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Some apriori estimates for the solutions of a degenerate nonlinear elliptic equations

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Abstract. *We establish weighted Hölder estimate for the solutions of a degenerate nonlinear elliptic equations of non-divergence type.*

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1 Introduction

Let in some ball $B_{2r} \subset \mathbb{R}^n$ with radius $2r, r \geq 1$, and center is 0 a solution $u(x)$ in $C(\overline{B}_{2r}) \cap W^{2,n}_{loc}(B_{2r})$ of nonlinear elliptic equation of non-divergence type

$$\sum_{i,j=1}^{n} a_{ij}(x, u(x), Du(x))D^2u(x) + f(x, u, Du(x)) = 0 \tag{1.1}$$

for a.e. $x \in B_{2r}$ is considered. Here $a_{ij} = a_{ji}$, i.d. $A(x, u, \xi)$ set of symmetric matrices of size $n \times n$ and $\forall y \in \mathbb{R}$, $\forall x, \xi \in \mathbb{R}^n$ coefficients satisfying

$$\begin{cases} \Lambda^{-1}\lambda(\xi)\omega(x)|\xi|^2 \leq (\xi, A(x, u, \xi)\xi) \leq \Lambda\lambda(\xi)\omega(x)|\xi|^2 \\ f(x, u, \xi) \leq \frac{1}{k}\Lambda(1 + \lambda(\xi))(1 + |\xi|) \end{cases} \tag{1.2}$$

for some $\Lambda \geq 1, k > 1$ and some continuous mapping $\lambda : \mathbb{R}^n \to \mathbb{R}_+$ for which there exist λ_0 and $M > 0$ such that $\lambda(\xi) \geq \lambda_0$ for $|\xi| \geq M$, $\omega(x)$ is Muckenhoupt weight function (see [1]). Let $u : \overline{B}_{2r} \to \mathbb{R}$ be a bounded and continuous solution of (1.1).

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Firstly results of Holder estimates for solution of divergence form equation De Giorgi and Nash (see[2],[3]) was obtained. In paper [4],[5] for non-divergence equations this type result by Krylov and Safanov was obtained. Serrin [6] and Ladyzhenskaya, Uraltseva [7], De Giorgi and Nash Hölder's estimates was been shown for quasilinear elliptic equations of divergence type.

The goal of this paper is to prove a similar result for degenerate quasilinear elliptic equations of non-divergence form.

Our results new and expands in a non-divergence form as $|Du|^{p-2}\omega(x)\sum\limits_{i,j=1}^{n}[I_n +(p-2)\left(|Du|^{-2}Du\right)D^2u]$, here I_n is the identity matrix of size n.

2 Main result

The main result is

Theorem 2.1 *Let conditions* (1.2) *be satisfies for the solution* (1.1)*. Then* $u(x)$ *is a weighted Hölder functions on* $\overline{B}_{2r}$*. Moreover, there exists constants* β, C*, only depending on* n, Λ, λ_0 *and* M*, such that*

$$|\omega(x)u(x)-\omega(y)u(y)|\le C|x-y|^{\beta}(1+\sup_{\bar{B}_{2r}}(|u|)),$$

for any $x, y\in \bar{B}_{2r}$*.*

Theorem 2.2 *Let* $\sigma:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ *be a Lipschitz continuous mapping such that* $\forall x,\xi\in\mathbb{R}^n$

$$\Lambda^{-1}\bar{\lambda}(x)\omega(x)|\xi|^2\le(\xi,a(x)\xi)\le\Lambda\bar{\lambda}(x)\omega(x)|\xi|^2, \tag{2.1}$$

