k+2r_k}-1\right)}. \qquad (2.7)
\end{aligned}$$

For any $s_1, s_2, \ldots, s_k \in \mathbb{C}$,

$$\begin{aligned}
\left|\alpha^{s_1+s_2+\cdots+s_k+2(r_1+r_2+\cdots+r_k)}-1\right| &\geq \alpha^{\sigma_1+\sigma_2+\cdots+\sigma_k+2(r_1+r_2+\cdots+r_k)}-1\\
&> \alpha^{\sigma_1+\sigma_2+\cdots+\sigma_k+r_1+r_2+\cdots+r_k}
\end{aligned}$$

for $r_i \geq r_i'^1$, where $r_i'^1 = r_i'^1(\sigma_1, \sigma_2, \ldots, \sigma_k, \alpha) \gg 0$ for $i = 1, 2, \ldots, k$.
Similarly,

$$\left|\alpha^{s_2+s_3+\cdots+s_k+2(r_2+r_3+\cdots+r_k)}-1\right| > \alpha^{\sigma_2+\sigma_3+\cdots+\sigma_k+r_2+r_3+\cdots+r_k}$$

for $r_i \geq r_i'^2$, where $r_i'^2 = r_i'^2(\sigma_2, \sigma_3, \ldots, \sigma_k, \alpha) \gg 0$ for $i = 2, 3, \ldots, k$. Continuing in this way,

$$\left|\alpha^{s_k+2r_k}-1\right| > \alpha^{\sigma_k+r_k}$$

for $r_i \geq r_i'^k$, where $r_i'^k = r_i'^k(\sigma_k, \alpha) \gg 0$, for $i = k$. Define $r_i^* = \max\{r_i'^1, r_i'^2, \ldots, r_i'^i\}$ for $i = 2, 3, \ldots, k$.

Now

$$\sum_{r_1>r_1'^1}^{\infty}\sum_{r_2>r_2^*}^{\infty}\sum_{r_2>r_2^*}^{\infty}\cdots\sum_{r_k>r_k^*}\left|\binom{-s_1}{r_1}(-1)^{r_1}\binom{-s_2}{r_2}(-1)^{r_2}\cdots\binom{-s_k}{r_k}(-1)^{r_k}\right.$$
$$\times\frac{1}{\left(\alpha^{s_1+s_2+\cdots+s_k+2(r_1+r_2+\cdots+r_k)}-1\right)}\frac{1}{\left(\alpha^{s_2+s_3+\cdots+s_k+2(r_2+r_3+\cdots+r_k)}-1\right)}\cdots$$
$$\left.\cdots\frac{1}{\left(\alpha^{s_k+2r_k}-1\right)}\right|$$
$$\leq\sum_{\substack{r_1>r_1'^1,r_2>r_2^*,\\ \ldots,r_k>r_k^*}}^{\infty}\binom{-|s_1|}{r_1}(-1)^{r_1}\binom{-|s_2|}{r_2}(-1)^{r_2}\cdots\binom{-|s_k|}{r_k}(-1)^{r_k}\frac{1}{\alpha^{\sigma_1+\sigma_2+\cdots+\sigma_k+r_1+r_2+\cdots+r_k}}$$
$$\times\frac{1}{\alpha^{\sigma_2+\sigma_3+\cdots+\sigma_k+r_2+r_3+\cdots+r_k}}\cdots\frac{1}{\alpha^{\sigma_k+r_k}}$$

$$\leq\alpha^{-(\sigma_1+2\sigma_2+\cdots+k\sigma_k)}\sum_{\substack{r_1>r_1'^1,r_2>r_2^*,\\ \ldots,r_k>r_k^*}}^{\infty}\binom{-|s_1|}{r_1}(-1)^{r_1}\alpha^{-r_1}\binom{-|s_2|}{r_2}(-1)^{r_2}\alpha^{-2r_2}\cdots$$
$$\cdots\binom{-|s_k|}{r_k}(-1)^{r_k}\alpha^{-kr_k}$$
$$\leq\alpha^{-(\sigma_1+2\sigma_2+\cdots+k\sigma_k)}\left(1-\alpha^{-1}\right)^{-|s_1|}\left(1-\alpha^{-2}\right)^{-|s_2|}\cdots\left(1-\alpha^{-k}\right)^{-|s_k|}.$$

