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On the basicity of a perturbed system of exponents with a unit in Morrey-type spaces

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Abstract. *In this paper a perturbed system of exponents with a unit, whose phase is a linear function depending on a real parameter is considered. The necessary and sufficient conditions for the parameter for the basis properties (completeness, minimality and basicity) of this system are found in the subspace of Morrey space in which continuous functions are dense.*

Keywords. Exponential system · Basicity · Morrey space

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1 Introduction

Consider the following exponential system with a unit

$$1\bigcup\left\{e^{i(n-\beta\, sign\, n)t}\right\}_{n\neq0}\equiv1\bigcup E^0_\beta, \tag{1.1}$$

where

$$E_\beta\equiv\left\{e^{i(n-\beta\, sign\, n)t}\right\}_{n\in Z},E_{\beta0}=E_\beta\backslash\left\{1\right\}, \tag{1.2}$$

$\beta\in C-$is a complex parameter. The study of basis properties of E_β (such as completeness, minimality and basicity) has a long history. It dates back to the works by Paley-Wiener [1] and N. Levinson [2,3]. Basicity (Riesz basicity) criterion for the system E_β in $L_2(-\pi,\pi)$ with respect to the real parameter $\beta\in R$ follows from the results obtained by N. Levinson [2,3] and M.I. Kadets [4], and this criterion is the inequality $|\beta|<\frac{1}{4}$. Basicity criterion for the system E_β in the Lebesgue spaces $L_p(-\pi,\pi)$, $1<p<+\infty$, with respect to the parameter β has been obtained later by A.M. Sedletski [5,6] and E.I.Moiseev [7]. These results were transferred to the complex case of the parameter $\beta\in C$ by G. G. Devdariani

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[8]. Basis properties of E_β are closely related to the similar properties of perturbed sine systems

$$\{\sin(n-\beta)t\}_{n\in N}, \tag{1.3}$$

and cosine systems

$$1\bigcup\{\cos(n-\beta)t\}_{n\in N}, \tag{1.4}$$

$$\{\cos(n-\beta)t\}_{n\in Z_+}, \left(Z_+=\{0\}\bigcup N\right), \tag{1.5}$$

in corresponding Banach spaces of functions on $[0,\pi]$. These systems arise when solving partial differential equations of mixed (or elliptic) type by Fourier method in special domains. To justify the formally constructed solution, it is very important to study the basis properties of these systems in appropriate spaces of functions (see, e.g., [9–13]). Many authors have studied the basis properties of systems in various functional spaces (mainly Lebesgue spaces and their weighted versions; see, e.g., [14–33]). The works which consider the approximation properties of the systems (1.2)-(1.5) can be divided into two groups. The first one includes the works which used the methods of the theory of entire functions (see, e.g., [1–6,33]) and the second group consists of those which used the methods of boundary value problems for analytic functions (see, e.g., [9,10,14–28,31,32]). The latter idea originated from A.V. Bitsadze [34], later to be successfully used in [9,10,7,14]. Further development of this approach, used in establishing basis properties of perturbed trigonometric systems and power systems, is due to B.T. Bilalov [16,17,21–24,35,36].

Note that, in the context of applications to some problems of mechanics and mathematical physics, since recently there arose great interest in the non-standart spaces of functions. As examples to this kind of spaces, we can mention Lebesgue space with the variable summability index, Morrey space, Companato space, etc. The theory of differential equations and its relationship with the harmonic analysis requires the study of many cornerstone issues of analysis in these spaces. A lot of classical facts about harmonic analysis have been extended to these spaces (for most detailed information about these matters see O. Kovacik, J. Rakosnik [37], F. Xianling, Z. Dun [38], C.T. Zorko [39], C.B. Morrey [40] and etc. Along with this, of course you have to study approximation matters in suchlike spaces. Approximation matters have been (and are being) relatively well studied in generalized Lebesgue spaces by I.I. Sharapudinov [29], D. Israfilov [43,44], B.T. Bilalov and Z.G. Huseynov [22,23] and etc. (see e.g. [28,55]). The situation is different with the case of Morrey-type spaces. Only recently the approximation matters began to be studied in these spaces, and many problems in this field still remain to be solved. Apparently the works by D. Israfilov [43,44], B.T. Bilalov and A.A. Guliyeva [45,53], T.B. Gasymov and A.A. Guliyeva have been pioneers in this field.Some aspects of the approximation theory in Morrey spaces are studied in the papers [53,56].

