Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **39** (1), 162-169 (2019).

Parabolic nonsingular integral in Orlicz spaces

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Received: 20.12.2018 / Revised: 22.03.2019 / Accepted: 29.04.2019

Abstract. *We show continuity in Orlicz spaces of parabolic nonsingular integral operator.*

Keywords. Orlicz space; parabolic nonsingular integral.

Mathematics Subject Classification (2010): 42B20, 42B35

1 Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators etc, from one Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Lebesgue spaces, Orlicz spaces also play an important role.

Orlicz space was first introduced by Orlicz in [8,9] as a generalizations of Lebesgue spaces L^p. Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L^1 space when L^1 space does not work.

Throughout this paper the following notations will be used:

$x = (x', t), y = (y', \tau) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$;
$x = (x'', x_n, t) \in \mathbb{D}^{n+1}_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$, $\mathbb{D}^{n+1}_- = \mathbb{R}^{n-1} \times \mathbb{R}_- \times \mathbb{R}_+$;
$|\cdot|$ is the Euclidean metric, $|x| = \left(\sum_{i=1}^n x_i^2 + t^2\right)^{1/2}$;
$\mathcal{E}_r(x) = \{y \in \mathbb{R}^{n+1} : \frac{(x_1-y_1)^2}{r^2} + \ldots + \frac{(x_n-y_n)^2}{r^2} + \frac{(t-\tau)^2}{r^4} < 1\}$;
$\mathcal{E}^c_r(x) = \mathbb{R}^{n+1} \setminus \mathcal{E}_r(x)$;
$2\mathcal{E}_r(x)$ is an ellipsoid centered at the same point as $\mathcal{E}_r(x)$ of radius $2r$;
$\mathbb{S}^n = \{y \in \mathbb{R}^{n+1} : |x - y| = 1\}$ is a unit sphere in $\mathbb{R}^{n+1}$ centered in $x \in \mathbb{R}^{n+1}$;

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The research of M. Omarova was partially supported by the Grant of 1st Azerbaijan-Russia Joint Grant Competition (Agreement Number No. EIF-BGM-4-RFTF-1/2017-21/01/1).

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for any $f \in L_p(A)$, $A \subset \mathbb{R}^{n+1}$ write

$$\|f\|_{p,A} \equiv \|f\|_{L_p(A)} = \left(\int_A |f(y)|^p dy\right)^{1/p}.$$

Let $x = (x', t) = (x'', x_n, t) \in \mathbb{D}_+^{n+1}$. Now we define the parabolic nonsingular integral (see [2]) by

$$\mathcal{R}f(x) = \int_{\mathbb{D}_+^{n+1}} \frac{|f(y)|}{\rho(\widetilde{x} - y)^{n+2}}\, dy, \quad \widetilde{x} = (x'', -x_n, t). \tag{1.1}$$

In this work we present the characterization for parabolic nonsingular integral operator $\mathcal{R}$ (Theorem 3.2) in Orlicz spaces.

The standard summation convention on repeated upper and lower indexes is adopted. The letter C is used for various positive constants and may change from one occurrence to another.

2 Some preliminaries on Orlicz spaces

Definition 2.1 *A function $\Phi : [0, +\infty] \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to+\infty} \Phi(r) = \Phi(\infty) = \infty$.*

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, +\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \geq s$.

We say that $\Phi \in \Delta_2$, if for any $a > 1$, there exists a constant $C_a > 0$ such that $\Phi(at) \leq C_a \Phi(t)$ for all $t > 0$.

The following two indices

$$q_\Phi = \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)}, \qquad p_\Phi = \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)},$$

of Φ, where $\varphi(t)$ is the right-continuous derivative of Φ, are well known in the theory of Orlicz spaces. As is well known,

$$p_\Phi < \infty \quad \Longleftrightarrow \quad \Phi \in \Delta_2,$$

and the function Φ is strictly convex if and only if $q_\Phi > 1$. If $0 < q_\Phi \leq p_\Phi < \infty$, then $\frac{\Phi(t)}{t^{q_\Phi}}$ is increasing and $\frac{\Phi(t)}{t^{p_\Phi}}$ is decreasing on $(0, \infty)$.