The above bound is uniform when $(s_1, s_2, \ldots, s_k)$ varies over a compact subsets of $\mathbb{C}^k$. Hence the infinite series (2.6) converges absolutely and uniformly on the compact subsets of $\mathbb{C}^k$ not containing any poles. Moreover, the series (2.6) determines the holomorphic function on $\mathbb{C}^k$ except for the poles derived from the functions $\alpha^{s_1+s_2+\cdots+s_k+2r_1+2r_2+\cdots+2r_k}-1=0$, $\alpha^{s_2+s_3+\cdots+s_k+2r_2+2r_3+\cdots+2r_k}-1=0,\cdots$, and $\alpha^{s_k+2r_k}-1=0$. Therefore, $\zeta_{EZC}(s_1, s_2, \ldots, s_k)$ can be analytically continued to a meromorphic function on $\mathbb{C}^k$ and its simple poles are at

$$s_1+s_2+\cdots+s_k=-2(r_1+r_2+\cdots+r_k)+\frac{2\pi i a_1}{\log\alpha},$$
$$s_2+s_3+\cdots+s_k=-2(r_2+r_3+\cdots+r_k)+\frac{2\pi i a_2}{\log\alpha},$$
$$\vdots$$
$$s_k=-2r_k+\frac{2\pi i a_k}{\log\alpha},$$

where $r_1, r_2, \ldots, r_k \in \mathbb{Z}_{\geq 0}$ and $a_1, \ldots, a_k \in \mathbb{Z}$. This completes the proof.

3 Poles and residues of Euler-Zagier multiple Lucas-balancing zeta functions

For $1 \leq d \leq k$, let $s_k(d) = s_d+\cdots+s_k,\ r_k(d) = r_d+\cdots+r_k,\ r_k'(d) = r_d'+\cdots+r_k'$ with $s_k(k+1) = 0,\ r_k(k+1) = 0,\ r_k'(k+1) = 0$. We define the residue of the Euler-Zagier multiple Lucas-balancing zeta functions $\zeta_{EZC}(s_1, s_2, \ldots, s_k)$ along the hyperplanes (2.2) to be the restriction of the meromorphic function

$$\left(s_k(d) + 2r_k(d) - \frac{2\pi i a_d}{\log\alpha}\right)\zeta_{EZC}(s_1, s_2, \ldots, s_k)$$

to the hyperplanes (2.2). Now we compute the corresponding residues for a fixed hyperplanes as follows.

Theorem 3.1 *Let r'_k be a non-negative integer. Let $f_k(k) = -2r'_k(k) + \frac{2\pi i a_k}{\log\alpha}$ and $\zeta_{EZC}(s_1, s_2, \ldots, s_{k-1}) = 1$ for $k = 1$. Then*

$$Res_{s_k=f_k(k)}\zeta_{EZC}(s_1, s_2, \ldots, s_k) = \zeta_{EZC}(s_1, s_2, \ldots, s_{k-1})\frac{(-1)^{r'_k}\ 2^{f_k(k)}}{\log\alpha}\binom{-f_k(k)}{r'_k}.$$

Proof. The case $k = 1$ was done by Behera et al. [4]. Let us assume that $k \geq 2$ and $s_k(k) = s_k$. It is clear that the function $\alpha^{s_k+2r'_k} - 1$ is an analytic function with simple poles at $f_k(k)$. Then

$$\lim_{s_k\to f_k(k)}\frac{s_k - f_k(k)}{\alpha^{s_k+2r'_k} - 1} = \text{Res}_{\ s_k=f_k(k)}\frac{1}{(\alpha^{s_k+2r'_k} - 1)} = \frac{1}{\log\alpha}.$$

Thus the residue of $\zeta_{EZC}(s_1, s_2, \cdots s_k)$ along the hyper plane $f_k(k)$ is given by

$$\begin{aligned}
&\text{Res}_{s_k=f_k(k)}\zeta_{EZC}(s_1, s_2, \ldots, s_k)\\
&= \lim_{s_k\to f_k(k)}(s_k - f_k(k))\ 2^{s_k(1)}\sum_{r_1=0}^{\infty}(-1)^{r_1}\binom{-s_1}{r_1}\cdots\sum_{r_k=0}^{\infty}(-1)^{r_k}\binom{-s_k}{r_k}\\
&\quad\times\frac{1}{(\alpha^{s_k(1)+2r_k(1)} - 1)}\cdots\frac{1}{(\alpha^{s_k+2r_k} - 1)}.\\
&= 2^{s_{k-1}(1)}\sum_{r_1=0}^{\infty}(-1)^{r_1}\binom{-s_1}{r_1}\cdots\sum_{r_{k-1}=0}^{\infty}(-1)^{r_{k-1}}\binom{-s_{k-1}}{r_{k-1}}\\
&\quad\times\left.\frac{1}{(\alpha^{s_k(1)+2r_k(1)} - 1)}\right|_{s_k(k)=f_k(k)}\times\left.\frac{1}{(\alpha^{s_k(1)+2r_k(1)} - 1)}\right|_{s_k(k)=f_k(k)}\cdots\\
&\quad\cdots\lim_{s_k\to f_k(k)}2^{f_k(k)}(-1)^{r_k}\binom{-s_k}{r_k}\frac{s_k - f_k(k)}{(\alpha^{s_k+2r_k} - 1)}.
\end{aligned}$$

It is noticed that, after applying limit, the terms containing only r_k but not $r_1, \cdots, r_{k-1}$ survives for $r_k = r'_k$ and the remaining of the terms vanish.

