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On bases of trigonometric systems in Hardy-Orlicz spaces and Riesz theorem

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Abstract. *In this article, problems of basicity of classical trigonometric systems in Orlicz spaces are considered. Hardy-Orlicz classes corresponding to a unit ball on the complex plane are defined. An analog of the classical Riesz theorem of the theory of analytic functions for Hardy-Orlicz classes is considered. The validity of Cauchy formula for functions from these classes is established.*

Keywords. Orlicz space, basicity, Hardy-Orlicz classes, Riesz and Smirnov theorems, Cauchy formula.

Mathematics Subject Classification (2010): 33B10, 46E30, 54D70

1 Introduction

In connection with applications in various fields of mathematics and mechanics (also mathematical physics) interest in non-standard spaces has recently greatly increased. These includes Orlicz space, Lebesgue space with variable summability exponent, Morrey space, grand-Lebesgue space and etc. Solution of differential equations in these spaces dictates the study of corresponding problems of harmonic analysis in the spaces under consideration. These problems have been studied to varying degrees in these spaces. Numerous articles, review papers, and monographs of various mathematicians are devoted to these directions (see, e.g. [1, 9, 10, 12–19, 21, 25–28]. Some information related to these problems we can refer from monographs of M.A. Krasnosel'skii, L.B. Rutitskii [14], M.M.Rao, Z.D. Ren [21], D.V.Cruz-Uribe, A. Fiorenza [10], D.R.Adams [1], V.Kokolashvili, A.Meshki, H. Rafeiro, S.Samko [26, 27], R.E.Castillo, H.Rafeiro [9], J.Musielak [28], W.M.Kozlowski [15] and etc.

Orlicz space was introduced by W.Orlicz and Z.Birnbaum in the beginning of 1930 s in connection with orthogonal decomposition. Their approach consisted of the consideration of a functional space that provides properties of increasing differ by power L_p-norm case.

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Orlicz spaces have wide applications in different fields of mathematics such as an approximation, stochastic analysis, nonlinear differential equations, Fourier analysis and etc. Numerous facts of classical analysis are transferred to these spaces. More information about these problems we can refer from monographies [14,15,19,21,28]. But in spite of this, many questions remain to be studied.

It should be noted that Hardy classes play a special role in the theory of harmonic analysis. With the advent of new functional spaces, the Hardy classes associated with them are defined and the natural question arises of the validity of classical facts in these classes. The problems of conjugation of the theory of analytic functions in these classes are studied (see, for example, [2–8, 11, 22–24]). As the authors know, these questions regarding the Hardy Orlicz spaces are left out of consideration. And for this, firstly we need to establish the validity of many classical facts in these classes. This article is devoted to precisely these problems. In this article, basicity problems of trigonometric systems in Orlicz spaces are considered. We give well-known and easy-to-obtain facts in this direction. We consider Hardy-Orlicz spaces, as well as Riesz and Smirnov theorems, the Cauchy formula for analytic functions associated with these spaces. The basicity of parts of the system of exponentials in Hardy-Orlicz spaces is proved.

2 Necessary information

Further, we will use following concepts related to Orlicz spaces. Firstly, let us take some standard notations. N is the set of natural numbers, $Z_+ = \{0\} \cup N$; $Z = \{-N\} \cup Z_+$; R is the set of real numbers; $\chi_M(\cdot)$ is the characteristic function of the set M; C is the complex plane; $\omega = \{z \in C : |z| < 1\}$ is a unit ball in C; $\gamma = \partial\omega$ is a unit circle. $\overline{M}$ is the closure of the set M according to the corresponding norm; $\bar{(\cdot)}$ is the complex conjugate.

Definition 2.1 *Continuous convex function $M(\cdot) : R \to R$ is called N function if it is even and satisfies the condition*

$$\lim_{u\to 0} \frac{M(u)}{u} = 0; \lim_{u\to\infty} \frac{M(u)}{u} = \infty.$$

Definition 2.2 *Let M be a N -function. Function*

$$M^*(\nu) = \max_{u\geq 0}[u(\nu) - M(u)],$$

is called N -function complement to $M(\cdot)$.

Function $M^*(\cdot)$ can be described as follows. Let $p(\cdot) : R_+ \to R_+ = [0; +\infty)$ be right continuous for $t \geq 0$, non-decreasing function that satisfies the condition $p(0) = 0, p(\infty) = \lim_{t\to 0} p(t) = \infty$.