Lemma 2.1 *([5], Lemma 1.3.2) Let $\Phi \in \Delta_2$. Then there exist $p > 1$ and $b > 1$ such that*

$$\frac{\Phi(t_2)}{t_2^p} \leq \frac{b\Phi(t_1)}{t_1^p}$$

for $0 < t_1 < t_2$.

Lemma 2.2 *([11], Proposition 62.20) Let Φ be a Young function with canonical representation*

$$\Phi(t) = \int_0^t \varphi(s)ds, \quad t \geq 0.$$

(1) Assume that $\Phi \in \Delta_2$. More precisely $\Phi(2t) \leq A\Phi(t)$ for some $A \geq 2$. If $p > 1 + \log_2 A$, then

$$\int_t^\infty \frac{\varphi(s)}{s^p} ds \lesssim \frac{\Phi(t)}{t^p}, \quad t > 0.$$

(2) Assume that $\Phi \in \nabla_2$. Then

$$\int_0^t \frac{\varphi(s)}{s} ds \lesssim \frac{\Phi(t)}{t}, \;\; t > 0.$$

Recall that a function Φ is said to be quasiconvex if there exist a convex function ω and a constant $c > 0$ such that

$$\omega(t) \leq \Phi(t) \leq c\omega(ct), \; t \in [0, \infty).$$

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \qquad \text{for} \qquad 0 < r < +\infty \tag{2.1}$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, +\infty)$ and bijective from $[0, +\infty)$ to itself.

In the following, besides the standard parabolic metric $\varrho(x) = \max(|x'|, |t|^{1/2})$ we use the equivalent one $\rho(x) = \left(\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}\right)^{1/2}$ introduced by Fabes and Riviére in [4]. The induced by it topology consists of ellipsoids

$$\mathcal{E}_r(x) = \left\{ y \in \mathbb{R}^{n+1} : \; \frac{|x' - y'|^2}{r^2} + \frac{|t - \tau|^2}{r^4} < 1 \right\}, \; |\mathcal{E}_r| = Cr^{n+2}.$$

It is easy to see that $\mathcal{E}_1(x)$ and $\mathbb{S}^n$ are the unit ball and the unit sphere, respectively, with respect to the both metrics and $\rho(x)$. On the other hand, the equivalence between the both parabolic metrics $\varrho(x)$ and $\rho(x)$ follows by the inclusion: for each $\mathcal{E}_r$ there exist parabolic cylinders $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ with measure comparable with r^{n+2} such that $\underline{\mathcal{C}} \subset \mathcal{E}_r \subset \overline{\mathcal{C}}$. In what follows all estimate obtained over ellipsoids hold true also over parabolic cylinders and we shall use this property without explicit references.

Definition 2.2 *(Orlicz Space). For a Young function Φ, the set*

$$L_\Phi(\mathbb{R}^{n+1}) = \left\{ f \in L_1^{loc}(\mathbb{R}^{n+1}) : \int_{\mathbb{R}^{n+1}} \Phi(k|f(x)|)dx < +\infty \, \textit{for some } k > 0 \right\}$$

is called Orlicz space. The space $L_\Phi^{\mathrm{loc}}(\mathbb{R}^{n+1})$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_{\mathcal{E}} \in L_\Phi(\mathbb{R}^{n+1})$ for all ellipsoids $\mathcal{E} \subset \mathbb{R}^{n+1}$.

Note that, $L_\Phi(\mathbb{R}^{n+1})$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^{n+1}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \leq 1 \right\},$$

see, for example, [10], Section 3, Theorem 10, so that

$$\int_{\mathbb{R}^{n+1}} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_\Phi}}\Big) dx \leq 1.$$

For a measurable set $Q = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$, a measurable function f and $r > 0$, let

$$m(Q, \; f, \; r) = |\{x \in Q : |f(x)| > r\}|.$$

In the case $Q = \mathbb{R}^{n+1}$, we shortly denote it by $m(f, \; r)$.