Therefore, from the above calculation, we have

$$\begin{aligned}
&\text{Res}_{s_k=f_k(k)}\zeta_{EZC}(s_1, s_2, \ldots, s_k)\\
&= 2^{f_k(k)}2^{s_{k-1}(1)}\sum_{r_1=0}^{\infty}(-1)^{r_1}\binom{-s_1}{r_1}\cdots\sum_{r_{k-1}=0}^{\infty}(-1)^{r_{k-1}}\binom{-s_{k-1}}{r_{k-1}}\\
&\quad\times\frac{1}{(\alpha^{s_{k-1}(1)+2r_{k-1}(1)} - 1)}\cdots\frac{(-1)^{r'_k}}{\log\alpha}\binom{-f_k(k)}{r'_k}\\
&= \zeta_{EZC}(s_1, s_2, \ldots, s_{k-1})2^{f_k(k)}\binom{-f_k(k)}{r'_k}(-1)^{r'_k}\frac{1}{\log\alpha}.
\end{aligned}$$

This ends the proof.

In the following theorem, we calculate the residues along the other hyperplanes for $1 \leq d \leq k-1$.

Theorem 3.2 *Let $k \geq 2$ and d be an positive integer such that $1 \leq d \leq k-1$. Let $r'_d \cdots r'_k$ be a non-negative integers. Let $f_k(d) = -2r'_k(d) + \frac{2\pi i a_d}{\log \alpha}$. Then*

$$\begin{aligned} &\mathit{Res}_{s_k(d)=f_k(d)} \zeta_{EZC}(s_1, s_2, \ldots, s_k) \\ &= 2^{f_k(d)} \zeta_{EZC}(s_1, s_2, \ldots, s_{d-1}) \sum_{\substack{r_d \geq 0, \ldots r_k \geq 0 \\ r_k(d) = r'_k(d)}} \binom{-s_d}{r_d} (-1)^{r_d} \cdots \\ &\cdots \binom{-s_k}{r_k} (-1)^{r_k} \frac{1}{\left(\alpha^{s_k(d+1)+2r_k(d+1)} - 1\right)} \cdots \frac{1}{\left(\alpha^{s_k+2r_k} - 1\right)} \frac{1}{\log \alpha}. \end{aligned}$$

Proof. By proceeding as in the proof of Theorem 3.1, we have

$$\lim_{s_k(d) \to f_k(d)} \frac{s_k(d) - f_k(d)}{\alpha^{s_k(d)+2r_k(d)} - 1} = \text{Res}_{\, s_k(d)=f_k(d)} \frac{1}{(\alpha^{s_k(d)+2r_k(d)} - 1)} = \frac{1}{\log \alpha}.$$

Now we evaluate the limit as follows:

$$\lim_{s_k(d) \to f_k(d)} 2^{s_k(d)} \sum_{r_d, \ldots, r_k = 0}^{\infty} (-1)^{r_d} \binom{-s_d}{r_d} \cdots (-1)^{r_k} \binom{-s_k}{r_k} \frac{s_k(d) - f_k(d)}{\alpha^{s_k(d)+2r_k(d)} - 1} \cdots \frac{1}{\alpha^{s_k(k)+2r_k(k)} - 1}.$$

In the above calculation after applying the limit, only those terns containing $r_d, \ldots, r_k$ will survive when $r_d + \cdots + r_k = r'_d + \cdots + r'_k$ and rest of the terms will vanish. Therefore, the above limit reduces to

$$2^{f_k(d)} \sum_{\substack{r_d \geq 0, \ldots, r_k \geq 0 \\ r_k(d) = r'_k(d)}} (-1)^{r_d} \binom{-s_d}{r_d} \cdots (-1)^{r_k} \binom{-s_k}{r_k} \frac{1}{\alpha^{s_k(d+1)+2r_k(d+1)} - 1} \cdots \frac{1}{\alpha^{s_k+2r_k} - 1} \frac{1}{\log \alpha}.$$