where $a(x)=\sigma\sigma^*(x)$*, for some* $\Lambda\ge 1$ *and some mapping* $\bar{\lambda}:\mathbb{R}^n\to[0,1]$*. Let* $(\Omega, F, (F_t)_{t\ge 0}, P)$ *also denote a filtered probability space satisfying the usual conditions endowed with* $a_n(F_t)_{t\ge0}$ *Brownian motion* $(W_t)_{t\ge 0}$*,* α *be a positive real and* Q_r *be some hypercube of* $\mathbb{R}^n$ *of radius* r*. Then for any* μ *in* $(0,1)$ *there exists some positive constants* $\varepsilon(\mu)$*,* $R(\mu)$ *and* $(\Gamma_p(\mu))_{1\le p<2}$ *only depending on* d, α, Λ *and* μ*, such that for any* $\rho\in(0,1)$ *and* $x_0\in Q_{\rho/8}$*, we can find an integrable* n*-dimensional* $(F_t)_{t\ge0}$ *progressively measurable process* $(b_t)_{t\ge0}$ *such that both* $(b_t)_{t\ge0}$ *and the process* X*, solution to the SDE*

$$X_t=x_0+\int\limits_0^t b_s ds+\int\limits_0^t\sigma(X_s)dB_s, t\ge 0$$

fulfill $\forall t\ge 0,\ \bar{\lambda}(X_t)\ge\alpha\Rightarrow b_t=0,$

$$\forall p\in[1,2), E\int\limits_0^{+\infty}|b_t|^p\le\Gamma_p(\mu)\rho^{p-2},$$

and for any Borel subset $V\subset Q_\rho$

$$|Q_\rho\backslash V|\le\mu|Q_\rho|\Rightarrow P\{T_V<(R(\mu)\rho^2)\cap S_{Q_\rho}\}\ge\varepsilon(\mu),$$

T_V *being the first hitting time of* V *and* S_{Q_ρ} *the first exit time from* Q_ρ *by* X*. (*$|.|$*-Lebesgue measure).*

For proof our theorem we use some auxillary results.

Now we define for any bounded and uniformly continuous function υ in $\mathbb{R}^n$. Let $\varepsilon > 0$ and $x \in \mathbb{R}^n$

$$\upsilon^{\varepsilon}(x) = \sup_{y\in\mathbb{R}^n}\left[\upsilon(y) - \frac{1}{2\varepsilon}|x-y|^2\right],$$

$$\upsilon_{\varepsilon}(x) = \inf_{y\in\mathbb{R}^n}\left[\upsilon(y) + \frac{1}{2\varepsilon}|x-y|^2\right].$$

We doing remark, that for any positive δ and ε, $(\upsilon^{\varepsilon+\delta})_{\delta}$ belong to classes of functions, which is continuously differentiable on $\mathbb{R}^n$ with Lipschitz continuous derivatives and converges ω as δ, ε tend to zero. Hear υ is satisfies equation (1.1), then $(\upsilon^{\varepsilon+\delta})_{\delta}$ is subsolution of equation (1.1).

Lemma 2.1 *Let A and f coefficients satisfies* (1.2) *to some $\Lambda \geq 1$, $\lambda : \mathbb{R}^n \to [0,1]$, $\lambda_0 \in [0,1]$, $M > 0$ and weight function $\omega \in A_p$. Let $u : \bar{B}_{2r} \to \mathbb{R}$ be also a continuous solution of equation* (1.1). *Setting $\upsilon = (\bar{u}^{\varepsilon+\delta})_{\delta}$ for $\delta > 0$, $\varepsilon > 0$ and for some arbitrarily chosen bounded, uniformly continuous extension $\bar{u}$ of u to the whole $\mathbb{R}^n$. Then there exists $\theta \in (0,1)$ such that, for $\delta = \theta\varepsilon$ and for ε small enough, υ satisfies*

$$\sum_{i,j=1}^{n} a_{ij}^{\varepsilon}(x, \upsilon(x), D\omega(x)D^2\upsilon(x) + f(x + \varepsilon D\upsilon(x), \upsilon, D\upsilon(x))$$

$$\leq 2(\omega(x)\Lambda + M), \quad \text{a.e.} \quad x \in B_{\frac{3}{2}r},$$