In this paper a perturbed system of exponents with a unit, whose phase is a linear function depending on a real parameter is considered. The necessary and sufficient conditions for the parameter for the basis properties (completeness, minimality and basicity) of this system are found in the subspace of Morrey space in which continuous functions are dense. It should be noted that basis properties of the system E_β in the Morrey-type spaces is studied by B.T.Bilalov [54]. Earlier, the basis properties of system (1.1) in classical Lebesgue spaces were studied in [7].

2 Necessary Information

In this section we state some notations and facts which will be used to obtain our main results. First let us give some standard notation. $N-$ is the set of natural numbers; $Z-$is the set of all integers; $Z_+=\{0\}\bigcup N$. Let's define the Morrey space on the unit circle

$\gamma = \{z \in C : |z| = 1\}$ on the complex plane C. Next, $\omega = int\gamma$ will denote the unit ball in C. By $L_0(-\pi, \pi)$ we denote the linear space of all (Lebesgue-) measurable functions on $(-\pi, \pi)$.

$L^{p,\alpha}(\gamma)$, $1 \le p < +\infty$, $0 \le \alpha \le 1$, will denote the normed space of all measurable functions $f(\cdot)$ on γ with the finite norm

$$\|f\|_{L^{p,\alpha}(\gamma)} = \sup_B \left(\left| B \bigcap \gamma \right|_\gamma^{\alpha-1} \int_{B \bigcap \gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty,$$

($|B \bigcap \gamma|_\gamma$ – is the linear measure of intersection $B \bigcap \gamma$), where $\sup$ is taken over all balls centered at γ with an arbitrary positive radius. $L^{p,\alpha}(\gamma)$is a Banach space with respect to this norm. We also define the space $L^{p,\alpha}(-\pi, \pi)$, $1 \le p < +\infty$, $0 \le \alpha \le 1$, which consists of measurable functions $f(\cdot)$ on $(-\pi, \pi)$ with the finite norm

$$\|f\|_{L^{p,\alpha}(-\pi,\pi)} = \sup_{I \subset [-\pi,\pi]} \left(|I|^{\alpha-1} \int_I |f(t)|^p \, |dt| \right)^{1/p} < +\infty,$$

where $\sup$ is taken over all intervals $I \subset [-\pi, \pi]$. It is not difficult to see that the correspondence $f(t) =: F(e^{it})$, $t \in (-\pi, \pi)$, $F(\cdot) \in L^{p,\alpha}(\gamma)$, establishes an isometric isomorphism between the spaces $L^{p,\alpha}(\gamma)$ and $L^{p,\alpha}(-\pi, \pi)$. Therefore, in what follows we will equate these spaces and denote $L^{p,\alpha}$ with the norm $\|\cdot\|_{p,\alpha}$.

It is not difficult to see that for $0 \le \alpha_1 \le \alpha_2 \le 1$ the following continuous embedding holds: $L^{p,\alpha_1} \subset L^{p,\alpha_2}$. Moreover, it is clear that $L^{p,1} = L_1(-\pi, \pi)$ and $L^{p,0} = L_\infty(-\pi, \pi)$. We also have $L^{p,\alpha} \subset L_1(-\pi, \pi)$, $\forall \alpha \in [0, 1]$, $\forall p \ge 1$.

The lemma below was proved in [54].

Lemma 2.1 *The space L_∞ (and so $C[-\pi, \pi]$ too) is not dense in $L^{p,\alpha}$ for $1 \le p < +\infty$ and $\forall \alpha \in (0, 1)$.*

It follows that the sequence of bounded functions cannot be complete in $L^{p,\alpha}$. In what follows, we will assume, if needed, that the function $f \in L^{p,\alpha}$ is periodically (with period 2π) extended to the whole real axis R. Following Lemma 2.1, we will consider the subspace $M^{p,\alpha}$ of functions $f(\cdot)$ the shifts of which are continuous in $L^{p,\alpha}$, i.e. $\|f(\cdot + \delta) - f(\cdot)\|_{p,\alpha} \to 0$, $\delta \to 0$.

