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Transformation operators for the Schrödinger equation with a linearly increasing potential

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Abstract. *The Schrödinger equation with a linearly growing potential is considered. Using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. Estimates for the kernels of the transformation operators are obtained.*

Keywords. Schrödinger equation · transformation operator · Airy functions · Stark equation.

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1 Introduction

In many aspects of the theory of inverse problems of spectral analysis, an important role is played by so-called transformation operators (see [8], [10] and the references therein). These operators arose from the general ideas of the theory of generalized shift operators created by Delsarte [3]. For arbitrary SturmLiouville equations, transformation operators were constructed by Povzner [10]. Marchenko [8] used transformation operators for studying inverse spectral problems and the asymptotic behavior of the spectral function of the singular SturmLiouville operator. Levin [7] introduced transformation operators of a new form that preserve the asymptotic expansions of solutions at infinity. Marchenko [8] used them to solve the inverse problem of scattering theory. Similar problems for the Schrödinger equation with unbounded potentials were considered in [2], [4], [9], [12].

We consider the differential equation

$$-y'' + |x|y + q(x)y = \lambda y, -\infty < x < \infty \ , \lambda \in C. \tag{1.1}$$

where the real potential $q(x)$ satisfies the conditions

$$q(x) \in C(-\infty, +\infty), \quad \int_{-\infty}^{\infty} |xq(x)|\, dx < \infty. \tag{1.2}$$

In the present paper, using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. The results obtained can be used to solve

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inverse spectral problems for an equation (1.1). Note that for $x \geq 0$, equation (1.1) turns into the one-dimensional Stark equation. Some questions of the spectral theory of the one-dimensional Stark equation were studied in [5], [6], [9].

2 The transformation operators

In what follows, we deal with special functions satisfying the Airy equation

$$-y'' + zy = 0.$$

It is well known (e.g., see [12]) that this equation has two linearly independent solutions $Ai(z)$ and $Bi(z)$ with the initial conditions

$$Ai\left(0\right) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}, Ai'\left(0\right) = \frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)},$$

$$Bi\left(0\right) = \frac{1}{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)}, Bi'\left(0\right) = \frac{3^{\frac{1}{6}}}{\Gamma\left(\frac{1}{3}\right)}.$$

The Wronskian $\{Ai(z), Bi(z)\}$ of these functions satisfies

$$\{Ai(z), Bi(z)\} = Ai(z)Bi'(z) - Ai'(z)Bi(z) = \pi^{-1}.$$

Both functions are entire functions of order $\frac{3}{2}$ and type $\frac{2}{3}$. We have (see [1]) asymptotic equalities for $|z| \to \infty$

$$Ai(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{-\zeta}\left[1 + O\left(\zeta^{-1}\right)\right], |\arg z| < \pi,$$

$$Ai\left(-z\right) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \sin\left(\zeta + \frac{\pi}{4}\right)\left[1 + O\left(\zeta^{-1}\right)\right], |\arg z| < \frac{2\pi}{3},$$

$$Bi(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{\zeta}\left[1 + O\left(\zeta^{-1}\right)\right], |\arg z| < \frac{\pi}{3},$$

$$Bi\left(-z\right) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \cos\left(\zeta + \frac{\pi}{4}\right)\left[1 + O\left(\zeta^{-1}\right)\right], |\arg z| < \frac{2\pi}{3}.$$

where $\zeta = \frac{2}{3} z^{\frac{3}{2}}$. In what follows we will need special solutions of the unperturbed equation

$$-y'' + |x|y = \lambda y, -\infty < x < \infty,\ \lambda \in C. \tag{2.1}$$

Lemma 2.1 *For any λ from the complex plane, equation (2.1) has solutions $\psi_{\pm}(x, \lambda)$ in the form*

$$\psi_{+}(x, \lambda) = \begin{cases} Ai\left(x - \lambda\right), x \geq 0, \\ -\pi\left(Ai\left(-\lambda\right)Bi\left(-\lambda\right)\right)' Ai\left(-x - \lambda\right) - \\ 2\pi Ai\left(-\lambda\right)Ai'\left(-\lambda\right)Bi\left(-x - \lambda\right), x < 0, \end{cases} \tag{2.2}$$

$$\psi_{-}(x, \lambda) = \begin{cases} -\pi\left(Ai\left(-\lambda\right)Bi\left(-\lambda\right)\right)' Ai\left(x - \lambda\right) - \\ -2\pi Ai\left(-\lambda\right)Ai'\left(-\lambda\right)Bi\left(x - \lambda\right), x \geq 0, \\ Ai\left(-x - \lambda\right), x < 0. \end{cases} \tag{2.3}$$

Proof. Obviously, when $x \geq 0$ one of the solutions of equation (2.1) is function $Ai(x-\lambda)$. On the other hand, for $x \leq 0$ any solution of equation (2.1) can be represented as

$$\alpha Ai(-x-\lambda) + \beta Bi(-x-\lambda).$$

If we glue these solutions at a point $x = 0$, we get

$$\alpha = -\pi(Ai(-\lambda)Bi(-\lambda))',\ \beta = -2\pi Ai(-\lambda)\, Ai'(-\lambda).$$

Thus, formula (2.2) is established. Formula (2.3) is derived similarly.

The lemma is proved.

We shall use the following notation

$$\sigma_{\pm}(x) = \pm \int_x^{\pm\infty} |-t+|t|+q(t)|dt.$$

In the following theorem the representation of solution from the equation (1.1) is found by means of transformation operator.