Let us define

$$q(s) = \sup_{p(t)\leq s} t, \quad s \geq 0.$$

The function $q(\cdot)$ has the same properties as the function $p(\cdot)$: for $s > 0$ it is positive, for $s \geq 0$ it is right continuous, non-decreasing and satisfies the conditions

$$q(t) = 0, q(\infty) = \lim_{s\to\infty} q(s) = \infty.$$

N-functions

$$M(u) = \int_0^{|u|} p(t)dt; M^*(\nu) = \int_0^{|\nu|} q(s)ds,$$

are complement of each other.

Definition 2.3 *N -functions $M(\cdot)$ satisfies the Δ_2 -condition for large values of u, if $\exists k > 0 \wedge \exists u_0 \geq 0$:*

$$M(2u) \leq kM(u), \forall u \geq u_0.$$

Δ_2 -condition is equivalent to that, for $\forall l > 1, \exists k(l) > 0 \wedge \exists u_0 \geq 0$:

$$M(lu) \leq k(l) M(u), \forall u \geq u_0.$$

Now let us define Orlicz space. Let $M(\cdot)$ be some N-function, $G \subset R$ is a Lebesgue measurable set with finite measure.

Denote by $L_0(G)$ the set of all measurable functions on G. Accept

$$\rho_M(u) = \int_G M[u(x)]dx,$$

and let

$$L_M(G) = \{u \in L_0(M) : \rho_M(u) < +\infty\}.$$

$L_M(G)$-is called Orlicz class.

Let $M(\cdot)$ and $M^*(\cdot)$ be complement for each other N-functions. Assume

$$L_M^*(G) = \{u \in L_0(M) : |(u, \nu)| < +\infty, \forall \nu(\cdot) \in L_{M^*}(G)\}$$

here

$$(u, \nu) = \int u(x)\overline{\nu(x)}dx.$$

$L_M^*(G)$ is called Orlicz space. According to the norm $\|\cdot\|_M$:

$$\|u\|_M = \sup_{\rho_M^*(\nu)\leq 1} |(u; \nu)|,$$

$L_M^*(G)$ is a Banach space. It should be noted that in $L_M^*(G)$ we can define equivalent norm to $\|\cdot\|_M$:

$$\|u\|_{(M)} = \inf\left\{k > 0 : \rho_M\left(\frac{u}{k}\right) \leq 1\right\}.$$

$\|\cdot\|_{(M)}$ is called the Luxemburg norm The following fact is well known.

Statement 2.1. *If N-function $M(\cdot)$ satisfies the Δ_2 -condition, then $L_M^*(G) = L_M(G)$ and the closure of the set of bounded (including continuous) functions coincides with $L_M^*(G)$.*

More information about these and other facts we can refer from monographs [14, 21].

Further, as G we will take the segment $G = [\pi, \pi]$ and for simplicity everywhere letter G will be omitted (for example $L_M^*(G) = L_M^*$) and etc.. In the following presentation, we will need some facts of Fourier analysis in Orlicz spaces.

Let $M(\cdot)$ be some N-function satisfying the Δ_2 -condition. Let us take $f \in L_M$ and consider

$$S_n[f](x) = \sum_{|k|\leq n} c_k e^{ikx},$$

where

$$c_k = c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx, k \in Z,$$

are Fourier coefficients of $f(\cdot)$. We have

$$S_n[f](x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt,$$

where

$$D_n(x) = \frac{1}{2}\sum_{|k|\leq n} e^{ikx} = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}, n\in Z_+$$

is Dirichlet kernel of order n.

We will also need the following concepts.

Definition 2.4 *We will say that function $M(\cdot)$ satisfies the ∇_2-condition, if*

$$\lim_{u\to\infty}\inf\frac{M(2u)}{M(u)} > 2, \;\; i.e. \;\; \exists k>2\wedge\exists u_0>0:$$

$$M(2u)\geq kM(u), \forall u\geq u_0.$$

Denote by $\Delta_2(\infty)$ ($\nabla_2(\infty)$) the set of N-functions satisfying the Δ_2 -condition(the ∇_2-condition).

Definition 2.5 *Operator $T: L_0\to L_0$, is called quasilinear if $|T(\lambda f)| = |\lambda|\,|T(f)|$, and $\exists c\geq 1: |T(f_1+f_2)|\leq c(|T(f_1)|+|T(f_2)|), \forall f; f_1; f_2\in L_0, \forall\lambda\in C$.*

Case of $c=1$ is called sublinear.

