Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **39** (4), 96-106 (2019).

Unique weak solvability of the first boundary value problem for a Hilbarg-Serrin parabolic equation in non-cylindrical domains

Nazim J. Jafarov ⋆

Received: 06.12.2018 / Revised: 21.08.2019 / Accepted: 03.11.2019

Abstract. The first boundary value problem for a Hilbarg-Serrin parabolic equation is considered in the paper. Its unique weak solvability in corresponding weight spaces of Sobolev is established.

Keywords. boundary value problem, weak solvability, parabolic operator.

Mathematics Subject Classification (2010): 35K05, 35K08, 34B05

1 Introduction.

Let G be a bounded n-dimensional domain containing the origin of coordinates. Consider in G the first boundary-value problem for the Hilbarg-Serrin equation

$$\begin{cases} \Delta u + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x), \quad x \in G, \\ u\,|_{\partial G} = 0, \end{cases} \tag{1.1}$$

where $\lambda > -1$ is a constant. The problems of a weak and strong solvability of the problem (1.1) in corresponding Sobolev weight spaces have been studied in [1-3]. The goal of this work is the obtaining of analogies of these results concerning the weak solvability for the case of Hilbarg-Serrin parabolic equation in the so-called P-domains. Note that in the general case, the Hilbarg-Serrin parabolic equation doesn't satisfy the parabolic condition by Cardes [4]. As to the problem of solvability of boundary value problems for general parabolic equations of second order, we shall indicate them in monographs [5-6].

2 Notations, definitions and subsidiary statements

Mention the notations and definitions used in this paper.

Let $\mathbb{R}_n$ be an n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$, D be a bounded domain in $\mathbb{R}_n$ with boundary ∂D, moreover $0 \in D$. By $\mathbb{R}_{n+1}$ we denote $(n+1)$-dimensional Euclidean space of points $(x,t) = (x_1, ..., x_n, t)$,

$$\mathbb{R}_{n+1}^{-} = \mathbb{R}_{n+1} \cap \{(x,t) : t < 0\} .$$

⋆ Corresponding author

N.J. Jafarov
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan,
E-mail: nazim.jafarov.math@gmail.com

We call $Q \subset R^-_{n+1}$, the domain of paraboloid type (or P-domain), ([7]) if its section with any hyper-plane of $t=\tau(\tau<0)$ has the form:

$$\left\{x : \frac{x}{2\sqrt{-\tau}} \in D\right\}.$$

Let

$$Q_T = Q \cap \{(x,t) : -T < t < 0\}\ ,\ S_T = \partial Q \cap \{(x,t) : -T < t < 0\}\ ,$$

$$D_T = Q \cap \{(x,t) : t = -T\}\ ,$$

$\Gamma(Q_T)$ be a parabolic boundary of the domain Q_T ([8]).

Consider a parabolic operator with a coefficient determined on Q_T

$$L \equiv \Delta + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} \cdot \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t}\,, \tag{2.1}$$

where Δ is a Laplace operator, λ is a numerical parameter that satisfies the condition

$$-\frac{1}{d^2} < \lambda < \infty\,, \tag{2.2}$$

$$d = \sup_{y \in D} |y|\ .$$

Note that the condition (2.2) is not otherwise than a condition of uniform parabolicity of the operator L on the domain Q_T .

By the analogy with elliptic case we shall call the operator L the Hilberg-Serrin operator.

The symbols u_i , u_{ij} everywhere denote the derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $u_x = (u_1, ..., u_n)$;

$$u_{xx} = (u_{ij})\,;\ \ i,j = \overline{1,n}\,;\ \ u_x^2 = \sum_{i=1}^{n} u_i^2\,;\ \ u_{xx}^2 = \sum_{i,j=1}^{n} u_{ij}^2\,.$$

Let a numerical number γ satisfy the condition:

$$\gamma \in \left(\frac{n^2\left(\lambda - \frac{1}{d^2}\right) + 2\lambda n}{8}\,;\, +\infty\right)\,, \tag{2.3}$$