where $B_{\frac{3}{2}r}$ is the ball of same center as B_{2r}, and a_{ij}^{ε} -elements of matrix $A_{\varepsilon} : \mathbb{R}^n \times \mathbb{R}^n \to S_n(R)$ is a smoth function satisfying $\inf\{(\xi, A_{\varepsilon}(x,u,\xi)\xi) : x, \xi \in \mathbb{R}^n, |\xi| = 1\} > 0$ as well as (1.2) *with respect to Λ and to some mapping $\lambda_{\varepsilon} : \mathbb{R}^n \to (0,2]$ such that $\lambda_{\varepsilon}(\xi) \geq \lambda_0$ for $|\xi| \geq M + 1$.*

Proof. By [9] there exists some constant $K \geq 0$ depends on u but is independent of δ and ε, such that $|D\upsilon(x)| \leq K \cdot \varepsilon^{-1/2}$, $x \in B_{\frac{3}{2}r}$ and for ε small enough $x + \varepsilon D\upsilon(x) \in B_{2r}$. Now choose $\hat{x} \in B_{\frac{3}{2}r}$ at which $D^2\upsilon(\hat{x})$ exists. We assume $\upsilon(\hat{x}) = \bar{u}^{\varepsilon}(\hat{x})$. Since $\upsilon \geq \bar{u}^{\varepsilon}$ on the $\mathbb{R}^n$, $D\upsilon(\hat{x})D^2\upsilon(\hat{x}) \in D^{2,+}\bar{u}^{\varepsilon}(\hat{x})$. By [6], we have

$$D\upsilon(\hat{x})D^2\upsilon(\hat{x}) \in D^{2,+}u(\hat{x} + \varepsilon D\upsilon(\hat{x}), \upsilon(\hat{x}), D\upsilon(\hat{x})),$$

so that by (1.1)

$$\sum_{i,j=1}^{n} a_{ij}^{\varepsilon}(\hat{x} + \varepsilon D\upsilon(\hat{x}), \upsilon(\hat{x}), D\upsilon(\hat{x}))D^2\upsilon(\hat{x})$$

$$+ f(\hat{x} + \varepsilon D\upsilon(\hat{x}), \upsilon(\hat{x}), D\upsilon(\hat{x})) = 0 \tag{2.2}$$

Let now $\upsilon(\hat{x}) > \bar{u}^{\varepsilon}(\hat{x})$. By [6], $\frac{1}{\delta}$ is an eigenvalue of $D^2\upsilon(\hat{x})$ and the other eigenvalues are greater than or equal to $-\frac{1}{\varepsilon}$. In particular, for any $y \in B_{\frac{3}{2}r}$

$$\sum_{i,j=1}^{n} a_{ij}(y, \upsilon(\hat{x}), D\upsilon(\hat{x}))D^2\upsilon(\hat{x}) + f(y, \upsilon(\hat{x}), D\upsilon(\hat{x}))$$

$$\leq \lambda(D\upsilon(\hat{x}))(-\Lambda^{-1}\upsilon^{-1}(x)\delta^{-1} + (d-1)\upsilon(x)\Lambda\varepsilon^{-1}) + \omega(x)\Lambda(1 + |D\upsilon(\hat{x})|).$$

Now choose $\delta = \frac{\varepsilon}{n \cdot \Lambda^2}$ for any $y \in B_{\frac{3}{2}r}$, so that

$$\sum_{i,j=1}^{n} a_{ij}(y, \upsilon(\hat{x}), D\upsilon(\hat{x}))D^2\upsilon(\hat{x}) + f(y, \upsilon(\hat{x}), D\upsilon(\hat{x}))$$

$$\leq \Lambda\omega(x)(D\upsilon(\hat{x}))(\lambda(D\upsilon(\hat{x})\varepsilon^{-1}) + 1 + |D\upsilon(\hat{x})|). \tag{2.3}$$