Definition 2.3 *The weak Orlicz space*

$$WL_\Phi(\mathbb{R}^{n+1}) = \{f \in L^1_{\rm loc}(\mathbb{R}^{n+1}) : \|f\|_{WL_\Phi} < +\infty\}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf\Big\{\lambda > 0 \ : \ \sup_{r>0} \Phi(r)\, m\Big(\frac{f}{\lambda},\, r\Big) \ \le 1\Big\}.$$

For Young functions Φ and Ψ, we write $\Phi \approx \Psi$ if there exists a constant $C \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \qquad \text{for all } r \ge 0$$

If $\Phi \approx \Psi$, then $L_\Phi(\mathbb{R}^{n+1}) = L_\Psi(\mathbb{R}^{n+1})$ with equivalent norms. We note that, for Young functions Φ and Ψ, if there exist $C, R \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \qquad \text{for } r \in (0, R^{-1}) \cup (R, \infty),$$

then $\Phi \approx \Psi$.

For a Young function Φ and $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = +\infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \le r < +\infty.$$

A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \qquad r \ge 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & , \ r \in [0,\infty) \\ +\infty & , \ r = +\infty. \end{cases} \tag{2.2}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$.

Remark 2.1 Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. Also, if Φ is a Young function, then $\Phi \in \nabla_2$ if and only if Φ^γ be quasiconvex for some $\gamma \in (0,1)$ (see, for example, [5], p. 15).

It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0. \tag{2.3}$$

The following analogue of the Hölder inequality is known, see [12].

Theorem 2.1 *[12] For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\|fg\|_{L_1(\mathbb{R}^{n+1})} \le 2\|f\|_{L_\Phi}\|g\|_{L_{\widetilde{\Phi}}}.$$

Note that Young functions satisfy the property

$$\Phi(\alpha t) \leq \alpha \Phi(t) \tag{2.4}$$

for all $0 < \alpha < 1$ and $0 \leq t < \infty$, which is a consequence of the convexity: $\Phi(\alpha t) = \Phi(\alpha t + (1-\alpha)0) \leq \alpha\Phi(t) + (1-\alpha)\Phi(0) = \alpha\Phi(t)$.

The following lemma is valid.

Lemma 2.3 *[1, 7] Let Φ be a Young function and $\mathcal{E}$ a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_{\mathcal{E}}\|_{WL_\Phi(\mathbb{R}^{n+1})} = \|\chi_{\mathcal{E}}\|_{L_\Phi(\mathbb{R}^{n+1})} = \frac{1}{\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)}.$$

In the next sections where we prove our main estimates, we use the following lemma, which follows from Theorem 2.1 and Lemma 2.3.

Lemma 2.4 *For a Young function Φ and for all parabolic balls $\mathcal{E}$ in $\mathbb{R}^{n+1}_+$, the following inequality is valid*

$$\|f\|_{L_1(\mathcal{E})} \leq 2\,|\mathcal{E}|\,\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)\,\|f\|_{L_\Phi(\mathcal{E})}.$$

3 Parabolic nonsingular integral operators in the Orlicz space $L_\Phi(\mathbb{D}^{n+1}_+)$

The following theorem were proved in [2].

Theorem 3.1 *Let $1 \leq p < \infty$ and $f \in L_p(\mathbb{D}^{n+1}_+)$. Then there exists a constant C independent of f, such that*

$$\|\mathcal{R}f\|_{L_p(\mathbb{D}^{n+1}_+)} \leq C_p\|f\|_{L_p(\mathbb{D}^{n+1}_+)}, \;\; 1 < p < \infty$$

and

$$\|\mathcal{R}f\|_{WL_1(\mathbb{D}^{n+1}_+)} \leq C\|f\|_{L_1(\mathbb{D}^{n+1}_+)}.$$

Theorem 3.2 *Let Φ be a Young function and R be a nonsingular integral operator, defined by* (1.1)*. If $\Phi \in \Delta_2 \bigcap \nabla_2$, then the operator R is bounded on $L_\Phi(\mathbb{D}^{n+1}_+)$ and if $\Phi \in \Delta_2$, then the operator R is bounded from $L_\Phi(\mathbb{D}^{n+1}_+)$ to $WL_\Phi(\mathbb{D}^{n+1}_+)$.*

Proof. At first proved that for $\Phi \in \Delta_2$ the nonsingular integral operator $\mathcal{R}$ is bounded from $L_\Phi(\mathbb{D}^{n+1}_+)$ to $WL_\Phi(\mathbb{D}^{n+1}_+)$.