$C_0^\infty(Q_T)$ is a space of all infinitely differentiable functions with a compact support in Q_T .
$A_0^\infty(Q_T)$
be a space of infinitely differentiable finite functions in Q_T, for which the integral

$$\int\limits_{Q_T} (-t)^{\gamma-1} u^2 dxdt$$

is finite. $L_{2,\gamma}(Q_T)$ is a class of measurable functions $u(x,t)$ given in Q_T with a finite norm

$$\|u\|_{L_{2,\gamma}(Q_T)} = \left(\int\limits_{Q_T} (-t)^{\gamma} u^2 dxdt\right)^{1/2}\,,$$

$W^{1,0}_{2,\gamma}(Q_T)$ is a class of measurable functions $u(x,t)$ given in Q_T with a finite norm

$$\|u\|_{W^{1,0}_{2,\gamma}(Q_T)} = \left(\int\limits_{Q_T} (-t)^{\gamma}(u^2 + u_x^2)dxdt \right)^{1/2},$$

$\overset{\circ}{W}{}^{1,0}_{2,\gamma}(Q_T)$ is a closed subspace of $W^{1,0}_{2,\gamma}(Q_T)$, where $A^{\infty}_0(Q_T)$ is a dense set.
$W^{1,1}_{2,\gamma}(Q_T)$ is a class of measurable functions $u(x,t)$, given in Q_T with a finite norm

$$\|u\|_{W^{1,1}_{2,\gamma}(Q_T)} = \left(\int\limits_{Q_T} (-t)^{\gamma}(u^2 + u_x^2 + u_t^2)dxdt \right)^{1/2},$$

$\overset{\circ}{W}{}^{1,1}_{2,\gamma}(Q_T)$ is a closed subspace of $W^{1,1}_{2,\gamma}(Q_T)$, where $A^{\infty}_0(Q_T)$ is a dense set.

In the domain Q_T we consider the first boundary value problem

$$\begin{cases} Lu = \mathit{\Delta} u + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = f + \sum_{k=1}^{n} \frac{\partial f^k}{\partial x_k}, \\ u \left|_{\mathit{\Gamma}(Q_T)} \right. = 0 \end{cases} \tag{2.4}$$

where $f \in L_{2,\gamma}(Q_T)$, $f^k \in L_{2,\gamma}(Q_T)$, $k = \overline{1,n}$.

Definition 2.1 *Under the weak solution of Hilbarg-Serrin equation with a right hand side $f + \sum_{k=1}^{n} \frac{\partial f^k}{\partial x_k}$ we shall understand such a function $u(x,t) \in W^{1,0}_{2,\gamma}(Q_T)$ that satisfies the integral identity*

$$\int\limits_{Q_T} (-t)^{\gamma} u \vartheta_t dxdt - \int\limits_{Q_T} (-t)^{\gamma} \sum_{i,j=1}^{n} \left(\delta_{ij} + \lambda \frac{x_i x_j}{4(-t)} \right) \vartheta_i u_j dxdt$$
$$+\lambda(n+1) \int\limits_{Q_T} (-t)^{\gamma} \sum_{i=1}^{n} \frac{x_i}{4(-t)} \vartheta_i u \, dxdt$$
$$+\frac{\lambda n(n+1)}{4} \int\limits_{Q_T} (-t)^{\gamma-1} u \vartheta dxdt - \gamma \int\limits_{Q_T} (-t)^{\gamma-1} \vartheta u dxdt$$
$$= \int\limits_{Q_T} (-t)^{\gamma} f \vartheta dxdt - \int\limits_{Q_T} (-t)^{\gamma} \sum_{k=1}^{n} f^k \vartheta_k dxdt \tag{2.5}$$

for any $\vartheta(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\gamma}(Q_T)$.

Clarify briefly the derivation principle of (2.5). For this, it is necessary to cut off the domain Q_T from above by a hyperplane $t = -\varepsilon$ for sufficiently small positive ε and write the Hilbarg-Serrin operator in the form of a sum of a divergent part and small terms. Then, the equation is considered in the layer $Q_{T,\varepsilon} = Q_T \backslash \overline{Q}_{\varepsilon}$ and both parts of the equation are multiplied by the function $(-t)^{\gamma} \cdot \vartheta(x,t)$, where $\theta \in A^{\infty}_0(Q_T)$ and they vanish near the parabolic boundary $\mathit{\Gamma}(Q_{T,\varepsilon})$. Integrating both sides of the equation with respect to $Q_{T,\varepsilon}$ and using Ostrogodsky's formula we are led to expressions that uniformly depend on ε.