As so $\lambda(D\upsilon(\hat{x}) \geq \lambda_0$ for $|D\upsilon(\hat{x})| \geq M$, so that the above right-hand side is less than $\omega\Lambda(-\lambda_0\varepsilon^{-1} + 1 + K\varepsilon^{-\frac{1}{2}})$. Otherwise, it is less than $\omega(x)\Lambda(1 + M)$. Choosing ε small enough, from (2.5),(2.6) that in any case

$$\sum_{i,j=1}^{n} a_{ij}(\hat{x} + \varepsilon D\upsilon(\hat{x}), \upsilon(\hat{x}), D\upsilon(\hat{x}))D^2\upsilon(\hat{x}) + f(\hat{x} + \varepsilon D\upsilon(\hat{x}), \upsilon(\hat{x}), D\upsilon(\hat{x})) \leq$$

$$\leq \omega(x)\Lambda(1 + M).$$

Now we using smooth coefficient. The norm of $D^2\upsilon(\hat{x})$ is less than some constant $c(\varepsilon) \geq 1$. Then can find a smooth matrical function $A_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to S_n(R)$ such that

$$\sup_{|x| \leq 2, |\xi| \leq K\varepsilon^{-\frac{1}{2}}} |A_\varepsilon(x, u, \xi) - A(x, u, \xi)| \leq \frac{\varepsilon}{C(\varepsilon)}.$$

It is clear that A_ε satisfies conditions (1.2) with Λ and to some mapping $\lambda_\varepsilon : \mathbb{R}^n \to [0, 1]$. For ε small enough $\lambda_\varepsilon(\xi) \geq \lambda_0$ if $|\xi| > M + 1$. After some calculation completes the proof.

Lemma 2.2 *Let satisfying assuptions of Lemma 2.1. Then there exist constants* $\gamma \in (0, 1)$ *and* $C_1 \leq 0$, $r \in (0, 1)$ *for* $\Omega_{r/4} \subset \Omega_r \subset B_{\frac{3r}{2}}$

$$\underset{\Omega_{r/4}}{osc}\ u \leq \gamma \underset{\Omega_r}{osc}\ u + c_1 r(1 + \sup_{\Omega_r} |u|), \tag{2.4}$$

where $\Omega_r = \Omega \cap B_r$, $\underset{\Omega_r}{osc}\ u = \sup_{\Omega_r} u - \inf_{\Omega_r} u$.

Lemma 2.2 is proved by [8] and after some calculations.

Proof of Theorem 2.2. The proof relies on a probabilistic interpretation of the quasilinear equations. For linear equation in case A and f are independent of y and p, the orginal proof consists in introducing a diffusion process X, solution to the stochastic differential equation (SDE)

$$dX_t = \sigma(X_t)dW_t,\ t \geq 0$$

where W is a n-dimensional Wiener process and σ a continuous version of the square root of the matrical mapping $2A$. The basic idea that the generator of in the a diffusion process having some smoothing property in the surrounding space with a nontrivial probability. Let f vanishes and u is smooth, $(u(X_t))_{t \geq 0}$ is matringale. In this case $u(x)$ may be expressed as the expectation $E[u(X_\tau^x)]$ for any well-controlled stopping time τ and the exponent x indicated the initial position of the diffusion process. As a consequence, $u(x)$ may be understood as a mean over the values of u in a neighborhood of x, since X visits the surrounding space around x almost all the values of only u in the neighboorhood of x. In linear case the probabilistic representation formula has the form

$$u(x) = E\left[u(X_\tau^x) + \int_0^\tau f(X_s^x)ds\right]. \tag{2.5}$$

The probability that the diffusion process X hits a Borel subset of non-zero Lebesgue measure included in B_{2r}.

In our non-linear case, when $A(x,y,p)$ and $u(x)$ are smooth, we can define X similarly by setting

$$dX_t = \sigma(X_t, u(X_t), Du(X_t))dW_t, t \geq 0 \tag{2.6}$$

$(x,y,p) \to \sigma(x,y,p)$ being a smooth version of the square root of $2A(x,y,p)$.