We take $f \in L_\Phi(\mathbb{D}^{n+1}_+)$ satisfying $\|f\|_{L_\Phi} = 1$. Fix $\lambda > 0$ and define $f_1 = \chi_{\{|f|>\lambda\}} \cdot f$ and $f_2 = \chi_{\{|f|\leq\lambda\}} \cdot f$. Then $f = f_1 + f_2$. We have

$$|\{|\mathcal{R}f| > \lambda\}| \leq |\{|\mathcal{R}f_1| > \lambda/2\}| + |\{|Rf_2| > \lambda/2\}|$$

and

$$\Phi(\lambda)|\{|\mathcal{R}f| > \lambda\}| \leq |\Phi(\lambda)\{|\mathcal{R}f_1| > \lambda/2\}| + \Phi(\lambda)|\{|\mathcal{R}f_2| > \lambda/2\}|.$$

We know that from the weak (1,1) boundedness and L_p, $p > 1$ boundedness of $\mathcal{R}$

$$\{|\mathcal{R}(\chi_{\{|f|>\lambda\}} \cdot f)| > \lambda\}| \lesssim \frac{1}{\lambda}\int_{\{|f|>\lambda\}} |f|$$

$$\{|\mathcal{R}f(\chi_{\{|f|\leq\lambda\}} \cdot f)| > \lambda\}| \lesssim \frac{1}{\lambda^p}\int_{\{|f|\leq\lambda\}} |f|^p.$$

Since $f_1 \in WL_1(\mathbb{D}_+^{n+1})$ and $\frac{\Phi(\lambda)}{\lambda}$ increasing we have

$$\begin{aligned}\Phi(\lambda)\big|\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}f_1(x)| > \frac{\lambda}{2}\}\big| &\lesssim \frac{\Phi(\lambda)}{\lambda}\int_{\mathbb{D}_+^{n+1}} |f_1(x)|dx \\ &= \frac{\Phi(\lambda)}{\lambda}\int_{\{x\in\mathbb{D}_+^{n+1}:|f(x)|>\lambda\}} |f(x)|dx \\ &\lesssim \int_{\mathbb{D}_+^{n+1}} |f(x)|\frac{\Phi(|f(x)|)}{|f(x)|}dx = \int_{\mathbb{D}_+^{n+1}} \Phi(|f(x)|)dx.\end{aligned}$$

By Lemma 2.1 and $f_2 \in L_p(\mathbb{D}_+^{n+1})$ we have

$$\begin{aligned}&\Phi(\lambda)\,\big|\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}f_2(x)| > \frac{\lambda}{2}\}\big| \lesssim \frac{\Phi(\lambda)}{\lambda^p}\int_{\mathbb{D}_+^{n+1}} |f_2(x)|^p dx \\ &= \frac{\Phi(\lambda)}{\lambda^p}\int_{\{x\in\mathbb{D}_+^{n+1}:|f(x)|\le\lambda\}} |f(x)|^p dx \\ &\lesssim \int_{\mathbb{D}_+^{n+1}} |f(x)|^p\frac{\Phi(|f(x)|)}{|f(x)|^p}dx = \int_{\mathbb{D}_+^{n+1}} \Phi(|f(x)|)dx.\end{aligned}$$

Thus we get

$$|\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}f(x)| > \lambda\}| \le \frac{C}{\Phi(\lambda)}\int_{\mathbb{D}_+^{n+1}} \Phi(|f(x)|)dx \le \frac{1}{\Phi\left(\frac{\lambda}{C\|f\|_{L_\Phi}}\right)}.$$

Since $\|\cdot\|_{L_\Phi}$ norm is homogeneous this inequality is true for every $f \in L_\Phi(\mathbb{D}_+^{n+1})$.

Now proved that for $\Phi \in \Delta_2 \bigcap \nabla_2$ the nonsingular integral operator $\mathcal{R}$ is bounded in $L_\Phi(\mathbb{D}_+^{n+1})$.