Tending ε to zero we derive the integral identity (2.5).

3 Fridrichs type inequality

Let Q_T be a domain described above and $u(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\gamma}(Q_T)$. Then it is valid the inequality

$$\int_{Q_T} (-t)^\gamma u^2(x,t)dxdt \le C \cdot \int_{Q_T} (-t)^\gamma u_x^2(x,t)dxdt\,, \tag{3.1}$$

where a constant $C > 0$ depends only on the domain Q_T.

Proof. Since the domain Q_T is bounded, then there exists a parallelepiped

$$K = \{(x,t) : -R \le x_i \le R,\ -T \le t \le 0\}$$

inside of which we can arrange Q_T. Let $u(x,t) \in A_0^\infty(Q_T)$. Continue the function $u(x,t)$ by zero in K.

We have

$$u(t,x_1,x') = u(t,-R,x') + \int_{-R}^{x_1} u_1(t,y,x')dy = \int_{-R}^{x_1} u_1(t,y,x')dy\,,$$

where $x' = (x_2,...,x_n)$.

Thus,

$$u^2(t,x_1,x') = \left(\int_{-R}^{x_1} u_1(t,y,x')dy\right)^2 \le 2R\int_{-R}^{x_1} u_1^2 dy\,. \tag{3.2}$$

Multiply the both sides of the latter inequality by $(-t)^\gamma$ and integrate with respect to the domain K. Since in $K\backslash Q_T$ $u = 0$, we get

$$\int_{Q_T} (-t)^\gamma u^2 dxdt \le 2R\int_{Q_T} (-t)^\gamma \int_{-R}^{x_1} u_1^2(t,y,x')dydxdt$$

$$\le 2R\int_{-R}^{R} ... \int_{-R}^{R}\int_{-T}^{0} (-t)^\gamma \int_{-R}^{R} u_1^2(t,y,x')dydxdt$$

$$= 2R\int_{-R}^{R} dy \int_{Q_T} (-t)^\gamma u_1^2 dxdt = 4R^2 \int_{Q_T} (-t)^\gamma u_1^2 dxdt\,. \tag{3.3}$$

(3.1) is obtained from (3.3) with the help of the passage to the limit.

4 Main a priori estimate

By deriving the main a priori estimate we use the same scheme as in deriving the main integral identity.

Let $Q_{T,\varepsilon}$ have the same meaning that above, but $A_p^\infty(Q_{T,\varepsilon})$ be a totality of all functions from $A_0^\infty(Q_T)$, vanishing near the parabolic boundary $\Gamma(Q_{T,\varepsilon})$. It is easy to see

$$Lu = \Delta u + \lambda \sum_{i,j=1}^{n}\left(\frac{x_i x_j}{4(-t)}u_j\right)_i - \lambda(n+1)\sum_{j=1}^{n}\frac{x_j}{4(-t)}u_j - u_t\,. \tag{4.1}$$

For any function $u(x,t) \in A_p^{\infty}(Q_{T,\varepsilon})$ we have

$$-\int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} uLudxdt = -\int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} u\Delta udxdt - \lambda \sum_{i,j=1}^{n} \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} u \left(\frac{x_i x_j}{4(-t)} u_j\right)_i dxdt$$

$$+\lambda \cdot (n+1) \sum_{j=1}^{n} \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} \frac{x_j}{4(-t)} u_j \cdot u\, dxdt + \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} u \cdot u_t dxdt$$

$$= J_{1,\varepsilon} + J_{2,\varepsilon} + J_{3,\varepsilon} + J_{4,\varepsilon}\,. \tag{4.2}$$