Thus the residue of $\zeta_{EZC}(s_1, s_2, \cdots s_k)$ along the hyper plane $s_k(d) = f_k(d)$ is given by

$$
\begin{aligned}
&\mathrm{Res}_{s_k(d)=f_k(d)}\zeta_{EZC}(s_1, s_2, \ldots, s_k)\\
&= \lim_{s_k(d)\to f_k(d)} (s_k(d) - f_k(d))\zeta_{EZC}(s_1, \ldots, s_k)\\
&= 2^{s_k(d)}2^{s_1+\cdots+s_{d-1}} \sum_{r_1,\ldots,r_{d-1}=0}^{\infty} (-1)^{r_1}\binom{-s_1}{r_1}\cdots(-1)^{r_{d-1}}\binom{-s_{d-1}}{r_{d-1}}\\
&\quad\times \sum_{r_d,\ldots,r_k=0}^{\infty} (-1)^{r_d}\binom{-s_d}{r_d}\cdots(-1)^{r_k}\binom{-s_k}{r_k}\\
&\quad\times \frac{1}{\left(\alpha^{s_k(1)+2r_k(1)}-1\right)}\cdots\frac{1}{\left(\alpha^{s_k(d-1)+2r_k(d-1)}-1\right)}\Bigg|_{s_k(d)=f_k(d)}\\
&\quad\times \lim_{s_k(d)\to f_k(d)} \frac{s_k(d)-f_k(d)}{\left(\alpha^{s_k(d)+2r_k(d)}-1\right)}\cdots\frac{1}{\left(\alpha^{s_k(k)+2r_k(k)}-1\right)}\\
&= 2^{f_k(d)}\zeta_{EZC}(s_1, s_2, \ldots, s_{d-1}) \sum_{\substack{r_d\geq 0,\ldots,r_k\geq 0\\ r_k(d)=r'_k(d)}} \binom{-s_d}{r_d}(-1)^{r_d}\cdots\\
&\quad\cdots\binom{-s_k}{r_k}(-1)^{r_k}\frac{1}{\left(\alpha^{s_k(d+1)+2r_k(d+1)}-1\right)}\cdots\frac{1}{\left(\alpha^{s_k+2r_k}-1\right)}\frac{1}{\log\alpha}.
\end{aligned}
$$

This ends the proof.

4 Values of Euler-Zagier multiple Lucas-balancing zeta functions at negative integers

In this section we consider the values of $\zeta_{EZC}(s_1, \ldots, s_k)$ at negative integers. First we give a sufficient condition for $\zeta_{EZC}(s_1, \ldots, s_k)$ to be holomorphic at $(s_1, \ldots, s_k) = (-n_1, \ldots, -n_k)$ where $n_i \in \mathbb{N}$ for $i = 1, 2, \ldots, k$. Let us denote $n_k(d) = n_d + n_{d+1} + \cdots + n_k,\ d = 1, 2, \ldots, k$.

Lemma 4.1 *Let* $(n_1, \ldots, n_k) \in \mathbb{N}^k$. $\zeta_{EZC}(s_1, \ldots, s_k)$ *is holomorphic at*

$$(s_1, \ldots, s_k) = (-n_1, \ldots, -n_k)$$

if and only if $n_k(1) \not\equiv 0 \pmod 2,\ n_k(2) \not\equiv 0 \pmod 2, \ldots, n_k(k) \not\equiv 0 \pmod 2.$

Proof. The infinite series (2.6) is holomorphic except the poles derived from

$$\left(\alpha^{s_1+s_2+\cdots+s_k+2(r_1+r_2+\cdots+r_k)}-1\right)\times\cdots\times\left(\alpha^{s_k+2r_k}-1\right)=0$$

This is true if and only if, one of the following equations holds:

$$s_k(1) = -2r_k(1),\ s_k(2) = -2r_k(2),\ \cdots s_k(k) = -2r_k(k),$$

with $r_k(1) \equiv 0 \pmod 2,\ r_k(2) \equiv 0 \pmod 2, \ldots, r_k(k) \equiv 0 \pmod 2.$

For $1 \leq d \leq k$, let us denote

$$\Delta_d(r_d, \ldots r_k; n_d) = (-1)^{r_d}\frac{1}{\alpha^{-n_k(d)+2r_k(d)}-1}. \tag{4.1}$$

In particular,

$$\Delta_1(r_1,\dots r_k;n_1)=(-1)^{r_1}\frac{1}{\alpha^{-n_k(1)+2r_k(1)}-1},$$
$$\vdots$$
$$\Delta_k(r_k;n_k)=(-1)^{r_k}\frac{1}{\alpha^{-n_k(k)+2r_k(k)}-1}.$$

By replacing r_d by n_d-r_d in the above notation,we have

$$\Delta_d(\hat{r}_d,\dots r_k;n_d)=(-1)^{n_d-r_d}\frac{1}{\alpha^{-n_k(d)+2(n_d-r_d+r_k(d+1))}-1}.$$