The following lemma holds.

Lemma 2.2 *The space $M^{p,\alpha}$, $1 \le p < +\infty$, $0 < \alpha \le 1$, is a Banach space and $C_0^\infty[-\pi, \pi]$ is dense in it.*

It is not difficult to see that the system E_β belongs to $M^{p,\alpha}$: $E_\beta \subset M^{p,\alpha}$. Therefore it is clear that the closure of the linear span of E_β also belongs to $M^{p,\alpha}$, i.e. $\overline{spanE_\beta} \subset M^{p,\alpha}$. So it is quite natural to study basis properties of the system E_β in the space $M^{p,\alpha}$.

We will use in this work the following result obtained in [46].

Lemma 2.3 *Let $f(\cdot) \in L_\infty(-\pi, \pi)$ and $g(\cdot) \in M^{p,\alpha}$, $1 \le p < +\infty$, $0 < \alpha \le 1$. Then $f(\cdot)\, g(\cdot) \in M^{p,\alpha}$.*

Finally, we state the following easy-to-prove lemma which will be frequently used throughout this work.

Lemma 2.4 *Let* $\{\tau_k\}_{k=\overline{1,m}} \subset \gamma-$ *be different points. Then the finite product*

$$\omega(\tau) = \prod_{k=1}^{m} |\tau - \tau_k|^{\alpha_k}, \quad \tau \in \gamma,$$

belongs to the space $L^{p,\alpha}$, $1 \le p < +\infty$, $0 < \alpha < 1$, *if and only if the inequalities* $\alpha_k \ge -\frac{\alpha}{p}$, $\forall k = \overline{1,m}$, *hold.*

When treating basis properties of systems, one often has to use a conjugate space. As we don't yet have a description for a Morrey-Lebesgue space conjugate to $L^{p,\alpha}$, in the form of functional space (see, e.g., [42]), it suffices to consider some subspace of $(L^{p,\alpha})^*$, denoted $(L^{p,\alpha})'$ and defined by

$$(L^{p,\alpha})' = \left\{ g \in L_0(-\pi,\pi) : \sup_{f \in S_{p,\alpha}} \|fg\|_{L_1(-\pi,\pi)} < +\infty \right\},$$

with the norm

$$\|g\|_{(p,\alpha)'} = \sup_{f \in S_{p,\alpha}} \|fg\|_{L_1}, \tag{2.1}$$

where $S_{\rho;\alpha} = \left\{ f \in L^{p,\alpha} : \ \|f\|_{p,\alpha} = 1 \right\}$ – is a unit sphere in $L^{p,\alpha}$.

The following theorem is true.

Theorem 2.1 *[54] The space* $(L^{p,\alpha})'$, $1 \le \rho < +\infty$, $0 \le \alpha \le 1$, *is a Banach space with respect to the norm* (2.1).

We will also use the following lemma

Lemma 2.5 *Let* $t_0 \in [-\pi,\pi]-$ *be some point. Then the function* $g(t) = |t - t_0|^{\beta}$ *belongs to the space* $(L^{p,\alpha})'$, $1 \le \rho < +\infty$, $0 < \alpha \le 1$, *if and only if* $\beta \in \left(\frac{\alpha}{p} - 1, \infty\right)$.

Remark 2.1 Let $J \subset [-\pi;\pi]$ be an arbitrary interval and $t_0 \in J$. It is not difficult to see that the proof of Lemma 2.5 is also applicable if J is taken instead of $[-\pi;\pi]$, i.e. $|t - t_0|^{\beta} \in (L^{p,\alpha}(J))'$ holds if and only if $\beta \in \left(-1 + \frac{\alpha}{p}, +\infty\right)$.

Using the results of Lemma 2.5, it is easy to prove the validity of the following lemma.