On the other hand,

$$J_{1,\varepsilon} = -\int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} u\Delta udxdt = -\int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} \sum_{i=1}^{n} u \cdot u_{ii} dxdt$$

$$= \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} \sum_{i=1}^{n} u_i^2 dxdt = \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} u_x^2 dxdt\,; \tag{4.3}$$

$$J_{2,\varepsilon} = -\lambda \sum_{i,j=1}^{n} \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} u \left(\frac{x_i x_j}{4(-t)} u_j\right)_i dxdt = \lambda \sum_{i,j=1}^{n} \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} \frac{x_i x_j}{4(-t)} u_i u_j dxdt\,; \tag{4.4}$$

$$J_{3,\varepsilon} = \lambda(n+1) \sum_{j=1}^{n} \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} \frac{x_j}{4(-t)} u_j u\, dxdt$$

$$= \frac{\lambda(n+1)}{2} \sum_{j=1}^{n} \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} \frac{x_j}{4(-t)} (u^2)_j dxdt = \frac{-\lambda n(n+1)}{2} \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} \frac{u^2}{4(-t)} dxdt\,; \tag{4.5}$$

$$J_{4,\varepsilon} = \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} u\, u_t dxdt = \frac{1}{2} \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} (u^2)_t dxdt\,. \tag{4.6}$$

Briefly we can transform (4.6). We have

$$\int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma} (u^2)_t dxdt = \int\limits_{Q_{T,\varepsilon}} \frac{d}{dt} \left((-t)^{\gamma} u^2\right) dxdt - \int\limits_{Q_{T,\varepsilon}} \left[(-t)^{\gamma}\right]_t \cdot u^2 dxdt\,. \tag{4.7}$$

We estimate the integral $\int\limits_{Q_{T,\varepsilon}} \frac{d}{dt}\left((-t)^{\gamma} u^2\right) dxdt$.

To this end, arrange the layer $Q_{T,\varepsilon}$ in a parallelepiped $D_T \times (-T, \varepsilon)$ and continue the function $(-t)^{\gamma} u^2$ by zero to this parallepiped. Denote a new function by $F(t,x)$. Then

$$\left| \int\limits_{Q_{T,\varepsilon}} \frac{d}{dt}\left((-t)^{\gamma} u^2\right) dxdt \right| = \left| \int\limits_{D_T \times (-T,\varepsilon)} \frac{d}{dt} F(t,x) dxdt \right| = \left| \int\limits_{-T}^{-\varepsilon} \int\limits_{D_T} \frac{d}{dt} F(t,x) dxdt \right|$$

$$= \left| \int\limits_{P_\varepsilon} F(-\varepsilon, x)dx - \int\limits_{D_T} F(-T, x)dx \right| ,$$

where P_ε is an upper edge of the parallelepiped, and $P_\varepsilon \supset D_\varepsilon$.

We see from the construction of $F(t,x)$ that

$$\int\limits_{D_T} F(-T, x)dx = 0 ,$$

$$\left| \int\limits_{P_\varepsilon} F(-\varepsilon, x)dx \right| = \left| \int\limits_{D_\varepsilon} (-\varepsilon)^\gamma (u^2)\,|_{t=-\varepsilon}\, dx \right| \le |(-\varepsilon)^\gamma| \cdot \sup_{x \in D_\varepsilon} \left|u^2\right| \cdot mes_n D_\varepsilon = o(\varepsilon),\ \varepsilon \to 0+ .$$

After transforming (4.7) we finally arrive at the expression

$$J_{4,\varepsilon} = \frac{\gamma}{2} \int\limits_{Q_{T,\varepsilon}} (-t)^\gamma u^2 dxdt + o(\varepsilon),\ \ \varepsilon \to 0+\ . \tag{4.8}$$

In (4.2) we consider (4.3), (4.4), (4.5), (4.8). We get

$$- \int\limits_{Q_{T,\varepsilon}} (-t)^\gamma uLudxdt = \int\limits_{Q_{T,\varepsilon}} (-t)^\gamma u_x^2 dxdt + \lambda \sum_{i,j=1}^{n} \int\limits_{Q_{T,\varepsilon}} (-t)^\gamma \frac{x_i x_j}{4(-t)} u_i u_j dxdt$$

$$- \frac{\lambda n(n+1)}{2} \int\limits_{Q_{T,\varepsilon}} (-t)^\gamma \frac{u^2}{4(-t)} dxdt + \frac{\gamma}{2} \int\limits_{Q_{T,\varepsilon}} (-t)^{\gamma-1} u^2 dxdt + o(\varepsilon)\ ,\ \ \varepsilon \to 0+\ . \tag{4.9}$$