Further, we denote

$$\sigma_0(r_1,\dots,r_k)=\Delta_1(r_1,\dots,r_k;n_1)\times\Delta_2(r_2,\dots,r_k;n_2)\times\cdots\times\Delta_k(r_k;n_k),$$

$$\begin{aligned}\sigma_1(r_1,\dots,\hat{r}_d,\dots,r_k)=&\ \Delta_1(r_1,\dots,\hat{r}_d,\dots,r_k;n_1)\times\cdots\times\Delta_d(\hat{r}_d,\dots,r_k;n_d)\\&\times\Delta_{d+1}(r_{d+1},\dots,r_k;n_d)\times\cdots\times\Delta_k(r_k;n_k),\end{aligned}$$

$$\begin{aligned}&\sigma_2(r_1,\dots,\hat{r}_c,\dots,\hat{r}_d,\dots,r_k)\\&=\Delta_1(r_1,\dots,\hat{r}_c,\dots,\hat{r}_d,\dots,r_k;n_1)\times\cdots\times\Delta_c(\hat{r}_c,\dots,r_k;n_d)\\&\quad\times\Delta_{c+1}(r_{c+1},\dots,\hat{r}_d\dots,r_k;n_d)\times\cdots\times\Delta_d(\hat{r}_d,\dots,r_k;n_d)\\&\quad\times\cdots\times\Delta_k(r_k;n_k),\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}&\sigma_p(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{c_P},\dots,r_k)\\&=\Delta_1(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{c_P},\dots,r_k;n_1)\times\cdots\times\Delta_{c_1}(\hat{r}_{c_1},\dots,\hat{r}_{c_p},\dots,r_k;n_{c_1})\\&\quad\times\ \Delta_{c_2}(\hat{r}_{c_2},\dots,\hat{r}_{c_p},\dots,r_k;n_{c_2})\times\cdots\times\ \Delta_{c_p}(\hat{r}_{c_p},\dots,r_k;n_{c_p})\times\cdots\times\Delta_k(r_k;n_k).\end{aligned}$$

It is observed that $\sigma_p(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{c_p},\dots,r_k)$ expression precisely represents that p number of integers in the tuple $(r_1,\dots,\dots,r_k)$ are replaced from r_t to n_t-r_t for $1\le t\le p$, whenever these terms appears in that corresponding $\Delta_i(r_i,\dots r_k;n_i)$ as in the above expression.

Using the above notations, we prove the following proposition which is very essential to prove our main result.

Proposition 4.1 *Let Ψ be the non-trivial automorphism of Gal$(\mathbb{Q}\sqrt{2}/\mathbb{Q})$ and σ_p as in the above expression. Then*

$$\begin{aligned}&\Psi\big(\sigma_p(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{c_p},\dots,r_k)\big)\\&=(-1)^{n_k(1)}\big[\sigma_{k-p}(\hat{r}_1,\dots,\hat{r}_{c_1-1},r_{c_1},\dots,r_{c_p},\hat{r}_{c_{p+1}},\dots,\hat{r}_k)\big]\end{aligned}\tag{4.3}$$

for any $0\le p\le k$.

Proof. Now,

$$
\begin{aligned}
\sigma_p(r_1,\dots,\hat r_{c_1},\dots,\hat r_{c_P},\dots,r_k) &= \Delta_1(r_1,\dots,\hat r_{c_1},\dots,\hat r_{c_P},\dots,r_k;n_1)\\
&\quad\times\cdots\times \Delta_{c_1}(\hat r_{c_1},\dots,\hat r_{c_p},\dots,r_k;n_{c_1})\\
&\quad\times\cdots\times \Delta_{c_p}(\hat r_{c_p},\dots,r_k;n_{c_p})\times\cdots\times\Delta_k(r_k;n_k)\\
&= \frac{(-1)^{r_1}}{\alpha^{-n_k(1)+2(r_{c_1-1}(1)+r_k(c_p+1)+\sum_{t=1}^{p}(n_{c_t}-r_{c_t}))}-1}\\
&\quad\times\cdots\times \frac{(-1)^{n_{c_1}-r_{c_1}}}{\alpha^{-n_k(c_1)+2(r_k(c_p+1)+\sum_{t=1}^{p}(n_{c_t}-r_{c_t}))}-1}\\
&\quad\times\cdots\times \frac{(-1)^{n_{c_p}-r_{c_p}}}{\alpha^{-n_k(c_p)+2(r_k(c_p+1)+(n_{c_p}-r_{c_p}))}-1}\\
&\quad\times\cdots\times \frac{(-1)^{r_k}}{\alpha^{-n_k(k)+2r_k(k)}-1}.
\end{aligned}
\tag{4.4}
$$