The following Ryan theorem (see, for example, [21, page 193] plays an important role in theory of Fourier analysis in Orlicz spaces.

Theorem 2.1 *R1 Let $M\in\Delta_2(\infty)\cap\nabla_2(\infty)$. If quasilinear operator T is bounded as a operator $T: L_p(-\pi,\pi)\to L_p(-\pi,\pi)$, for $\forall p: 1<p<+\infty$, then it is bounded as a operator $T: L_M\to L_M$.*

From the following Ryan theorem immediately follows the basicity criterion of exponential systems $\left\{e^{\mathrm{int}}\right\}_{n\in Z}$ in L_M.

Theorem 2.2 *R2 Let $M(\cdot)$ be a N-function. Then following properties are equivalent:*
i) L_M is reflexive $\Leftrightarrow M\in\Delta_2(\infty)\cap\nabla_2(\infty)$;
ii) $\exists c>0: \left\|\tilde{f}\right\|_M\leq c\left\|f\right\|_M, \forall f\in L_M$,
where $\tilde{f}$ conjugate function to function f:

$$\tilde{f}(x) = -\frac{1}{\pi}\int_0^{\pi}\frac{f(x+t)-f(x-t)}{2\tan\frac{t}{2}}dt;$$

iii)$\exists c>0$:

$$\left\|S_n[f]\right\|_M\leq c\left\|f\right\|_M, \forall f\in L_M.$$

From this theorem follows following

Corollary 2.1 *Let M be some N-function. Exponential system $\left\{e^{\mathrm{int}}\right\}_{n\in Z}$ forms a basis for L_M if and only if $M\in\Delta_2(\infty)\cap\nabla_2(\infty)$.*

In fact, if system $\left\{e^{\mathrm{int}}\right\}_{n\in Z}$ forms a basis in L_M, then from the basicity criterion it follows that condition iii) of Theorem R2 is true and from this theorem follows that condition i) holds.

The converse also follows from Corollary 9 [14] (see page 197). Similarly to L_p-case the validity of following statement is established.

Statement 2.2. *Let $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$ be some N -function. Then the system of sines $\{\sin nt\}_{n\in N}$ and cosines $\{\cos nt\}_{n\in Z}$ form bases in $L_M\,(0,\pi)$.*

More information about these and other results we can refer from monographs [14].

3 Hardy-Orlicz classes and bases in these classes

Let $M(\cdot)$ be some N -function. Further as $M(f)$ we will understand $M(|f|)$, i.e. $M(f) =: M(|f|)$ (and also for a complex-valued function $f(\cdot)$). As usual, by H_M^+ we denote the Hardy-Orlicz class of functions $F\,(\cdot)$ that are analytic inside ω with the norm

$$\|F\|_{H_M^+} = \sup_{0<r<1} \sup_{\rho_M^*(\nu)\le 1} \left|(F_r\,(\cdot)\,;\nu\,(\cdot))\right| = \sup_{0<r<1} \|F_r\,(\cdot)\|_M\,,$$

where $F_r(t) = F(re^{it})$.

These classes began to be studied in [11, 17–19, 22–25]. The papers [2–8] are devoted to some problems of approximation in these classes. We will consider problems of basicity of parts of the system of exponents in these classes. In the subsequent presentation, we will need some concepts and facts in this direction.

Denote by $\mathcal{A}$ the set of functions $F(\cdot)$ that are analytic in ω for which

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log^+ \left|F(re^{it})\right| dt < +\infty,$$

where $\log^+ u = \log\max\{1; u\}, u \ge 0$.

It is known that a nonzero function $F(\cdot)$ belongs to the class $\mathcal{A}$ if and only if it is representable in the form

$$F(z) = B(z)\exp\Big(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} dz(t)\Big), \tag{3.1}$$

where $B(\cdot)$ is the Blaschke function, $h(\cdot)$ is a function with bounded variation on $[0, 2\pi]$.

Denote by $\mathcal{A}'$ class of functions $F \in \mathcal{A}$ for which the function $h(\cdot)$ in the representation (3.1) is absolutely continuous on $[0, 2\pi]$.

For $F \in \mathcal{A}$ assume

$$\rho_M(\mathcal{A}; F) = \sup_{0<r<1} \rho_M\,(F_r) = \sup_{0<r<1} \int_0^{2\pi} M(F(re^{it}))dt.$$

The following theorem is true.