We pass to the limit for $\varepsilon \to 0+$ and write the obtained expression in the form

$$\int\limits_{Q_{T,\varepsilon}} (-t)^\gamma \left(|\nabla u|^2 + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j \right) dxdt$$

$$= \frac{\lambda n(n+1) - 4\gamma}{2} \int\limits_{Q_T} (-t)^\gamma \frac{u^2}{4(-t)} dxdt - \int\limits_{Q_T} (-t)^\gamma uLudxdt\,. \tag{4.10}$$

We see from (4.10) that for

$$\gamma \ge \frac{\lambda n(n+1)}{4}\,, \tag{4.11}$$

$$\int\limits_{Q_T} (-t)^\gamma \left(u_x^2 + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j \right) dxdt \le - \int\limits_{Q_T} (-t)^\gamma uLudxdt\,. \tag{4.12}$$

At $\frac{\lambda n(n+1)-4\gamma}{2} > 0$ for any $\varepsilon_1 > 0$ we have

$$- \sum_{i=1}^{n} \int\limits_{Q_T} (-t)^\gamma \frac{1}{2\sqrt{-t}} \cdot \frac{x_i}{2\sqrt{-t}} u \cdot u_i dxdt \le \frac{\varepsilon_1}{2} \int\limits_{Q_T} (-t)^\gamma \frac{u^2}{4(-t)} dxdt$$

$$+\frac{1}{2\varepsilon_1}\int\limits_{Q_T}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_jdxdt\,. \tag{4.13}$$

On the other hand,

$$-\sum_{i=1}^{n}\int\limits_{Q_T}(-t)^{\gamma}\frac{1}{2\sqrt{-t}}\cdot\frac{x_i}{2\sqrt{-t}}u\cdot u_idxdt=-\frac{1}{2}\sum_{i=1}^{n}\int\limits_{Q_T}(-t)^{\gamma}\frac{x_i}{4(-t)}(u^2)_idxdt$$

$$=\frac{n}{2}\int\limits_{Q_T}(-t)^{\gamma}\frac{u^2}{4(-t)}dxdt\,. \tag{4.14}$$

From (4.13) and (4.14) we get

$$\frac{n-\varepsilon_1}{2}\int\limits_{Q_T}(-t)^{\gamma}\frac{u^2}{4(-t)}dxdt\leq\frac{1}{2\varepsilon_1}\int\limits_{Q_T}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_jdxdt\,. \tag{4.15}$$

For $\varepsilon_1=\frac{n}{2}$, from (4.15) it follows

$$\int\limits_{Q_T}(-t)^{\gamma}\frac{u^2}{4(-t)}dxdt\leq\frac{4}{n^2}\int\limits_{Q_T}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_jdxdt\,. \tag{4.16}$$

Taking into account (4.16) in (4.10), we conclude

$$\int\limits_{Q_T}(-t)^{\gamma}\left(u_x^2+\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\right)dxdt\leq\frac{2\lambda n(n+1)-8\gamma}{n^2}$$

$$\times\int\limits_{Q_T}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_jdxdt-\int\limits_{Q_T}(-t)^{\gamma}uLudxdt\,. \tag{4.17}$$

We solve the inequality

$$\frac{2\lambda n(n+1)-8\gamma}{n^2}<\frac{1}{d^2}+\lambda\,,$$

and obtain

$$\gamma>\frac{n^2\left(\lambda-\frac{1}{d^2}\right)+2\lambda n}{8}=\gamma_1\,. \tag{4.18}$$

On the other hand, $\gamma<\frac{\lambda n(n+1)}{4}=\gamma_2$. Hence we see that for $\lambda\geq-\frac{1}{d^2}\ \gamma_1<\gamma_2$. Return to (4.12). As we had shown above (4.12) is satisfied when

$$\gamma\geq\frac{\lambda n(n+1)}{4}\,.$$

But

$$\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j=\lambda\left(\sum_{i=1}^{n}\frac{x_i}{2\sqrt{-t}}u_i\right)^2\,. \tag{4.19}$$