It is the same as before that we use the distribution functions.

$$\begin{aligned}\int_{\mathbb{D}_+^{n+1}} \Phi\left(\frac{\mathcal{R}f(x)}{\Lambda}\right)dx &= \frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{\lambda}{\Lambda}\right)|\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}f(x)| > \lambda\}|d\lambda \\ &= \frac{2}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)|\{x \in \mathbb{D}_+^{n+1} : |Rf(x)| > 2\lambda\}|d\lambda.\end{aligned}$$

What is different from the estimate for the maximal operator is the point that $\mathcal{R}$ is not $L_\infty(\mathbb{D}_+^{n+1})$ bounded. Let $p > 1$ be sufficiently large. Then

$$\begin{aligned}|\{x \in \mathbb{D}_+^{n+1} : \mathcal{R}f > 2\lambda\}| \le\, &|\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}(\chi_{\{|f|>\lambda\}} \cdot f)| > \lambda\}| \\ &+ |\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}(\chi_{\{|f|\le\lambda\}} \cdot f)(x)| > \lambda\}|.\end{aligned}$$

By the weak $(1,1)$ boundedness and L_p-boundedness of $\mathcal{R}$ (see Theorem 3.1) gives us

$$|\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}(\chi_{\{|f|>\lambda\}} \cdot f)(x)| > \lambda\}| \lesssim \frac{1}{\lambda}\int_{\{x\in\mathbb{D}_+^{n+1}:|f(x)|>\lambda\}} |f(x)|dx$$

and

$$|\{x \in \mathbb{D}_+^{n+1} : |\mathcal{R}(\chi_{\{|f|\le\lambda\}} \cdot f)(x)| > \lambda\}| \lesssim \frac{1}{\lambda^p}\int_{\{x\in\mathbb{D}_+^{n+1}:|f(x)|\le\lambda\}} |f(x)|^p dx.$$

The same calculation as we used for the maximal operator works for the first term to obtain

$$\frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\{|\mathcal{R}(\chi_{\{|f|>\lambda\}}\cdot f)|>\lambda\}|d\lambda \le \int_{\mathbb{D}_+^{n+1}} \Phi\left(\frac{c|f|}{\Lambda}\right). \tag{3.1}$$

As for the second term a similar computation still works but we use the fact that $\Phi \in \Delta_2$.

$$\begin{aligned}&\frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\{|\mathcal{R}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)|>\lambda\}|d\lambda\\ &\lesssim \frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\left(\int_{\{x\in\mathbb{D}_+^{n+1}:|f(x)|\le\lambda\}}|f(x)|^p dx\right)\frac{d\lambda}{\lambda^p}\\ &\lesssim \frac{1}{\Lambda}\int_{\mathbb{D}_+^{n+1}}|f(x)|^p\left(\int_{|f(x)|}^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\frac{d\lambda}{\lambda^p}\right)dx.\end{aligned}$$

Using Lemma 2.2 (1), we have

$$\begin{aligned}&\frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\{|\mathcal{R}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)|>\lambda\}|d\lambda\\ &\lesssim \int_{\mathbb{D}_+^{n+1}}\Phi\left(\frac{2|f(x)|}{\Lambda}\right)dx \le \int_{\mathbb{D}_+^{n+1}}\Phi\left(\frac{c|f(x)|}{\Lambda}\right)dx.\end{aligned} \tag{3.2}$$

Thus, putting together (3.1) and (3.2), we obtain

$$\int_{\mathbb{D}_+^{n+1}}\Phi\left(\frac{\mathcal{R}f(x)}{\Lambda}\right)dx \le \int_{\mathbb{D}_+^{n+1}}\Phi\left(\frac{c_0|f(x)|}{\Lambda}\right)dx.$$

Again we shall label the constant we want to distinguish from other less important constants. As before, if we set $\Lambda = c_2\|f\|_{L_\Phi}(\mathbb{D}_+^{n+1})$, then we obtain

$$\int_{\mathbb{D}_+^{n+1}}\Phi\left(\frac{Rf(x)}{\Lambda}\right)dx \le 1.$$

Hence the operator norm of $\mathcal{R}$ is less than c_2.

Acknowledgments

The author would like to express their gratitude to the referees for their valuable comments and suggestions.

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