Further simplification gives

$$
\begin{aligned}
&-n_k(1)+2\big(r_{c_1-1}(1)+r_k(c_p+1)+\sum_{t=1}^{p}(n_{c_t}-r_{c_t})\big)\\
&= -\big[n_1+\cdots+n_{c_1-1}+n_{c_1}+\cdots+n_{c_p}+\cdots+n_k\big]+2\big[r_1+\cdots+r_{c_1-1}+r_{c_p+1}+\cdots+r_{c_k}\\
&\quad+\sum_{t=1}^{p}(n_{c_t}-r_{c_t})\big]\\
&= \big(n_{c_1}+n_{c_2}\cdots+n_{c_p}\big)-\big(n_1+\cdots+n_{c_1-1}+n_{c_p+1}+\cdots+n_k\big)\\
&\quad+2\big[r_1+\cdots+r_{c_1-1}+r_{c_p+1}+\cdots+r_{c_k}+\sum_{t=1}^{p}(-r_{c_t})\big]\\
&= \big(n_1+\cdots+n_{c_1-1}+n_{c_1}+\cdots+n_{c_p}+\cdots+n_k\big)-2\Bigg[\sum_{t=1}^{c_1-1}n_t-r_t+\sum_{t=c_p+1}^{k}n_t-r_t+\sum_{t=1}^{p}r_{c_t}\Bigg]\\
&= n_k(1)-2\Bigg[\sum_{t=1}^{c_1-1}(n_t-r_t)+\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\Bigg].
\end{aligned}
\tag{4.5}
$$

Similarly,

$$
\begin{aligned}
&-n_k(c_1)+2\big(r_k(c_p+1)+\sum_{t=1}^{p}(n_{c_t}-r_{c_t})\big)\\
&\quad= n_k(c_1)-2\Bigg[\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\Bigg]
\end{aligned}
\tag{4.6}
$$

and

$$
-n_k(c_p)+2\big(r_k(c_p+1)+n_{c_p}-r_{c_p}\big)=n_k(c_p)-2\Bigg[\sum_{t=c_p+1}^{k}(n_t-r_t)+r_{c_p}\Bigg]. \tag{4.7}
$$

Using (4.5), (4.6) and (4.7) in (4.4), we have

$$\begin{aligned}\sigma_p(r_1,\ldots,\hat{r}_{c_1},\ldots,\hat{r}_{c_P},\ldots,r_k) = & \frac{(-1)^{r_1}}{\alpha^{n_k(1)-2\left(\sum_{t=1}^{c_1-1}(n_t-r_t)+\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p} r_{c_t}\right)}-1} \\ & \times\cdots\times \frac{(-1)^{n_{c_1}-r_{c_1}}}{\alpha^{n_k(c_1)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p} r_{c_t}\right)}-1} \\ & \times\cdots\times \frac{(-1)^{n_{c_p}-r_{c_p}}}{\alpha^{n_k(c_p)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+r_{c_p}\right)}-1} \\ & \times\cdots\times \frac{(-1)^{r_k}}{\alpha^{-n_k(k)+2r_k(k)}-1}.\end{aligned}$$

Similarly, we can deduce

$$\begin{aligned}&\sigma_{k-p}(\hat{r}_1,\ldots,\hat{r}_{c_1-1},r_{c_1},\ldots,r_{c_p},\hat{r}_{c_{p+1}},\ldots,\hat{r}_k) \\ = & \frac{(-1)^{n_1-r_1}}{\alpha^{-n_k(1)+2\left(\sum_{t=1}^{c_1-1}(n_t-r_t)+\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p} r_{c_t}\right)}-1} \\ & \times\cdots\times \frac{(-1)^{r_{c_1}}}{\alpha^{-n_k(c_1)+2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p} r_{c_t}\right)}-1} \\ & \times\cdots\times \frac{(-1)^{r_{c_p}}}{\alpha^{-n_k(c_p)+2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+r_{c_p}\right)}-1}\times\cdots\times\frac{(-1)^{n_k-r_k}}{\alpha^{-n_k(k)+2(n_k(k)-r_k(k))}-1}.\end{aligned}$$

Since $\alpha\beta = 1$, then

$$\begin{aligned}&\sigma_{k-p}(\hat{r}_1,\ldots,\hat{r}_{c_1-1},r_{c_1},\ldots,r_{c_p},\hat{r}_{c_{p+1}},\ldots,\hat{r}_k) \\ = & (-1)^{n_k(1)}\frac{(-1)^{r_1}}{\beta^{n_k(1)-2\left(\sum_{t=1}^{c_1-1}(n_t-r_t)+\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p} r_{c_t}\right)}-1} \\ & \times\cdots\times \frac{(-1)^{n_{c_1}-r_{c_1}}}{\beta^{n_k(c_1)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p} r_{c_t}\right)}-1} \\ & \times\cdots\times \frac{(-1)^{n_{c_p}-r_{c_p}}}{\beta^{n_k(c_p)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+r_{c_p}\right)}-1}\times\cdots\times\frac{(-1)^{r_k}}{\beta^{-n_k(k)+2r_k(k)}-1}. \\ = & (-1)^{n_k(1)}\Psi\big(\sigma_p(r_1,\ldots,\hat{r}_{c_1},\ldots,\hat{r}_{c_p},\ldots,r_k)\big).\end{aligned}$$

This completes the proof of the proposition.