If $\lambda \geq 0$, then

$$\lambda\left(\sum_{i=1}^{n}\frac{x_i}{2\sqrt{-t}}u_i\right)^2 \geq 0\,. \tag{4.20}$$

But if $-\frac{1}{d^2} < \lambda < 0$, then

$$\lambda\left(\sum_{i=1}^{n}\frac{x_i}{2\sqrt{-t}}u_i\right)^2 \geq \lambda d^2 u_x^2\,. \tag{4.21}$$

From (4.20) and (4.21) we get that there exists $\varepsilon_2 > 0$ such that

$$\varepsilon_2 \int\limits_{Q_T}(-t)^{\gamma}u_x^2 dxdt \leq -\int\limits_{Q_T}(-t)^{\gamma}u\,Ludxdt\,. \tag{4.22}$$

Return to the case when $\gamma \in \left(\frac{n^2\left(\lambda-\frac{1}{d^2}\right)+2\lambda n}{8}\,;\,\frac{\lambda n(n+1)}{4}\right)$. As we had shown above in this case (4.17) is fulfilled.

Then there exists $\varepsilon_3 > 0$ such that $\frac{2\lambda n(n+1)-8\gamma}{n^2} < \frac{1}{d^2} + \lambda - \frac{\varepsilon_3}{d^2}$, $0 < \varepsilon_3 < 1$. Then we obtain from (4.17)

$$\int\limits_{Q_T}(-t)^{\gamma}\left(u_x^2 + \left(\frac{\varepsilon_3}{d^2} - \frac{1}{d^2}\right)\sum_{i,j=1}^{n}\frac{x_i x_j}{4(-t)}u_i u_j\right)dxdt \leq -\int\limits_{Q_T}(-t)^{\gamma}u\,Ludxdt\,. \tag{4.23}$$

We get from (4.17) and Fridrichs type inequality that there exists such a positive constant ε_4 that

$$\varepsilon_4 \int\limits_{Q_T}(-t)^{\gamma}u^2 dxdt \leq -\int\limits_{Q_T}(-t)^{\gamma}u\,Ludxdt\,. \tag{4.24}$$

Thus, we arrive at the main a priori estimate which we formulate in the form of a theorem.

Theorem 4.1 *Let $Q_T - P$ be a domain, D be its basis, moreover $O \in D$ and $\gamma \in \in \left(\frac{n^2\left(\lambda-\frac{1}{d^2}\right)+2\lambda n}{8};\,+\infty\right)$. Then for the Hilbarg-Serrin operator on Q_T and for any function $u(x,t) \in \overset{\circ}{W}{}^{1,0}_{2,\gamma}(Q_T)$ there exists such a constant $C > 0$ not depending on the choice $u \in \overset{\circ}{W}{}^{1}_{2,\gamma}(Q_T)$ that it is valid the inequality*

$$\|u\|_{W^1_{2,\gamma}(Q_T)} \leq C\,\|Lu\|_{L_{2,\gamma}(Q_T)}\,. \tag{4.25}$$

5 Unique solvability of the first boundary value problem

Theorem 5.1. *Let* $\gamma \in \left(\frac{n^2\left(\lambda-\frac{1}{d^2}\right)+2\lambda n}{8}; \infty\right)$. *Then the problem* (2.4) *is uniquely solvable in the space* $\overset{\circ}{W}{}^{1,0}_{2,\gamma}(Q_T)$.

Proof. First we prove the existence of the solution. Assume again $Q_{T,\varepsilon} = Q_T \backslash \overline{Q}_\varepsilon,\ \varepsilon \in (0,T)$. Let $D_h \subset D,\ D_h \to D$ as $h \to 0$ and domains D_h have sufficiently smooth boundaries. Moreover, we smooth the functions $f^k,\ k = \overline{1,n}$ and f which are in right-hand side of (2.4). We denote the smoothed functions by $f^{k,\mu}$ and f^μ, where $\mu > 0$.