Let $m_- = \inf\limits_{\Omega_r} u$, $m_+ = \sup\limits_{\Omega_r} u$. Assume that $|\{x \in \Omega_r : u(x) \leq \frac{(m_+ + m_-)}{2}\}| \geq \frac{1}{2}|\Omega_r|$. Let consider W from Lemma 2.1, and λ_ε is $(0,\frac{1}{2}]$values. Then, we can use Theorem 2.2 to $2A_\varepsilon(x,\upsilon,D\upsilon(x)) = A_\varepsilon(x)$. Condition Theorem 2.2 easily checked with $\hat{\lambda}(x) = 2\lambda_\varepsilon(D\omega(x))$. Parameters α and μ are respectively chosen equal to λ_0 and $\frac{1}{2}$. The resulting processes are denoted by $(b_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$. Now consider $V = \{x \in \Omega_r : u(x) \leq \frac{(m_+ + m_-)}{2}\}$. By Theorem2.2 and also define τ as the sopping time $T \bigwedge (R_0 p^2) \cap S_{\Omega_r}$, where constant $R(\frac{1}{2})$ denote by R_0. We apply Ito;s formula to $(W(X_t))_{t\geq 0}$. W is not in $C^2(\mathbb{R}^n)$ but in $C^{1,1}(\mathbb{R}^n)$. Since the matrix of X is uniformly elliptic, we apply the Ito-Krylov formula that holds for functions with Sobolev derivatives. There is then another problem: it requires the drift $(b_t)_{t\geq 0}$ to be bounded. We thus define, for any $n \leq 1$, the Ito process

$$X_t^n = x_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b_s 1_{\{|b_s| \leq n\}} ds.$$

Since $(b_t)_{t\geq 0}$ belongs to $L^1(\Omega \times R_+)$, it is clear that $E(\sup\limits_{t\geq 0} |X_t^0 - X_t|)$ tends to 0 with n. Expanding $(W(X_t))_{t\geq 0}$ and taking expectation (τ is bounded);

$$\upsilon(x_0) = E(\upsilon(X_t^n)$$

$$-E\int_0^\tau \left[\frac{1}{2}(A_\varepsilon(X_s)D^2\upsilon(X_s^n)) + (b_s, D\upsilon(X_s^n)) \cdot 1_{\{|b_s|\leq n\}}\right] ds.$$

Since υ is a subsolution, then by Lemma 2.1

$$-\frac{1}{2}(A_\varepsilon(X_s)D^2\upsilon(X_s^n)) \leq -f(X_s^n + \varepsilon D\upsilon(X_s^n, D\upsilon(X_s^n) + 2(\Lambda\omega(x) + M).$$

From we get the bound of $D^2\upsilon$ as so

$$\upsilon(x_0) \leq E(\upsilon(X_\tau^n)$$

$$+E\int_0^\tau \left[3\omega\Lambda + 2M + \omega(x)\Lambda|D\upsilon(X_s^n)| - (b_s, D\upsilon(X_s^n)) \cdot 1_{\{|b_s|\leq n\}}\right.$$

$$\left. + K_\varepsilon|X_s^n - X_s|\right] ds, \tag{2.7}$$

where K_ε is a constant. It is then plain to let n tend to $+\infty$ in (2.7). We also estimate for the integral of $D\upsilon(X_s^n)$. After some calculations we is proved of Theorem 2.2.

Proof Theorem 2.1. The connection with Theorem 2.1 may be understood as follows: when $u(x)$ is a strong solution of the (1.1), we choose $a(x)$ in the statement of Theorem2.2 as $2A(x,u(x),Du(x))$. The term $\lambda(Du(x))$ in Theorem2.1 then plays the role of $\bar{\lambda}(x)$ in

Theorem2.2. By choosing α in the Theorem 2.2 equal to λ_0 given in Theorem2.1, we deduce that $|Du(x_t)| \geq M \Rightarrow \lambda(Du(x_t)) \geq \lambda_0 \Rightarrow \bar{\lambda}(X_t) \geq \alpha \Rightarrow b_t = 0$. In other words the resulting drift $(b_t)_{t\geq 0}$ just acts when the gradient is small, i.e. is bounded by M.

