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Coefficient estimates for certain subclass of bi-univalent analytic functions

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Abstract. *In this paper, we introduce and investigate the initial coefficients of the subclass of bi-univalent functions of the form* $\Sigma D^m_{\alpha,\lambda}(p, q; \mu)$.

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1 Introduction

Let A be the class of normalized analytic functions f in the open unit disc $U = \{z \in C : |z| < 1\}$ with $f(0) = f'(0) = 0$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in C, \tag{1.1}$$

and S the class of all functions in A that are univalent in U. Such class of functions are one-to-one and onto, since it is defined on the entire unit disk of the complex plane and thus, the inverse exists, but the inverse may not be defined in the entire unit disc. The popular Koebe one quarter theorem established that every univalent function maps the unit disc to a disk of radius $\frac{1}{4}$. Thus, the inverse function for every univalent function $S \in A$ can be defined by

$$f^{-1}(f(z)) = z, \quad |z| < 1$$

and

$$f^{-1}(f(w)) = w, \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$$

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where, using (1), one obtains

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ... \qquad (1.2)$$

We denote $f^{-1}(w)$ by $f^{-1}(w) = g(w)$, and write

$$g(w) = w + \sum_{n=2}^{\infty} A_n w^n$$

A function of the form (1) is said to be bi-univalent in U if both f and $f^{-1} = g$ are univalent in U. We denote the class of bi-univalent functions by Σ. The study of the coefficients of bi-univalent functions started as far back as 1967 with the work of M.Lewin, see Lewin [1],[2] and the coefficient bounds of bi-univalent functions picked from the work of Brannan and Taha in 1986, see [3], and has gained the attention of several researchers in the recent years see [2], [4], [5], [6], [7], [8].
The author in [8] introduced a new linear differential operator for the analytic functions of the form (1) given by

Definition 1.1 *Let* $0 \le \lambda \le 1, \alpha \ge 1, m \in N \cup 0$. *Then for* $f \in A$, *the operator* $D_\alpha^m f : A \to A$ *is defined by:*

$$D_\alpha^0 f(z) = f(z)$$

$$D_\alpha f(z) = (1-\lambda)f(z) + z\lambda z f'(z) \qquad (1.3)$$

$$D_\alpha^m f(z) = D_\alpha(D_\alpha^{m-1} f(z)) \qquad (1.4)$$

$$D_\alpha^{m+1} f(z) = (1-\lambda)D_\alpha^m f(z) + z\lambda(D_\alpha^m f(z))'. \qquad (1.5)$$

such that $D_\alpha^m f$ *is given by*

$$D_\alpha^m f(z) = z + \sum_{n=2}^{\infty} \alpha \left(\frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^m a_n z^n. \qquad (1.6)$$

The importance of the coefficient estimates of the analytic functions cannot be overemphasized. Initial coefficient estimates is important in the evaluation of the Fekete-Szego functional and Hankel determinant among its other uses. Also, using the coefficient bounds, the author in [10] applied the generalized linear functions introduced in [9] to a disease control measure.. Motivated by the work of Zireh and Audegani [14] , using the differential operator in [8] we define a new subclass of bi-univalent functions and obtained the initial coefficient estimates for the subclass so obtained. For the differential operator of the form (6), with simple calculations, we obtain the inverse function given by

$$D_\alpha^m f^{-1}(w) = w - t_2 w^2 + t_3 w^3 - t_4 w^4 - ... \qquad (1.7)$$

where $D_\alpha^m f^{-1}(w) = D_\alpha^m g(w)$and
$t_2 = \left(\alpha \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)} \right)^m a_2 \right),$
$t_3 = \left(2\alpha^2 \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)} \right)^{2m} a_2^2 - \alpha \left(\frac{1+\lambda(\alpha-1)}{1+\lambda(\alpha-1)} \right)^m a_3 \right),$
$t_4 = \left(5 \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)} \right)^m a_2^3 - 5 \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)} \right)^m \left(\frac{1+\lambda(\alpha+1)}{1+\lambda(\alpha-1)} \right)^m a_2a_3 + 5 \left(\frac{1+\lambda(\alpha+2)}{1+\lambda(\alpha-1)} \right)^m a_4 \right)$
In what follows, we introduce and investigate the initial coefficient bounds of the subclass $\Sigma D_{\alpha,\lambda}^m(p, q; \mu)$.

2 Coefficient estimates for the subclass $\Sigma D^m_{\alpha,\lambda}(p,q;\mu)$

In this section we give the definition of our subclass of bi-univalent analytic functions denoted by $\Sigma D^m_{\alpha,\lambda}(p,q;\mu)$ and estimate the bounds of its initial coefficients.