Theorem 4.1 *Let k be a positive integer and $\boldsymbol{n} = (n_1,\ldots n_k) \in \mathbb{N}^k$. Then $\zeta_{EZC}(-\boldsymbol{n}) \in \mathbb{Q}$ if even number of n_i's in $\boldsymbol{n}$ are odd and $\zeta_{EZC}(-\boldsymbol{n}) \in \sqrt{2}\mathbb{Q}$ if odd number of n_i's in $\boldsymbol{n}$ are odd except for singularities.*

Proof. Consider $\mathbf{n} = (n_1,\ldots n_k) \in \mathbb{N}^k$, for some positive integer k. The binomial coefficient

$$\binom{n_i}{r_i} = 0 \text{ for } r_i > n_i, 1 \le i \le k.$$

Then using the notations $n_k(d), r_k(d)$ in (2.6), we have

$$\zeta_{EZC}(-\mathbf{n}) = 2^{-n_k(1)} \sum_{r_1=0}^{n_1} (-1)^{r_1} \binom{n_1}{r_1} \times \cdots \times \sum_{r_k=0}^{n_k} (-1)^{r_k} \binom{n_k}{r_k}$$
$$\frac{1}{\left(\alpha^{-n_k(1)+2r_k(1)} - 1\right)} \times \cdots \times \frac{1}{\left(\alpha^{-n_k+2r_k} - 1\right)}. \tag{4.8}$$

It is clear that $\sum_{i=0}^{n} z_i = \sum_{i=0}^{n} z_{n-i}$ for any finite sequence of complex numbers $\{z_i\}$ and

$$\binom{n}{r} = \binom{n}{n-r}.$$

Therefore, we can write (4.8) as

$$\begin{aligned}\zeta_{EZC}(-\mathbf{n}) = 2^{-n_k(1)} &\sum_{r_1=0}^{n_1} \binom{n_1}{r_1} (-1)^{r_1} \times \cdots \times \sum_{r_{k-1}=0}^{n_{k-1}} \binom{n_{k-1}}{r_{k-1}} (-1)^{r_{k-1}} \\ &\times \frac{1}{2}\Bigg[\sum_{r_k=0}^{n_k} \binom{n_k}{r_k} (-1)^{r_k} \frac{1}{\left(\alpha^{-n_k(1)+2r_k(1)} - 1\right)} \times \cdots \times \frac{1}{\left(\alpha^{-n_k+2r_k} - 1\right)} \\ &+ \sum_{r_k=0}^{n_k} \binom{n_k}{n_k - r_k} \frac{(-1)^{n_k - r_k}}{\left(\alpha^{-n_k(1)+2r_{k-1}(1)+2(n_k-r_k)} - 1\right)} \times \cdots \times \frac{1}{\left(\alpha^{n_k-2r_k} - 1\right)}\Bigg].\end{aligned}$$

By continuing in this process for each index r_t, where $t = k-1, k-2, \ldots, 1$ and using the notations in (4.1), we have

$$\begin{aligned}\zeta_{EZC}(-\mathbf{n}) = \frac{2^{-n_k(1)}}{2^k} &\sum_{r_1=0}^{n_1} \binom{n_1}{r_1} \times \cdots \times \sum_{r_k=0}^{n_k} \binom{n_k}{r_k} \\ &\Bigg[\prod_{t=1}^{k} \Delta_t(r_t, \ldots, r_k; n_t) + \sum_{c=1}^{k} \left(\prod_{t=1}^{k} \Delta_t(r_t, \ldots, \hat{r}_c, \ldots, r_k; n_t)\right) \\ &+ \sum_{1 \le c \le d \le k} \left(\prod_{t=1}^{k} \Delta_t(r_t, \ldots, \hat{r}_c, \ldots, \hat{r}_d, \ldots, r_k; n_t\right) \\ &+ \cdots + \prod_{t=1}^{k} \Delta_t(\hat{r}_t, \ldots, \hat{r}_k; n_t)\Bigg)\Bigg].\end{aligned}$$

Using notations in (4.2), the above equation reduces

$$\zeta_{EZC}(-\mathbf{n}) = \frac{2^{-n_k(1)}}{2^k}\Bigg[\sigma_0(r_1, \ldots, r_k) + \sum_{p=1}^{k} \sum_{1 \le c_1 < c_2 < \cdots < c_p \le k}^{k} \sigma_p(r_1, \ldots, \hat{r}_{c_1}, , \ldots, \hat{r}_{c_p}, \ldots, r_k)\Bigg].$$

From Proposition 4.1, for any $0 \le p \le k$,

$$\begin{aligned}&\Psi\big(\sigma_p(r_1, \ldots, \hat{r}_{c_1}, \ldots, \hat{r}_{c_p}, \ldots, r_k)\big) \\ &\qquad = (-1)^{n_k(1)}\big[\sigma_{k-p}(\hat{r}_1, \ldots, \hat{r}_{c_1-1}, r_{c_1}, \ldots, r_{c_p}, \hat{r}_{c_p+1}, \ldots, \hat{r}_k)\big].\end{aligned}$$