Theorem 3.1 *[19]. If for analytic function F in ω : $\rho_M(\mathcal{A}; F) < +\infty$, then $F \in \mathcal{A}'$ and conversely, if $F \in \mathcal{A}' \wedge F^+ \in L_M$, then $\rho_M(\mathcal{A}; F) < +\infty$, where $F^+(\cdot)$ are non-tangential boundary values of $F(\cdot)$ on γ.*

Let us consider following singular operator

$$S(f)(\tau) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\xi)}{\xi - \tau} d\xi, \;\; \tau \in \gamma,$$

where $f(\cdot) \in L_1(\gamma)$ is some function. The following theorem is also true.

Theorem 3.2 *[16] Let L_M be a reflexive Orlicz space. Then singular operator S is bounded in L_M, i.e. $\exists M > 0$:*

$$\|Sf\|_M \le M \|f\|_M, \forall f \in L_M.$$

The reflexivity of space L_M is equivalent to the condition $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$. It is clear that the inclusion $L_M \subset L_1$ is true and

$$\|f\|_{L_1} \le c\|f\|_M, \forall f \in L_M, \tag{3.2}$$

where $c > 0$ is an absolute constant.

Further, everywhere we will assume that $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$. Let $f \in H_M^+$. From (3.2) it follows $H_M^+ \subset H_1^+$. Denote by $f^+(\cdot)$ the non-tangential boundary values of f on $\gamma : f^+ = f/\gamma$. Let us expand the function $f(\cdot)$ in Taylor series in a neighborhood of point $z = 0$:

$$f(z) = \sum_{n=0}^{\infty} f_n^+ z^n, \;\; |z| < 1.$$

It is known that $f^+(\cdot) \in L_M$. By Riesz theorem, we have

$$\int_{-\pi}^{\pi} \left|f\left(re^{it}\right) - f^+(e^{it})\right| dt \to 0, \;\; r \to 1 - 0.$$

From here we directly get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^+\left(e^{it}\right) e^{-\mathrm{int}} dt = \begin{cases} f_n^+, n \ge 0, \\ 0, \;\; n < 0. \end{cases}$$

As a result from the basicity of system $\left\{e^{\mathrm{int}}\right\}_{n\in Z}$ in L_M the following decomposition follows :

$$f^+\left(e^{it}\right) = \sum_{n=0}^{\infty} f_n^+ e^{\mathrm{int}}. \tag{3.3}$$

Denote by L_M^+ a restriction of H_M^+ on γ: $L_M^+ = H_M^+/\gamma$. L_M^+ is a subspace of L_M. From the minimality of $\left\{e^{\mathrm{int}}\right\}_{n\in Z}$ in L_M follows the minimality of system $\left\{e^{\mathrm{int}}\right\}_{n\in Z_+}$ in L_M^+, and it means that the decomposition (3.3) is unique.

Therefore, the following statement is true.

Statement 3.1. *Let $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$. Then the system $\{z^n\}_{n\in Z_+}$ $\left(\left\{e^{int}\right\}_{n\in Z_+}\right)$ forms a basis for H_M^+ (for L_M^+.)*

By the uniqueness theorem for analytic functions from the Hardy classes, H_δ^+, $\delta > 0$ we can identify the spaces H_M^+ and L_M^+.

Similarly to the classical case, we us define the Hardy-Orlicz class ${}_mH_M^-$ of functions which are analytic outside of unit circle that have finite order at infinity. Let an analytic function $f(\cdot)$ outside of ω have a Laurent expansion in a neighborhood of infinitely remote point of the form

$$f(z)=\sum_{n=-\infty}^{m} a_n z^n, z\to\infty.$$

So, for $m>0$, the point $z=\infty$ is a pole of order m; for $m\le 0$ the point $z=\infty$ is a zero of order $(-m)$. Let $f(z)=f_0(z)+f_1(z)$, where $f_0(\cdot)$ is a regular part, $f(\cdot)$ is a principal part of the Laurent expansion in the neighborhood of the point $z=\infty$. If the function $g(z)=\overline{f_0\left(\frac{1}{z}\right)},\ |z|<1$, belongs to Hardy class H_M^+, then we will say that, the function $f(\cdot)$ belongs to ${}_mH_M^-$. Similarly H_M^+ case we can prove following