Let $Q_T - P$ be a domain with a basis D_h, $Q^h_{T,\varepsilon} = Q^h_T \backslash \overline{Q}^h_\varepsilon$, assume $\varepsilon > 0$ and $h \to 0,\ \mu > 0$. Consider the problem (2.4) in the domain $Q^h_{T,\varepsilon}$. It is known that this problem has a unique solution $u^{h,\mu}_\varepsilon \in C^\infty\left(\overline{Q}^h_{T,\varepsilon}\right)$. We have

$$\varDelta u + \lambda \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{x_i x_j}{4(-t)} u_j\right)_i - \lambda(n+1)\sum_{i=1}^{n} \frac{x_i}{4(-t)} u_i - u_t = f + \sum_{k=1}^{n} \frac{\partial f^k}{\partial x_k}\,, \tag{5.1}$$

where we omit the indices $u^{h,\mu}_\varepsilon$ for the reduction of notations.

We multiply the both sides of (5.1) by $(-t)^\gamma u(t,x)$ and integrate with respect to the domain $Q^h_{T,\varepsilon}$. We get

$$\int\limits_{Q^h_{T,\varepsilon}} (-t)^\gamma \varDelta u \cdot u dx dt + \lambda \int\limits_{Q^h_{T,\varepsilon}} (-t)^\gamma u \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{x_i x_j}{4(-t)} u_j\right) dx dt - \lambda(n+1)$$

$$\times \int\limits_{Q^h_{T,\varepsilon}} (-t)^\gamma u \sum_{i=1}^{n} \frac{x_i}{4(-t)} u_i u\, dx dt - \int\limits_{Q^h_{T,\varepsilon}} (-t)^\gamma u \cdot u_t dx dt = \int\limits_{Q^h_{T,\varepsilon}} (-t)^\gamma f \cdot u\, dx dt$$

$$+ \int\limits_{Q^h_{T,\varepsilon}} (-t)^\gamma \sum_{k=1}^{n} \frac{\partial f^k}{\partial x^k} u\, dx dt\,. \tag{5.2}$$

Further we have

$$\int\limits_{Q^h_{T,\varepsilon}} (-t)^\gamma \left(u_x^2 + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j\right) dx dt - \frac{\lambda(n+1) - 4\lambda}{2}$$

$$\times \int\limits_{Q^h_{T,\varepsilon}} (-t)^\gamma \cdot \frac{u^2}{4(-t)} dx dt + o(\varepsilon)$$

$$= \int\limits_{Q^h_{T,\varepsilon}} (-t)^\gamma \sum_{k=1}^{n} f^k \cdot u_k dx dt - \int\limits_{Q^h_{T,\varepsilon}} (-t)^\gamma f \cdot u\, dx dt,\ \varepsilon \to 0+\ . \tag{5.3}$$

Assume $h > 0,\ \mu > 0$ and pass to the limit for $\varepsilon \to 0$. Denote a limit function by $u^{h,\mu}(t,x)$.

Let $\frac{\lambda n(n+1)-4\gamma}{2} \leq 0$, i.e. $\gamma \geq \frac{\lambda n(n+1)}{2}$. Then, the second integral at the hand side of (5.3) is positive. Besides, if $\lambda > 0$, then

$$\lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j \geq 0\,.$$

But if $-\frac{1}{d^2} < \lambda < 0$, then

$$\lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j \geq \lambda\, d^2 u_x^2\,.$$

Let $\frac{\lambda n(n+1)-4\gamma}{2} > 0$, i.e. $\gamma < \frac{\lambda n(n+1)}{2}$. Then

$$\frac{\lambda\, n(n+1)-4\gamma}{2} \int\limits_{Q_T^h} (-t)^{\gamma} \cdot \frac{u^2}{4(-t)} dxdt$$

$$\leq \frac{2\lambda\, n(n+1)-8\gamma}{n^2} \int\limits_{Q_T^h} (-t)^{\gamma} \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j dxdt\,. \tag{5.4}$$

Note that for $\gamma \in \left(\frac{n^2\left(\lambda-\frac{1}{d^2}\right)+2\lambda n}{8};\ \frac{\lambda n(n+1)}{2}\right)$,

$$\frac{2\lambda n(n+1)-8\gamma}{n^2} < \frac{1}{d^2} + \lambda - \frac{\mu}{d^2}\,, \tag{5.5}$$

where $0 < \mu < 1$.