It is clear that, if even number of n_i's are odd in the tuple $(n_1,\dots,n_k)$, then $(-1)^{n_k(1)}=1$ and hence

$$\Psi\big(\sigma_p(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{c_p},\dots,r_k)\big)=\sigma_{k-p}(\hat{r}_1,\dots,\hat{r}_{c_1-1},r_{c_1},\dots,r_{c_p},\hat{r}_{c_p+1},\dots,\hat{r}_k).$$

If odd number of n_i's are odd in the tuple $(n_1,\dots,n_k)$, then $(-1)^{n_k(1)}=-1$ and therefore

$$\Psi\big(\sigma_p(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{c_p},\dots,r_k)\big)=-\sigma_{k-p}(\hat{r}_1,\dots,\hat{r}_{c_1-1},r_{c_1},\dots,r_{c_p},\hat{r}_{c_p+1},\dots,\hat{r}_k).$$

As Ψ is the non-trivial automorphism of Gal$(\mathbb{Q}\sqrt{2}/\mathbb{Q})$, then

$$\sigma_p+\psi(\sigma_p)\in\mathbb{Q} \text{ and } \sigma_p-\psi(\sigma_p)\in\sqrt{2}\mathbb{Q}.$$

Let $\chi(r_1,\dots,r_k)=\sigma_0(r_1,\dots,r_k)+\sum_{p=1}^{k}\sum_{1\le c_1<c_2<\cdots<c_p\le k}\sigma_p(r_1,\dots,\hat{r}_{c_1},,\dots,\hat{r}_{c_p},\dots,r_k)$, then

$$\zeta_{EZC}(-\mathbf{n})=\frac{2^{-n_k(1)}}{2^k}\chi(r_1,\dots,r_k).$$

The following two cases arise.

Case-I : (Even number of n_i's are odd in the tuple $(n_1,\dots,n_k)$.)

In this case, $2^{-n_k(1)-k}$ is a rational number. Now

$$\begin{aligned}\Psi\big(\chi(r_1,\dots,r_k)\big)&=\Psi\big(\sigma_0(r_1,\dots,r_k)\big)\\&\quad+\Psi\Bigg(\sum_{p=1}^{k-1}\sum_{1\le c_1<c_2<\cdots<c_p\le k-1}\sigma_p(r_1,\dots,\hat{r}_{c_1},,\dots,\hat{r}_{c_p},\dots,r_k)\Bigg)\\&\quad+\Psi\big(\sigma_k(\hat{r}_1,\dots,\hat{r}_k)\big)\\&=\sigma_k(\hat{r}_1,\dots,\hat{r}_k)+\sum_{p=1}^{k-1}\sum_{1\le c_1<c_2<\cdots<c_p\le k-1}\sigma_p(r_1,\dots,\hat{r}_{c_1},,\dots,\hat{r}_{c_p},\dots,r_k)\\&\quad+\sigma_0(r_1,\dots,r_k)\\&=\chi(r_1,\dots,r_k),\end{aligned}$$

which implies that $\chi(r_1,\dots,r_k)\in\mathbb{Q}$. Therefore, $\zeta_{EZC}(-\mathbf{n})\in\mathbb{Q}$.

Case-II : (Odd number of n_i's are odd in the tuple $(n_1,\dots,n_k)$.)

In this case, $2^{-n_k(1)-k}$ is also a rational number. Now

$$\begin{aligned}\chi(r_1,\dots,r_k)&=\sigma_0(r_1,\dots,r_k)+\sigma_1(\hat{r}_1,\dots,r_k)+\cdots+\sigma_{k-1}(\hat{r}_1,\hat{r}_2,\dots,\hat{r}_{k-1},r_k)\\&\quad+\sigma_k(\hat{r}_1,\dots,\hat{r}_k)\\&=\sigma_0(r_1,\dots,r_k)-\Psi\big(\sigma_0(r_1,\dots,r_k)\big)\\&\quad+\sigma_1(\hat{r}_1,\dots,r_k)-\Psi\big(\sigma_1(\hat{r}_1,\dots,r_k)\big)+\cdots\\&\quad+\sigma_{k-1}(\hat{r}_1,\hat{r}_2,\dots,r_k)-\Psi\big(\sigma_{k-1}(\hat{r}_1,\hat{r}_2,\dots,r_k)\big)\in\sqrt{2}\mathbb{Q}.\end{aligned}$$

Therefore, in this case $\zeta_{EZC}(-\mathbf{n})\in\sqrt{2}\mathbb{Q}$. This completes the proof of the theorem.

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