Lemma 2.6 *The finite product*

$$\nu(t) = \prod_{k=1}^{m} |t - t_k|^{\beta_k}, \{t_k\}_{k=\overline{1,m}} \subset [-\pi,\pi], \quad t_i \ne t_j, \quad i \ne j,$$

belongs to the space $(L^{p,\alpha})'$, $1 \le p < +\infty$, $0 < \alpha \le 1$, *if and only if* $\beta_k \in \left(-1 + \frac{\alpha}{p}, +\infty\right)$, $\forall k = \overline{1,m}$.

This lemma has the following immediate corollary.

Corollary 2.1 *Let* $-\pi = s_0 < s_1 < \ldots < s_r < \pi-$ *be different points. Then the finite product*

$$\mu(t) = \prod_{k=0}^{r} \left| \sin \frac{t - s_k}{2} \right|^{\alpha_k}, \ t \in (-\pi,\pi),$$

belongs to the space $(L^{p,\alpha})'$, $1 \le p < +\infty, 0 \le \alpha \le 1$, *if and only if* $\alpha_k \in \left(-1 + \frac{\alpha}{p}, +\infty\right)$, $\forall k = \overline{0,r}$.

In obtaining the main results, we will need some facts concerning the basis properties of the following system of exponents

$$E_\beta = \left\{ e^{i(n-\beta signn)t} \right\}_{n\in Z},$$

in Morrey-type space $M^{p,\alpha}, 0 < \alpha < 1, 1 < p < +\infty$. First we define Morrey-Hardy spaces. Define the Morrey-Hardy class $H_+^{p,\alpha}$, $0 \le \alpha \le 1$, $1 \le p < +\infty$, of functions $f(\cdot)$ analytic inside ω endowed with the norm

$$\|f\|_{H_+^{p,\alpha}} = \sup_{0<r<1} \|f_r(\cdot)\|_{p,\alpha},$$

where $f_r(t) = f\left(re^{it}\right)$. It is not difficult to see that the inclusion $H_+^{p,\alpha} \subset H_1^+$, $1 \le p < +\infty$, where $H_1^+ -$ is a usual Hardy class. Therefore, every function $f(\cdot) \in H_+^{p,\alpha}$ has non-tangential boundary values $f^+(\cdot)$ on γ.

The following theorem is true.

Theorem 2.2 *Let $f(\cdot) \in H_+^{p,\alpha}$, $0 < \alpha \le 1, 1 < p < +\infty$. Then $f^+(\cdot) \in L^{p,\alpha}$ $f^+(\cdot) \in L^{p,\alpha}$ and the Cauchy formula*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f^+(\tau)\, d\tau}{\tau - z}, \quad z \in \omega, \tag{2.2}$$

holds, where $f^+(\cdot)$ −are non-tangential boundary values of $f(\cdot)$ on γ. Conversely, if $f^+(\cdot) \in L^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \le 1$, then the function $f(\cdot)$, defined by the Cauchy-type integral (2.2)*, belongs to the class $H_+^{p,\alpha}$.*

Consider the space $H_+^{p,\alpha}$. Denote by $L_+^{p,\alpha}$ the subspace of $L^{p,\alpha}$ generated by the restrictions of the functions from $H_+^{p,\alpha}$to γ, i.e. $L_+^{p,\alpha} = H_+^{p,\alpha}/_\gamma$. From the uniqueness theorem for analytic functions and Theorem 2.2 it follows that the spaces $H_+^{p,\alpha}$ and $L_+^{p,\alpha}$ are isomorphic and the restriction operator $J{:}H_+^{p,\alpha} \leftrightarrow L_+^{p,\alpha}$; $(Jf)(\tau) = f^+(\tau)$, $\tau \in \gamma$, $\forall f \in H_+^{p,\alpha}$, performs the corresponding isomorphism. Let $M_+^{p,\alpha} = M^{p,\alpha} \bigcap L_+^{p,\alpha}$ with the norm $\|\cdot\|_{p,\alpha}$. It is clear that $M_+^{p,\alpha}$ is a subspace of $M^{p,\alpha}$ (because $M^{p,\alpha}$ and $L_+^{p,\alpha}$ both are the closed subspaces of $L^{p,\alpha}$). Let $MH_+^{p,\alpha} = J^{-1}\left(M_+^{p,\alpha}\right)$. Obviously, $MH_+^{p,\alpha}$ is a subspace of $H_+^{p,\alpha}$. It follows from the above considerations that for $\forall f \in H_+^{p,\alpha}$ the norm $\|f\|_{H_+^{p,\alpha}}$ can be defined also by the relation $\|f\|_{H_+^{p,\alpha}} = \|f^+\|_{p,\alpha}$, where $f^+ -$ are non-tangential boundary values of f on γ.