Later we can show how to deduce Theorem 2.1 from Theorem 2.2 when the proportion of V inside Q_ρ is large enough. Compared with the original argument given by Krylov and Safonov, the main difference in the application of the probabilistic estimate follows from the interpretation of the underlying PDE. In the work by Krylov and Safanov the PDE is understood in the strong sence, i.e. the solution $u(x)$ is assumed to be in $C(\bar{B}_{2r}) \cap W^{2,n}_{loc}(B_{2r})$. To complete the proof of Theorem 2.1 we also used to Gilbarg and Trudinger [[8], Lem.8.23] and Lemma 2.2.

Theorem 2.1 is proved.

3 Exceptional points of solutions

Let now the noise is too small. Towards which part of Borel subset V do we have to push the process? We in forcing the process X to go to the neighborhood of some exceptional point x in V.

Theorem 3.1 *There exists constants $q_0 > 0$ and $K_0 \leq 0$, only depending on d, such that, for any Borel subset $V \in \Omega_1(0)$ satisfying $|\Omega_1(0)\ V| \leq q_0$, there exists $x \in \Omega_{\frac{1}{8}}(0) \cap V$ such that, for any $\rho \in (0, \frac{3}{4})$,*

$$|\Omega_\rho(0)\ V| \leq |\Omega_1(0)\ V|^{\frac{1}{2}} \rho^n.$$

Proof. Lemma says that if the proportion of V inside $\Omega_1(0)$ is large enough, then we can find some point x close to zero such that the proportion of V inside any neighborhood of x is also large. This result is close to the Lebesgue theorem: for a.e. point $x \in V$, we know that $|\Omega_\rho(x)|^{-1} \cdot |\Omega_\rho(x) \cap V|$ tends to one as ρ tends to zero, so that the proportion of V inside any small neighborhood of x is large. Lemma2.2 is in fact a bit stronger: the lower bound for the proportion of V inside a given neighborhood of x does not depend on the radius of the neighborhood. Before we admit are small sets where as they are large sets in the statment of Theorem3.1: There exist universal constants $q_1 \in (0,1)$, $K_1 \leq 0$ and $\delta \in (0, \frac{1}{2})$, only depending and such that for any Borel subset $U \subset [0,1]^n$ satisfying $|U| \leq q_1$, there exists $y \in [\delta, 1-\delta]^n \cap U^c$ such that, for any $\rho > 0$, $|U \cap \Omega_\rho(y)| \leq K_1 |U|^{\frac{1}{2}} \rho^n$.

We apply this statement to $U = \{z \in [0,1]^n, \frac{1}{8}z \in V^c\}$, i.e. $U = 8([0, \frac{1}{8}]^n \cap V^c)$. Then $|U| \leq 8^n |\Omega_1(0)\ V|$. Therefore for $|\Omega_1(0)\ V| \leq 8^{-n} q_1$,

$$\exists y \in [\delta, 1-\delta]^n \cap V^c : \forall \rho > 0, |U \cap \Omega_\rho(y)| \leq K_1 |U|^{\frac{1}{2}} \rho^n \tag{3.1}$$

For $\rho < \frac{\delta}{8}$ and $|\Omega_1(0)\ V| \leq 8^{-n} q_1$ from (3.1) is obtain

$$|\Omega_\rho(x)\ V| \leq 8^{\frac{n}{2}} K_1 |\Omega_1(0)\ V|^{\frac{1}{2}} \rho^n. \tag{3.2}$$