Definition 2.1 *Let p, q be analytic functions such that $p, q : U \to C$, $p(0) = q(0) = 1$ and*

$$min\,\{Re(p(z)), Re(q(z))\} > 0, z \in U.$$

Also, let f be as in (1), then f is said be in $\Sigma D^m_{\alpha,\lambda}(p,q;\mu)$ if:

$$f \in \Sigma \quad and \quad (D^m_{\alpha,\lambda}f(z))' + \mu z(D^m_{\alpha,\lambda}f(z))'' \in p(U), (D^m_{\alpha,\lambda}g(w))' + \mu w(D^m_{\alpha,\lambda}g(w))'' \in p(U) \tag{2.1}$$

where $z \in U, w \in U$ $\mu > 0$ and $D^m_{\alpha,\lambda}$ is as given in (7)

Theorem 2.1 *Let $f \in \Sigma D^m_{\alpha,\lambda}(p,q;\mu)$ where $\mu \geq 0; \alpha \geq 1; 0 \leq \lambda \leq 1; m = 0, 1, 2, ...$ and p, q satisfy the condition in definition 2. Then*

$$|a_2| \leq min\left\{\frac{b^m}{2\alpha a^m(1+\mu)}\sqrt{\frac{|p'(0)|^2 + |q'(0)|^2}{2}}, \frac{b^m}{\alpha a^m}\sqrt{\frac{|p''(0)| + |q''(0)|}{12(1+2\mu)}}\right\},$$

and

$$|a_3| \leq min\left\{\frac{b^m(|p'(0)|^2 + |q'(0)|^2)}{8\alpha a^m(1+\mu)^2} + \frac{b^m(|p''(0)| + |q''(0)|)}{12\alpha c^m(1+2\mu)}, \frac{b^m(|p''(0)|}{6\alpha c^m(1+2\mu)}\right\}.$$

where $a = 1 + \alpha\lambda, \quad b = 1 + \lambda(\alpha - 1), \quad c = 1 + \lambda(\alpha + 1)$

Proof. Let $f \in \Sigma D^m_{\alpha,\lambda}(p,q;\mu)$ and $D^m_{\alpha,\lambda}g = (D^m_{\alpha,\lambda}f)^{-1}$, thus by definition, we have respectively

$$(D^m_{\alpha,\lambda}f(z))' + \mu z(D^m_{\alpha,\lambda}f(z))'' = p(z), \quad z \in u \tag{2.2}$$

and

$$(D^m_{\alpha,\lambda}g(w))' + \mu z(D^m_{\alpha,\lambda}g(w))'' = q(z), \quad z \in u \tag{2.3}$$

where p, q satisfies the condition of the definition 2 and

$$p(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ... \tag{2.4}$$

$$q(z) = 1 + c_1 w + c_2 w^2 + c_3 w^3 + ... \tag{2.5}$$

Using (6) in (9) and (7) in (10) respectively give

$$1 + 2\alpha h_{21} a_2 z + h_{31} a_3 z^2 + ... + h_{22} a_2 z + h_{32} a_3 z^2 + ... = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ... \tag{2.6}$$

and

$$1 - H_{21} w + H_{31} w^2 - H_{41} w^3 + .. - H_{22} w + H_{32} w^2 - H_{42} w^3 + ... = 1 + c_1 w + c_2 w^2 + c_3 w^3 + ... \tag{2.7}$$

where $h_{21} = 2\alpha\left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^m$,
$h_{31} = 3\alpha\left(\frac{1+\lambda(\alpha+1)}{1+\lambda(\alpha-1)}\right)^m$,
$h_{22} = 2\mu\alpha\left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^m$,
$h_{32} = 6\mu\alpha\left(\frac{1+\lambda(\alpha+1)}{1+\lambda(\alpha-1)}\right)^m$ and

$H_{21} = 2\alpha \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^m a_2,$
$H_{31} = (6\alpha^2 \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^{2m} a_2^2 - 3\alpha \left(\frac{1+\lambda(\alpha+1)}{1+\lambda(\alpha-1)}\right)^m a_3),$
$H_{41} = (20\alpha \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^m a_2^3 - 20a_2a_3 + 4a_4),$
$H_{22} = 2\mu\alpha \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^m a_2,$
$H_{32} = \mu(12\alpha^2 \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^{2m} a_2^2 - 6\alpha \left(\frac{\lambda(\alpha+1)}{1+\lambda(\alpha-1)}\right)^m a_3),$
$H_{42} = \mu(60\alpha \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^m a_2^3 - 60a_2a_3 + 12a_4)$
equating coefficients in (13) and (14), we obtain