Absolutely similar to the classical case, we define the Morrey-Hardy class outside the unit circle ω. Let $\omega^- = C\backslash\bar{\omega}\,(\bar{\omega} = \omega \bigcup \gamma)$. We will say that the function f analytic in ω^- has a finite order m at infinity, if its Laurent decomposition at infinitely remote point has the following form:

$$f(z) = \sum_{k=-\infty}^{m} a_k z^k, \quad a_m \ne 0. \tag{2.3}$$

Thus, for $m > 0$ the function f has a pole of order m at $z = \infty$; for $m = 0$, it is bounded in the vicinity of $z = \infty$; and in case $m < 0$ it has a zero of order $(-m)$ at $z = \infty$. Let $f(z) = f_0(z) + f_1(z)$, where $f_0(z)$ is the principal part (i.e. $f_0(z) = \sum_{k=0}^{m} a_k z^k$), and $f_1(z)$ is the regular part of decomposition (3.2). Consequently, $f_0(z) \equiv 0$, for $m < 0$, and f_0 is a polynomial of degree m, i.e. $\deg f_0 = m$, if $m \ge 0$. We will say that the function f belongs to the class ${}_mH_-^{p,\alpha}$, if $\deg f_0 \le m$ and $F \in H_+^{p,\alpha}$, where $F(z) = \overline{f_1\left(\frac{1}{\bar{z}}\right)}z \in \omega$.

Absolutely similar to the case of $MH_{+}^{p,\alpha}$, we define the class ${}_mMH_{-}^{p,\alpha}$. In other words, ${}_mMH_{-}^{p,\alpha}$is a subspace of functions from ${}_mH_{-}^{p,\alpha}$, whose shifts are continuous on γ with respect to the norm $\|\cdot\|_{p,\alpha}$.

Regarding the basicity of parts of the exponential system in these subspaces, the following theorem is true.

Theorem 2.3 *The system* $\left\{e^{int}\right\}_{n\in Z_+}\left(\left\{e^{-int}\right\}_{n\in N}\right)$ *forms a basis for* $M_{+}^{p,\alpha}$ *(for* ${}_{-1}M_{-}^{p,\alpha}$*),* $0<\alpha<1, 1<p<+\infty$.

Let $(z+1)_{-}^{-2\beta}$ be a branch of an analytic function $(z+1)^{-2\beta}$ on a complex plane cut along a part $(-\infty,-1]$ of the negative semi-axis. Let us introduce the following systems

$$h_n^{+}(t)=\frac{e^{i\beta t}}{2\pi}\left(e^{it}+1\right)_{-}^{-2\beta}\sum_{k=0}^{n}C_{2\beta}^{n-k}e^{-ikt},\ n\in Z_+;$$

$$h_n^{-}(t)=-\frac{e^{i\beta t}}{2\pi}\left(e^{it}+1\right)_{-}^{-2\beta}\sum_{k=1}^{n}C_{2\beta}^{n-k}e^{ikt},\ n\in N,$$

where $C_{2\beta}^{n}=\frac{2\beta(2\beta-1)\cdot\ldots\cdot(2\beta-n+1)}{n!}-$ are binomial coefficients. The following lemma is true.

Lemma 2.7 *Let inequality* $|Re\beta|<\frac{1}{2}$ *holds. Then the relations*

$$<x_k^{+},h_n^{+}>=<x_{k+1}^{-},h_{n+1}^{-}>=\delta_{nk},$$

$$<x_k^{+},h_{n+1}^{-}>=<x_{k+1}^{-},h_n^{+}>=0,\ \forall n,k\in Z_+,$$

are true, where the notation

$$<x,y>=\int_{-\pi}^{\pi}x(t)\,\overline{y(t)}dt, x_n^{\pm}=e^{\pm i(n-\beta)t},$$

are accepted.