Statement 3.2. *Let $M\in\Delta_2(\infty)\cap\nabla_2(\infty)$. Then the system $\{z_n\}_{-\infty}^{m}$ $\left(\left\{e^{\text{int}}\right\}_{-\infty}^{m}\right)$ forms a basis for ${}_mH_M^-$ (for ${}_mL_M^-$) where ${}_mL_M^-={}_m H_M^-/\gamma$.*

4 Riesz theorem for Hardy-Orlicz classes. Cauchy formula

Let $M\in\Delta_2(\infty)$ be some N-function. In this case the norm $\|\cdot\|\|_M$ is absolutely continuous, i.e. $\forall\varepsilon>0,\ \exists\delta>0:$

$$\|u\chi_e\|_M<\varepsilon,\forall e:|e|<\delta, \tag{4.1}$$

where χ_e is the characteristic function of the set $e, |e|$ is the Lebesgue measure of the set $e\subset[-\pi,\pi]$. For this reason, we can consider, for example, monograph [14, page 105]. The following analogue of the classical Riesz theorem in Hardy classes is true.

Theorem 4.1 *Let $M\in\Delta_2(\infty)$ be some N-function. Then for $\forall F\in H_M^+$:*
$\alpha)$ $\rho_M(F_r)\to\rho(F^+), r\to 1-0,$
$\beta)$ $\rho_M(F_r-F^+)\to 0, r\to 1-0;$
are true. Here $F_r(e^{it})=F(re^{it})$, F^+ are non-tangential boundary values of F on γ.

Proof. Part $\alpha)$ was proved in monograph [19, page 13; Theorem 1.1.2]. Let us prove part $\beta)$. Take $\forall F\in H_M^+$. It is clear that $F\in H_1^+$. Consequently, $F_r(e^{it})\to F^+(e^{it}), r\to 1-0$ a.e. on $[-\pi,\pi]$.

So

$$\rho_M\left(F_r\left(e^{it}\right)\right)\to\rho_M\left(F^+\left(e^{it}\right)\right),\ r\to 1-0, \tag{4.2}$$

is true. Let $r_n\to 1-0,\ n\to\infty$ be an arbitrary sequence. It is clear that $M\left(F_{r_n}(e^{it})\right)\to M\left(F^+(e^{it})\right), n\to\infty$, a.e. on $[-\pi,\pi]$. Let $\varepsilon>0$ be an arbitrary number. Proceeding from (4.1), $\exists\delta>0$:

$$\rho_M\left(F^+\chi_e\right)<\varepsilon, \tag{4.3}$$

where $|e|<\delta$. By Eqorov's theorem $\exists e_1, |e_1|<\delta : F_{r_n}(e^{it})\to F^+(e^{it}), n\to\infty$ uniformly in $E=[-\pi,\pi]\backslash e_1$. So, $M(\cdot)$ is continuous on R, then it is clear that $M\left(F_{r_n}(e^{it})\right)\to M\left(F^+(e^{it})\right), n\to\infty$ uniformly on E. It is obvious that $\rho_M\left(F^+\chi_{e_1}\right)<\varepsilon$ is true. Let us show that

$$\rho_M\left(F_{r_n}\chi_E\right)\to\rho_M\left(F^+\chi_E\right), n\to\infty. \tag{4.4}$$

We have

$$\begin{aligned}&|\rho_M\left(F_{r_n}\chi_E\right)-\rho_M\left(F^+\chi_E\right)|=\\&=\left|\int_{-\pi}^{\pi}M\left(F_{r_n}\left(e^{it}\right)\chi_E\left(t\right)\right)dt-\int_{-\pi}^{\pi}M\left(F_{r_n}^+\left(e^{it}\right)\chi_E\left(t\right)\right)dt\right|\le\\&\le\int_E\left|M\left(F_{r_n}\left(e^{it}\right)\right)-M\left(F^+\left(e^{it}\right)\right)\right|dt\to 0, n\to\infty.\end{aligned}$$

So, relation (4.4) was established. Let us show that

$$\rho_M\left[\left(F_{r_n}-F^+\right)\chi_E\right]\to 0, n\to\infty, \tag{4.5}$$

is true. From $\lim\limits_{x\to+0} M(x)=0$ it follows that for $\forall\varepsilon_1>0$, $\exists\delta_1>0$:

$$M(x)<\varepsilon_1, \forall x: 0\le x<\delta_1.$$

Let us take $n_2\in N$ such that for $\forall n\ge n_2$:

$$\left|F_{r_n}\left(e^{it}\right)-F^+\left(e^{it}\right)\right|<\delta_1, \forall t\in E.$$