$$2\alpha \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^m (1+\mu)a_2 = b_1 \tag{2.8}$$

$$3\alpha \left(\frac{1+\lambda(\alpha+1)}{1+\lambda(\alpha-1)}\right)^m (1+2\mu)a_3 = b_2 \tag{2.9}$$

respectively

$$-2\alpha \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^m (1+\mu)a_2 = c_1 \tag{2.10}$$

$$6\alpha^2 \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^{2m} (1+2\mu)a_2^2 - 3\alpha \left(\frac{1+\lambda(\alpha+1)}{1+\lambda(\alpha-1)}\right)^m (1+2\mu)a_3 = c_2 \tag{2.11}$$

Comparing (15) and (17), we obtain

$$c_1 = -b_1$$

and

$$4\alpha^2 \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^{2m} (1+\mu)^2 a_2^2 = b_1^2 \tag{2.12}$$

$$4\alpha^2 \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^{2m} (1+\mu)^2 a_2^2 = c_1^2 \tag{2.13}$$

From (19) and (20), we have

$$8\alpha^2 \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^{2m} (1+\mu)^2 a_2^2 = b_1^2 + c_1^2 \tag{2.14}$$

Also, from (16) and (18), we have

$$6\alpha^2 \left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^{2m} (1+2\mu)^2 a_2^2 = b_2 + c_2 \tag{2.15}$$

Now from (21) and (22), we have

$$a_2^2 = \frac{(1+\lambda(\alpha-1))^{2m}(b_1^2+c_1^2)}{8\alpha^2(1+\alpha\lambda)^{2m}(1+\mu)^2} \tag{2.16}$$

Respectively,

$$a_2^2 = \frac{(1+\lambda(\alpha-1))^{2m}(b_2+c_2)}{6\alpha^2(1+\alpha\lambda)^{2m}(1+2\mu)} \tag{2.17}$$

Thus, from (23) and (24), using (6) and (7) we have

$$|a_2|^2 \leq \frac{(1+\lambda(\alpha-1))^{2m}(|p'(0)|^2+|q'(0)|^2}{8\alpha^2(1+\alpha\lambda)^{2m}(1+\mu)^2} \tag{2.18}$$

Respectively,

$$|a_2|^2 \leq \frac{(1+\lambda(\alpha-1))^{2m}(|p''(0)|+|q''(0)|}{12\alpha^2(1+\alpha\lambda)^{2m}(1+2\mu)} \tag{2.19}$$

And from (25) and (26), we obtain and our desired result for the coefficient $|a_2|$.
Furthermore, from (16) and (18), we have

$$-6\alpha^2\left(\frac{1+\alpha\lambda}{1+\lambda(\alpha-1)}\right)^{2m}(1+2\mu)a_2^2+6\alpha\left(\frac{1+\lambda(\alpha+1)}{1+\lambda(\alpha-1)}\right)^{m}(1+2\mu)a_3=b_2-c_2 \tag{2.20}$$

Using (23) and (24) in (27), we obtain

$$|a_3| = \frac{(1+\lambda(\alpha-1))^m(b_2-c_2)}{6\alpha(1+\lambda(\alpha+1))^m(1+2\mu)}+\frac{(1+\lambda(\alpha-1))^m(b_1^2-c_1^2)}{8\alpha(1+\lambda(\alpha+1))^m(1+\mu)^2} \tag{2.21}$$

and

$$|a_3| = \frac{(1+\lambda(\alpha-1))^m b_2}{3\alpha(1+\lambda(\alpha+1))^m(1+2\mu)} \tag{2.22}$$

And from (28) and (29), our desired result for the coefficient $|a_3|$is obtained.
Hence, we conclude the proof of the Theorem 1.

Remarks 2.1 *(1.) For $m>0, \alpha>1$ and $\lambda\neq 0$, we have $|a_2|, |a_3|$ to be a refinement of the $|a_2|, |a_3|$ in [14], thus, Theorem 1 is a refinement of the Theorem 2.3 in [14].*
(2.) It is also noted that the linear differential operator in [8] generalize Al-Oboudi and Salagean differential operators. For $\alpha=1$ in [8], we have the Al-Oboudi differential operator [1] and for $\alpha,\lambda=1$ in [8], we have the Salagean differential operator [12]. Thus, Theorem 1 will give the result for Al-Oboudi and Salagean differential operators in place of the linear differential operator by the author in [8] when $\alpha=1$ respectively when $\alpha,\lambda=1$ in the Theorem. This is given in some of the corollaries below.