We consider (5.4), (5.5) in (5.3) and get

$$\int\limits_{Q_T^h} (-t)^{\gamma} \left(u_x^2 + \frac{\mu-1}{d^2} \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j \right) dxdt \leq \int\limits_{Q_T^h} (-t)^{\gamma} \left(\sum_{i=1}^{n} f^i \cdot u_i - fu \right) dxdt\,. \tag{5.6}$$

Since $\frac{\mu-1}{d^2} < 0$, then

$$\frac{\mu-1}{d^2} \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j \geq (\mu-1)\ |\nabla u|^2\ .$$

Taking (3.7) into account in (3.6), we deduce

$$\mu \cdot \int\limits_{Q_T^h} (-t)^{\gamma} u_x^2 dxdt \leq \int\limits_{Q_T^h} (-t)^{\gamma} \left(\sum_{k=1}^{n} f^k \cdot u_k - f \cdot u \right) dxdt\,. \tag{5.7}$$

If we estimate the right hand side of (5.7) in a standard way and use the Frederichs type inequality, we deduce

$$\left\| u^{h,\mu} \right\|_{W_{2,\gamma}^{1,0}(Q_T^h)} \leq M\,, \tag{5.8}$$

where a constant M doesn't depend on $u^{h,\mu}$. Continue the function $u^{h,\mu}$ by zero in $Q_T\backslash Q_T^h$. It is obvious that the continued function will be an element of the space $\overset{\circ}{W}{}^{1,0}_{2,\gamma}(Q_T)$ and the estimate (5.8) is valid in the norm $\overset{\circ}{W}{}^{1,0}_{2,\gamma}(Q_T)$. Hence it follows that there exist such sequences $\mu_k \to 0$, $h_m \to 0$ for $k \to \infty$, $m \to \infty$ that u^{h_m,μ_k} tends to some function $u \in \overset{\circ}{W}{}^{1,0}_{2,\gamma}(Q_T)$ weakly in $\overset{\circ}{W}{}^{1,0}_{2,\gamma}(Q_T)$. It is easy to see that the function $u(x,t)$ is the solution of the problem (2.4).

To prove the uniqueness of the solution it is sufficient to use the procedure in proving the existence of the solution with small alternations and to obtain the estimate

$$\|u\|_{W^{1,0}_{2,\gamma}(Q_T)} \le C\left(\|f\|_{L_{2,\gamma}(Q_T)} + \sum_{k=1}^{n}\left\|f^k\right\|_{L_{2,\gamma}(Q_T)}\right)$$

with a constant C not depending on a function u.

References

1. Bass, R.F.: *The Dirichlet problem for radially homogeneous elliptic operators*, Trans. Amer. Math. Soc. **2** (320), 593-61 (1990).
2. Mamtiev, T.R.: *On solvability of the first boundary value problem for second order elliptic equations with homogeneous coefficients*, Dep. In Az NIINTI, **2287**, Az., 26 p. (1995).
3. Mamedov, I.T., Mamtiev, T.R.: *Coercive estimation of second order elliptic operators with homogeneous coefficients*, Proc. I Repub. Conf. Math. Mech., Baku, "Elm", **II**, 140-148 (1995).
4. Alkhutov, Yu.A., Mamedov, I.T.: *The first boundary value problem for second order nondivergent parabolic equations with discontinuous coefficients*, Math. Sb. **131** (173), **4 (12)**, 477-500 (1986).
5. Ladyzhenskaya, O.A., Solonnikov V.A., Uraltseva N.N.: Linear and quasilinear equations of parabolic type, *M. Nauka*, 736 p. (1967).
6. Krylov, N.V.: Nonlinear elliptic and parabolic equations of second order, *M. Nauka*, 376 p. (1985).
7. Jafarov, N.J.: *On smoothness up to the boundary of the solutions of the first boundary value problem of second order in parabolic type domains.* Dokl. AN Azerb. **1-2** (55), 3-8 (1999).
8. Landis, E.M.: Second order equations of elliptic and parabolic types, *Moscow, Nauka*, 288 p. (1971).