We have

$$\rho_M\left[(F_{r_n}-F^+)_E\,\chi_E\right]=\int\limits_{-\pi}^{\pi} M\left[\left(F_{r_n}\left(e^{it}\right)-F^+\left(e^{it}\right)\right)\chi_E(t)\right]dt=$$
$$=\int\limits_E M\left(F_{r_n}\left(e^{it}\right)-F^+\left(e^{it}\right)\right)dt\le\varepsilon_1|E|\le 2\pi\varepsilon_1, \forall n\ge n_2.$$

This directly implies relation (4.5). Let us show that

$$\rho_M\left(F_{r_n}\chi e_1\right)\to\rho_M\left(F^+\chi e_1\right), n\to\infty.$$

We have

$$\left|\rho_M\left(F_{r_n}\left(e^{it}\right)\chi_{e_1}(t)\right)-\rho_M\left(F^+\left(e^{it}\right)\chi_{e_1}(t)\right)\right|=$$
$$=\left|\int\limits_{-\pi}^{\pi} M\left[F_{r_n}\left(e^{it}\right)\chi_{e_1}(t)\,dt-\int\limits_{-\pi}^{\pi} M\left[F^+\left(e^{it}\right)\chi e_1(t)\,dt\right]\right]\right|=$$
$$=\left|\int\limits_{e_1} M\left[F_{r_n}\left(e^{it}\right)\right]dt-\int\limits_{e_1} M\left[F^+\left(e^{it}\right)\right]dt\right|=$$
$$=\left|\int\limits_{-\pi}^{\pi} M\left[F_{r_n}\right]dt-\int\limits_E M\left[F_{r_n}\right]dt-\int\limits_{-\pi}^{\pi} M\left[F^+\right]dt+\int\limits_E M\left[F^+\right]dt\right|\le$$
$$\le\left|\rho_M\left(F_{r_n}\right)-\rho_M\left(F^+\right)\right|+\left|\rho_M\left(F_{r_n}\chi_E\right)-\rho_M\left(F^+\chi_E\right)\right|.$$

Paying attention to relations (4.2) and (4.4) immediately we have

$$\rho_M\left(F_{r_n}\left(e^{it}\right)\chi_{e_1}(t)\right)\to\rho_M\left(F^+\left(e^{it}\right)\chi_{e_1}(t)\right), n\to\infty.$$

Then from (4.3) it follows that $\exists n_1\in N$:

$$\rho_M\left(F_{r_n}\left(e^{it}\right)\chi_{e_1}(t)\right)<\varepsilon, \quad \forall n\ge n_1$$

We have

$$\rho_M\left(F_{r_n}\left(e^{it}\right)-F^+\left(e^{it}\right)\right)=\int\limits_{-\pi}^{\pi} M\left(F_{r_n}\left(e^{it}\right)-F^+\left(e^{it}\right)\right)dt=$$
$$=\rho_M\left(\left(F_{r_n}\left(e^{it}\right)-F^+\left(e^{it}\right)\right)\chi_E(t)\right)+\int\limits_{e_1} M\left(F_{r_n}\left(e^{it}\right)-F^+\left(e^{it}\right)\right)dt.$$

On the other hand, we have

$$M\left(F_{r_n}\left(e^{it}\right)-F^{+}\left(e^{it}\right)\right)\leq M\left(\left|F_{r_n}\left(e^{it}\right)\right|+\left|F^{+}\left(e^{it}\right)\right|\right)\leq$$
$$\leq\frac{1}{2}\left[M\left(2F_{r_n}\left(e^{it}\right)\right)+M\left(2F^{+}\left(e^{it}\right)\right)\right]\leq$$
$$\leq c\left(M\left(F_{r_n}\left(e^{it}\right)\right)+M\left(F^{+}\left(e^{it}\right)\right)\right).$$

Consequently

$$\int\limits_{e_1} M\left(F_{r_n}\left(e^{it}\right)-F^{+}\left(e^{it}\right)\right)dt\leq$$
$$\leq c\left(\int\limits_{e_1} M\left(F_{r_n}\left(e^{it}\right)\right)dt+\int\limits_{e_1} M\left(F^{+}\left(e^{it}\right)\right)dt\right)\leq 2c\varepsilon,\forall n\geq n_1.$$

Paying attention to (4.5), we get

$$\rho_M\left(F_{r_n}-F^{+}\right)\rightarrow 0, n\rightarrow\infty.$$

Theorem is proved.