Let's apply Lemma 2.5 to the system $\{h_n^{\pm}\}$. From the representations of the system $\{h_n^{\pm}\}$ and from this lemma it immediately follows that if the inequality $2Re\beta<-\frac{\alpha}{p}+1$, holds, then the system $\{h_n^{\pm}\}$ belongs to the space $\left(L^{p,\alpha}\right)'$ and so $\{h_n^{\pm}\}\subset\left(L^{p,\alpha}\right)^{*}$. Then from Lemma 2.7 it follows that the system $E_\beta=\left\{x_n^{+};x_{n+1}^{-}\right\}_{n\in Z+}$ is minimal in $L^{p,\alpha}$. Thus, the following lemma is also true.

Lemma 2.8 *Let the inequality* $-1<2Re\beta<-\frac{\alpha}{p}+1, 0<\alpha<1, 1<p<+\infty$, *holds. Then the system* E_β *is minimal in* $L^{p,\alpha}$.

So, with respect to the basis properties of the system E_β the following theorem is true.

Theorem 2.4 *Let* $2Re\beta+\frac{\alpha}{p}\notin Z$*. Then the system* E_β *forms a basis for* $M^{p,\alpha}$, $0<\alpha<1, 1<p<+\infty$, *if and only if* $\left[2Re\beta+\frac{\alpha}{p}\right]=0$ *(*$[\cdot]-$ *is the integer part). Its defect is equal to* $d\left(E_\beta\right)=\left[2Re\beta+\frac{\alpha}{p}\right]$*. For* $d\left(E_\beta\right)<0$*, it is not complete, but is minimal in* $M^{p,\alpha}$*; for* $d\left(E_\beta\right)>0$*, it is complete, but it is not minimal in* $M^{p,\alpha}$.

3 Main results

Suppose that the inequality

$$0 < 2Re\beta + \frac{\alpha}{p} < 1, \ \ 1 < p < +\infty, \ 0 < \alpha < 1,$$

holds. By the results of [54], in this case the system E_β forms a basis for $M^{p,\alpha}$ and $\left\{h_n^+, h_{n+1}^-\right\}_{n\in Z_+}$ is biorthogonal to this system. Let $|\beta| < \frac{1}{2}$. We have

$$c_0^+ = \int_{-\pi}^{\pi} h_0^+ (t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dt}{(1+e^{it})^{2\beta}(e^{-it})^{\beta}} =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dt}{\left(e^{i\frac{t}{2}}+e^{-i\frac{t}{2}}\right)^{2\beta}} = \frac{1}{2^{2\beta+1}\pi}\int_{-\pi}^{\pi} \frac{dt}{\left(\cos\frac{t}{2}\right)^{2\beta}} \neq 0.$$

Let us consider a new system $\left\{H_n^+; H_{n+1}^-\right\}_{n\in N}$:

$$H_0^+ = \frac{1}{c_0^+} h_0^+ ; \ \ H_n^\pm = h_n^\pm - \frac{c_n^\pm}{c_0^+} h_0^+,$$

where

$$c_n^\pm = \int_{-\pi}^{\pi} h_n^\pm dt \ , \ \ \forall n \in N.$$

We have

$$\int_{-\pi}^{\pi} e^{i(n-\beta)t}\overline{H_k^+ (t)}dt = \int_{-\pi}^{\pi} e^{i(n-\beta)t}\overline{h_k^+ (t)}dt - \frac{\overline{c_k^+}}{\overline{c_0^+}}\int_{-\pi}^{\pi} e^{i(n-\beta)t}\overline{h_0^+ (t)}dt = \delta_{nk}\, \forall n, k \in N;$$

$$\int_{-\pi}^{\pi} 1 \cdot \overline{H_k^\pm (t)}dt =$$

$$\int_{-\pi}^{\pi} \overline{h_k^\pm (t)}dt - \frac{\overline{c_k^+}}{\overline{c_0^+}}\int_{-\pi}^{\pi} \overline{h_0^+ (t)}dt = \overline{c_k^\pm} - \frac{\overline{c_k^\pm}}{\overline{c_0^+}}\overline{c_0^+} = 0, \forall k \in N;$$