Corollary 2.1 *Let $f\in D^m_{\alpha,\lambda}(p,q;1)$ where $\alpha\geq 1; 0\leq\lambda\leq 1; m=0,1,2,...$ and p,q satisfy the condition in definition 2. Then*

$$|a_2| \leq \min\left\{\frac{(1+\lambda(\alpha-1))^m}{4\alpha(1+\alpha\lambda)^m}\sqrt{\frac{|p'(0)|^2+|q'(0)|^2}{2}}, \frac{(1+\lambda(\alpha-1))^m}{6\alpha(1+\alpha\lambda)^m}\sqrt{|p''(0)|+|q''(0)|}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{a^m(|p'(0)|^2+|q'(0)|^2)}{32\alpha b^m}+\frac{a^m(|p''(0)|+|q''(0)|)}{36\alpha b^m}, \frac{a^m(|p''(0)|}{18\alpha b^m}\right\}.$$

$a=(1+\lambda(\alpha-1)), b=(1+\lambda(\alpha+1))$

Corollary 2.2 *Let $f\in D^m_{1,\lambda}(p,q;1)$ where $\lambda\geq 0;, m=0,1,2,...$ and p,q satisfy the condition in definition 2. Then*

$$|a_2| \leq \min\left\{\frac{1}{4(1+\lambda)^m}\sqrt{\frac{|p'(0)|^2+|q'(0)|^2}{2}}, \frac{1}{6(1+\lambda)^m}\sqrt{|p''(0)|+|q''(0)|}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{(|p'(0)|^2+|q'(0)|^2)}{32(1+2\lambda)^m}+\frac{(|p''(0)|+|q''(0)|)}{36(1+2\lambda)^m}, \frac{|p''(0)|}{18(1+2\lambda)^m}\right\}.$$

Remark 2.2 *The corollary 2 gives the Al-Oboudi differential operator for the subclass.*

Corollary 2.3 *Let* $f \in D^{m}_{1,1}(p,q;1)$ *where* $m = 0,1,2,...$ *and* p,q *satisfy the condition in definition 2. Then*

$$|a_2| \leq \min\left\{\frac{1}{4(2^m)}\sqrt{\frac{|p'(0)|^2+|q'(0)|^2}{2}}, \frac{1}{6(2^m)}\sqrt{|p''(0)|+|q''(0)|}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{(|p'(0)|^2+|q'(0)|^2)}{32(1+2)^m} + \frac{(|p''(0)|+|q''(0)|)}{36(1+2)^m}, \frac{|p''(0)|}{18(1+2)^m}\right\}.$$

Remark 2.3 *The corollary 3 gives the Salagean differential operator for the subclass.*

Corollary 2.4 *Let* $f \in D^{m}_{1,0}(p,q;1)$ *where* $\geq 0;, m = 0,1,2,...$ *and* p,q *satisfy the condition in definition 2. Then*

$$|a_2| \leq \min\left\{\frac{1}{4}\sqrt{\frac{|p'(0)|^2+|q'(0)|^2}{2}}, \frac{1}{6}\sqrt{|p''(0)|+|q''(0)|}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{(|p'(0)|^2+|q'(0)|^2)}{32} + \frac{(|p''(0)|+|q''(0)|)}{36}, \frac{|p''(0)|}{18}\right\}.$$

Remark 2.4 *The corollary 4 agrees with the Theorem 2.3 in [14].*

Corollary 2.5 *Let* $p(z) = q(z) = \left(\frac{1+\frac{z}{k}}{1-\frac{z}{k}}\right)^{\beta}, \quad 0 < \beta \leq 1, k \geq 1$. *Then*

$$|a_2| \leq \min\left\{\beta\frac{(1+\lambda(\alpha-1))^m}{\alpha(1+\alpha\lambda)^m(1+\mu)}, \frac{\beta(1+\lambda(\alpha-1))^m}{k\alpha(1+\alpha\lambda)^m}\sqrt{\frac{2}{3(1+2\mu)}}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{\beta^2 a^m}{k^2\alpha(b^m(1+\mu)^2} + \frac{2\beta^2 a^m}{3k^2\alpha b^m(1+2\mu)}, \frac{2a^m}{3k^2\alpha b^m(1+2\mu)}\right\}.$$

where $a = (1+\lambda(\alpha-1)), b = (1+\lambda(\alpha+1))$

3 Conclusion

The the linear transformation employed in this paper extends the Salagean and Al-Oboudi operators and the result is a generalization of the work of Zireh and Audegani in [14].

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