This theorem is corresponding analogue of classical Riesz theorem for Hardy classes H_p^+.

It is known that, if $M\in\Delta_2(\infty)$, then the convergence in terms of ρ_M is equivalent to the convergence by the norm $\|\cdot\|_M$. We can consider, for example, monography [14, page 93, Theorem 9.4]. Then from Theorem 4.1 it follows

Corollary 4.1 *Let $M\in\Delta_2(\infty)$. Then for $\forall F\in H_M^+$:*
$\alpha)$ $\|F_r\|_M\rightarrow\|F^+\|_M, r\rightarrow 1-0;$
$\beta)$ $\|F_r-F^+\|_M\rightarrow 0,\ \ r\rightarrow 1-0.$

Statement $\beta)$ of Corollary 4.1 follows from Theorem 4.1. Part $\alpha)$ of this corollary follows from $\beta)$. Let us consider Cauchy formula in Hardy- Orlicz classes. The following theorem is true.

Theorem 4.2 *Let M be some N-function. $F\in\mathcal{A}$ is some analytic function in ω. Then:*
1) if $F\in H_M^+$, then $F^+\in L_M$ and Cauchy formula

$$F(z)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{F^{+}(\xi)\,d\xi}{\xi-z},\ \ z\notin\gamma, \tag{4.6}$$

is true;
2) if $F\in L_M$, then the function $F(\cdot)$ defined by the formula (4.6) belongs to Hardy Orlicz class H_M^+.

Proof. It is clear that if M and M^* are complementary for each other N-function, then following Holders formula is true

$$\left|\int\limits_{-\pi}^{\pi} u(x)\,\nu(x)\,dx\right|\leq\|u\|_M\,\|\nu\|_M^*,\quad\forall u\in L_M,\forall\nu\in L_M^* \tag{4.7}$$

Considering the inequality

$$\|\nu\|_{M^*}\leq\rho_{M^*}(\nu)+1,$$

(see [14, page 89]) and putting in it $\nu(x) \equiv 1$, we have $|1|_M^* \le 1 + \rho_{M^*}(1) < +\infty$, as M^* is a continuous function on R (it means that $\rho_{M^*}(1) = \int\limits_{-\pi}^{\pi} M^*(1)\,dx = 2\pi M^*(1)$). As a result, we get the estimation

$$\|u\|_{L_1} \le c\,\|u\|_M\,,\ \forall u \in L_M, \tag{4.8}$$

where $c = \|1\|_M$. By the Holder inequality (4.7) (see [14, page 91]) from (4.8) it follows immediately

$$H_M^+ \subset H_1^+. \tag{4.9}$$

So, let $F \in H_M^+ \Rightarrow F \in H_1^+$, and as a result by Fikhtengol'ts theorem (see [20, page 97]) Cauchy formula is true

$$F(z) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{F^+(\xi)\,d\xi}{\xi - z},\ \ z \in \omega,$$

where $F^+(\cdot)$ is non-tangential boundary value of $F(\cdot)$ on γ.

Now, let $F \in \mathcal{A}'$ (Nevanlinna class) and $F^+ \in L_M$. It follows from theorem [19] that $\rho_M(\mathcal{A}, F) < +\infty$. From inequality

$$\|F_r(\cdot)\|_M \le \rho_M(F_r) + 1,$$

it immediately follows that

$$\|F\|_{H_M^+} \le \rho_M(\mathcal{A}; F) + 1,$$

as a result, $F \in H_M^+$. Let $f \in L_M$ be some function. Let us consider following Cauchy type integral

$$F(z) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{f(\xi)\,d\xi}{\xi - z},\ z \notin \gamma.$$

It is clear that, $f \in L_1$ Then it is known that $F \in H_\delta^+$, $\forall\delta \in (0,1)$ (see, for example, [20, page 94, V.I.Smirnov theorem]). Since, $H_\delta^+ \in \mathcal{A}$, $\forall\delta > 0$ (see [19], page 19), then $F \in \mathcal{A}$ and as a result, by Theorem 3.1 [19] we get

$$\rho_M(\mathcal{A}; F) < +\infty \Rightarrow \|F\|_{H_M^+} < +\infty \Rightarrow F \in H_M^+.$$

Theorem is proved.

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