For $\frac{\delta}{8} \leq \rho < \frac{3}{4}$, $\Omega_\rho(x) \subset \Omega_1(0)$ so that

$$|\Omega_\rho(x)\ V| \leq \left(\frac{8}{\delta}\right)^n |\Omega_1(0)\ V|^{\frac{1}{2}} \rho^n. \tag{3.3}$$

By (3.2)-(3.3), we take $q_0 = 8^{-n} q_1$, $K_0 = \max(8^{\frac{n}{2}} K_1, (\frac{8}{\delta})^n)$. Now let p be an integer greater than 3 and E be square $E = [\frac{1}{p}, \frac{1}{p} + \frac{1}{2}]^n \subset (0,1)^n$. For any integer $d \geq 1$, we denote by $\mathfrak{B}_d$ the collection of $\Omega_d(l_1, \ldots, l_n)$ included in E of the form $\mathfrak{B}_d(l_1, \ldots, l_n) =$

$\prod_{i=1}^{n}\left[\frac{1}{p}+\frac{l_i}{2^n},\frac{1}{p}+\frac{(l_i+1)}{2^d}\right]$, $0 \le l_i < 2^{d-1}$,$1 \le i \le n$, and we put $M_n(x) = \sum_{A\subset B_d}[((|U \cap A||A|)1_A(x)]$ for any $x \in \mathbb{R}^n$. ($M_d(x)$ is the proportion of U inside the Ω_d containing x). Then $|F| > \frac{1}{2^n} - |U|^{\frac{1}{2}}$, where $F = \{x \in E : \sup_{d\ge1} M_d(x) \le |U|^{\frac{1}{2}}\}$.

We endow E with the Borel σ-algebra $\mathfrak{B}(E)$ and with the probability measure $\mu = 2^n|\cdot|$. For any $d \ge 1$ we also denote by $\mathfrak{B}_d(E)$ the σ-subalgebra of $\mathfrak{B}(E)$ generated by the collection Ω_d. It is well seen that the sequence $(\mathfrak{B}_d)_{d\ge1}$ is a filtration and that on E, the sequence $(M_d)_{d\ge1}$ coincides with the martingale $(\varepsilon 1_{U\cap E}B_d)_{d\ge1}$ where ε stands for the expectation associated with μ. By Doob's maximal inequality, for every $\varepsilon > 0$ $\mu x \in E : \sup_{d\ge1} M_d(x) > \varepsilon \le \varepsilon^{-1}\mu(u \cap E)$. Choosing $\varepsilon = |U|^{\frac{1}{2}}$, we deduce that $|F| \ge |E| - |U|^{\frac{1}{2}}|U \cap E|$. This is complete the proof.

References

1. Chanillo, S., Wheeden, R.: *Weighted Poincare and Sobolev inequalites*, Nonlinear Anal. **107**, 1151–1226 (1986).
2. De Giorgi, E.: *Sulla differenziabilita e l'analiticita della estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **3**, 25–43 (1957).
3. Nash, J.: *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80**, 931–954 (1958).
4. Krylov, N., Safanov, M.: *An estimate for the probability of a diffusion process hitting a set of positive measure*, Dokl. Akad. Nauk SSSR. **245**, 18–20 (1979).
5. Krylov, N., Safonov, M.: *A property of the solutions of parabolic equations with measurable coefficients*, Izv. Ross. Akad. Nauk Ser. Mat. **44**, 161–175 (1980).
6. Serrin, J.: *Local behavior of solutions of quasilinear equations*. Acta Math. **111**, 247–302 (1964).
7. Ladyzhenskaya, O., Uraltseva, N., Solonnikov, V.: Linear and quasilinear equations of parabolic type. *Amer. Math. Society, Providence, R.I.* (1967).
8. Gilbarg, P.: Trudinger N. Elliptic partial differential equations of second order. *Springer-Verlag, Berlin* (1983).
9. Lasry, J., Lions, P.: *A remark on regularization in Hilbert spaces*, Israel J. Math. **55**, 257–266 (1986).