$$\int_{-\pi}^{\pi} e^{-i(n-\beta)t}\overline{H_k^- (t)}dt =$$

$$\int_{-\pi}^{\pi} e^{-i(n-\beta)t}\overline{h_k^- (t)}dt - \frac{\overline{c_k^-}}{\overline{c_0^+}}\int_{-\pi}^{\pi} e^{-i(n-\beta)t}\overline{h_0^+ (t)}dt = \delta_{nk} \ , \ \ \forall n, k \in N;$$

$$\int_{-\pi}^{\pi} e^{i(n-\beta)t}\overline{H_k^- (t)}dt =$$

$$\int_{-\pi}^{\pi} e^{i(n-\beta)t}\overline{h_k^- (t)}dt - \frac{\overline{c_k^-}}{\overline{c_0^+}}\int_{-\pi}^{\pi} e^{i(n-\beta)t}\overline{h_0^+ (t)}dt = \delta_{nk} \ , \ \ \forall n, k \in N;$$

$$\int_{-\pi}^{\pi} e^{-i(n-\beta)t}\overline{H_k^+ (t)}dt$$

$$= \int_{-\pi}^{\pi} e^{-i(n-\beta)t}\overline{h_k^+ (t)}dt - \frac{\overline{c_k^+}}{\overline{c_0^+}}\int_{-\pi}^{\pi} e^{-i(n-\beta)t}\overline{h_0^+ (t)}dt = 0 \ , \ \ \forall n, k \in N.$$

On the other hand
$$\int_{-\pi}^{\pi} 1\cdot\overline{H_0^+(t)}dt=\frac{1}{c_0^+}\int_{-\pi}^{\pi}\overline{h_0^+(t)}dt=1.$$
By the result of Lemma 2.8, the systems $\left\{h_n^+;h_{n+1}^-\right\}_{n\in N}$ and E_β satisfy the biorthogonality relations. Consequently, systems $\left\{H_n^+;H_{n+1}^-\right\}_{n\in Z_+}$ and (1.1) are biorthonormal. Thus, from the expressions of the biorthogonal system, it follows that for $2Re\beta+\frac{\alpha}{p}<1$ this system belongs to the space $\left(M^{p,\alpha}\right)'$, and as a result, the system (1.1) is minimal in $M^{p,\alpha}$. Take $\forall f\in M^{p,\alpha}$ and consider the partial sum
$$S_m\left[f\right]=f_0^+ +\sum_{n=1}^{m}\left[f_n^+e^{i(n-\beta)t}+f_n^-e^{-i(n-\beta)t}\right],$$
where $\{f_n^\pm\}-$ are the biorthogonal coefficients of the function f:
$$f_n^\pm=\int_{-\pi}^{\pi}f\left(t\right)\overline{H_n^\pm(t)}dt,\ \ \forall n. \tag{3.1}$$
Taking into account the expression for the biorthogonal system $\left\{H_n^+;H_{n+1}^-\right\}_{n\in Z_+}$ in (3.1), we obtain
$$f_n^\pm=\int_{-\pi}^{\pi}f\left(t\right)\overline{h_n^\pm(t)}dt-\frac{\overline{c_n^\pm}}{c_0^+}\int_{-\pi}^{\pi}f\left(t\right)\overline{h_0^+(t)}dt=d_n^\pm-\frac{\overline{c_n^\pm}}{c_0^+}d_0^+,$$
where $\left\{d_n^+;d_{n+1}^-\right\}_{n\in Z_+}$ −are biorthogonal coefficients of the function f by system E_β and
$$f_0^+=\int_{-\pi}^{\pi}f\left(t\right)\overline{H_0^+}\left(t\right)dt=\frac{1}{c_0^+}d_0^+.$$