Theorem 2.1 *If the potential $q(x)$ satisfies conditions (1.2), then for any λ from the complex plane equation (1.1) has a solution $f_+(x,\lambda)$ that can be represented in the form*

$$f_+(x,\lambda) = \psi_+(x,\lambda) + \int_x^{\infty} K_+(x,t)\psi_+(t,\lambda)dt, \tag{2.4}$$

where kernel $K_+(x,t)$ is continuous function and satisfies relations

$$K_+(x,t) = O\Big(\sigma_+\Big(\frac{x+t}{2}\Big)\Big),\ x+t \to \infty, \tag{2.5}$$

$$K_+(x,x) = \frac{1}{2}\int_x^{\infty} (|t|-t+q(t))dt. \tag{2.6}$$

Proof. We rewrite the perturbed equation (1.1) in the form

$$-y'' + xy + Q(x)y = \lambda y, -\infty < x < \infty. \tag{2.7}$$

where $Q(x) = |x| - x + q(x)$. Obviously, the $Q(x)$ function for all $x > a, a > -\infty$ satisfies the condition

$$Q \in C(-\infty,+\infty), \quad \int_a^{\infty} |x\,Q(x)|dx < \infty. \tag{2.8}$$

Let $f_+(x,\lambda)$ be solution of equation (2.7) with the asymptotic behavior

$$f_+(x,\lambda) = f_0(x,\lambda)\,(1+o\,(1))\,,\ x \to \infty,$$

where $f_0(x,\lambda) = Ai\,(x-\lambda)$. Subject to the conditions (2.8), such solution exist, is determined uniquely by its asymptotic behavior. With the aid of operator transformations, we have the representation

$$f_+(x,\lambda) = f_0(x,\lambda) + \int_x^{\infty} K(x,t)f_0(t,\lambda)dt. \tag{2.9}$$

Moreover, the kernel $K(x,t)$ is a continuous function and satisfies the following relations

$$K(x,t) = O\left(\sigma_+\left(\frac{x+t}{2}\right)\right), x+t \to \infty, \tag{2.10}$$

$$K\left(x,x\right)=\frac{1}{2}\int_x^{\infty}Q\left(t\right)dt. \tag{2.11}$$

In addition, rewriting the unperturbed equation (2.1) in the form

$$-y''+xy+Q_0(x)y=\lambda y,\ -\infty<x<\infty,$$

where $Q_0(x)=|x|-x$, we obtain

$$\psi_+(x,\lambda)=f_0(x,\lambda)+\int_x^{\infty}K_0(x,t)f_0(t,\lambda)dt.$$

Moreover, in this case, $K_0(x,t)$ satisfies the identity $K_0(x,t)\equiv 0, x\geq 0$. From the well-known properties of the transformation operators it follows that (see [8]) the function $f_0(x,\lambda)$ also admits the representation

$$f_0(x,\lambda)=\psi_+(x,\lambda)+\int_x^{\infty}\widetilde{K}_0(x,t)\psi_+(t,\lambda)dt, \tag{2.12}$$

where the kernels $K_0(x,t), \widetilde{K}_0(x,t)$ are connected by the equality

$$K_0(x,t)+\widetilde{K}_0(x,t)+\int_x^{t}\widetilde{K}_0\left(x,u\right)K_0\left(u,t\right)du=0. \tag{2.13}$$

Substituting the expression (2.12) from the $f_0(x,\lambda)$ in (2.8), we get

$$\begin{aligned}
&f_+(x,\lambda)=\psi_+(x,\lambda)+\textstyle\int_x^{\infty}K(x,t)\left[\psi_+(t,\lambda)+\int_t^{\infty}\widetilde{K}_0\left(t,u\right)\psi_+\left(u,\lambda\right)du\right]dt=\\
&=\psi_+(x,\lambda)+\textstyle\int_x^{\infty}K(x,t)\psi_+(t,\lambda)dt+\int_x^{\infty}K(x,t)\int_t^{\infty}\widetilde{K}_0\left(t,u\right)\psi_+\left(u,\lambda\right)dudt=\\
&=\psi_+(x,\lambda)+\textstyle\int_x^{\infty}K(x,t)\psi_+(t,\lambda)dt+\int_x^{\infty}\left(\int_x^{t}K\left(x,u\right)\widetilde{K}_0\left(u,t\right)du\right)\psi_+(t,\lambda)dt.
\end{aligned}$$

Setting

$$K_+(x,t)=K(x,t)+\int_x^{t}K\left(x,u\right)\widetilde{K}_0\left(u,t\right)du, \tag{2.14}$$

one can recast the last relation in the form

$$f_+(x,\lambda)=\psi_+(x,\lambda)+\int_x^{\infty}K_+(x,t)\psi_+(t,\lambda)dt.$$

Formula (2.5) is a straightforward consequence of (2.10), (2.13), (2.14). Taking $t=x$ in the equality (2.14), we find that $K_+(x,t)=K(x,t)$. Whence, by virtue of (2.11), formula (2.6) follows.

The theorem is proved.

The following theorem is proved in a similar way.

Theorem 2.2 *If the potential $q(x)$ satisfies condition (1.1), then, for all values of λ, equation (1.2) has a solution $f_-(x,\lambda)$ representable as*

$$f_-(x,\lambda)=\psi_-(x,\lambda)+\int_{-\infty}^{x}K_-(x,t)\psi_-(t,\lambda)dt,$$

where the kernel $K_-(x,t)$ is continuous function and satisfy the following conditions

$$\begin{aligned}
&K_-(x,t)=O\left(\sigma_-\left(\tfrac{x+t}{2}\right)\right), x+t\to\infty,\\
&K_-(x,x)=\tfrac{1}{2}\textstyle\int_{-\infty}^{x}(|t|+t+q(t))dt.
\end{aligned}$$

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