Substituting these values into (3.1) we have
$$S_m\left[f\right]=\tfrac{1}{c_0^+}d_0^+ +\textstyle\sum_{n=1}^m\left[\left(d_n^+-\frac{\overline{c_n^+}}{c_0^+}d_0^+\right)e^{i(n-\beta)t}+\ \left(d_n^--\frac{\overline{c_n^-}}{c_0^+}d_0^+\right)e^{-i(n-\beta)\,t}\right]=$$
$$=\textstyle\sum_{n=1}^m\left[d_n^+e^{i(n-\beta)t}+d_n^-e^{-i(n-\beta)t}\right]+\frac{d_0^+}{c_0^+}\left[1-\sum_{n=1}^m\left(\overline{c_n^+}e^{i(n-\beta)t}\right)+\overline{c_n^-}e^{-i(n-\beta)t}\right],$$
where
$$\overline{c_n^\pm}=\overline{\int_{-\pi}^{\pi}h_n^\pm\left(t\right)dt}=\int_{-\pi}1\cdot\overline{h_n^\pm(t)}dt\,,\ \forall n\neq 0.$$
Adding and subtracting $\overline{c_0^+}e^{-i\beta t}$ to the second term $S_m\left[f\right]$, we get
$$S_m\left[f\right]=\overline{c_0^+}e^{i\beta t}+\textstyle\sum_{n=1}^m\left[d_n^+e^{i(n-\beta)t}+d_n^-e^{-i(n-\beta)t}\right]+\frac{d_0^+}{c_0^+}\overline{c_0^+}e^{i\beta t}+$$
$$+\tfrac{d_0^+}{c_0^+}\left[1-\textstyle\sum_{n=0}^m\overline{c_n^+}e^{i(n-\beta)t}-\sum_{n=1}^m\overline{c_n^-}e^{-i(n-\beta)t}\right]=\sum_{n=0}^m d_n^+e^{i(n-\beta)t}+$$
$$\textstyle\sum_{n=1}^m d_n^-e^{-i(n-\beta)t}+\frac{d_0^+}{c_0^+}\left[1-\sum_{n=0}^m\overline{c_n^+}e^{i(n-\beta)t}-\sum_{n=1}^m\overline{c_n^-}e^{-i(n-\beta)t}\right]=$$
$$=\ S_m^0\left[f\right]-\tfrac{d_0^+}{c_0^+}\left(S_m^0\left[1\right]-1\right),$$
$$S_m^0\left[f\right]=\sum_{n=0}^m d_n^+e^{i(n-\beta)t}+\sum_{n=1}^m d_n^-e^{-i(n-\beta)t},$$

$$S_m^0[1] = \sum_{n=0}^{m} \overline{c_n^+} e^{i(n-\beta)t} + \sum_{n=1}^{m} \overline{c_n^-} e^{-i(n-\beta)t}.$$

Thus, $S_m^0[f]$ is a partial sum of a biorthogonal series on system (1.1) of the function f, and $S_m^0[1]$ – of a unit function. As a result we have

$$\|S_m[f] - f\|_{p_t} \le \left\|S_m^0[f] - f\right\|_{p_t} + \left|\frac{d_0^+}{c_0^+}\right| \left\|S_m^0[1] - 1\right\|_{p_t}. \tag{3.2}$$

As already known, in this case the system E_β forms a basis for $M^{p,\alpha}$. Then from (3.2) we obtain that $S_m[f] \to f$ in $M^{p,\alpha}$ as $m \to \infty$.

This proves the basicity of system (1.1) in $M^{p,\alpha}$. The basis properties for the remaining values $\beta : 2Re\beta + \frac{\alpha}{p} \notin Z$ are established analogously to the case of exponential systems E_β.

Thus, we proved the following main

Theorem 3.1 *Let $2Re\beta + \frac{\alpha}{p} \notin Z$. Then the system of exponents*

$$1 \bigcup \left\{ e^{i(n-\beta\, sign\, n)t} \right\}_{n\neq 0},$$

forms a basis for $M^{p,\alpha}, 0 < \alpha < 1, 1 < p < +\infty$, if and only if $d(\beta) = \left[2Re\beta + \frac{\alpha}{p}\right] = 0$. For $d(\beta) < 0$ it is not complete, but it is minimal in $M^{p,\alpha}$; for $d(\beta) > 0$ it is complete, but not minimal in $M^{p,\alpha}$.

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