Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **40** (1), 28-40 (2020).

Mixed problem for one-dimensional wave equation with dynamic boundary condition

Akbar B. Aliev[⋆] · **Samira O. Rustamova**

Received: 05.08.2019 / Revised: 16.03.2020 / Accepted: 14.04.2020

Abstract. *In this work we study a mixed problem for a nonlinear wave equation with dynamical boundary conditions. Using the nonlinear semigroup theory we investigate the existence and uniqueness of local and global solutions. We also investigate the question about solvability of these problems, when the dynamical boundary conditions deteriorate to the quasi-static boundary conditions.*

Keywords. semilinear wave equation, dynamical boundary conditions, mixed problem, global solvability, nonlinear dissipation.

Mathematics Subject Classification (2010): 35L05, 35L11, 35L35, 35L82, 35B40

1 Introduction

In this paper, we consider a nonlinear wave equation with dynamic boundary conditions and source term

$$u_{tt} - u_{xx} + B_1(u_t) + B_2(x,u) = f(t,x), \quad x \in (0,1), \quad t > 0, \tag{1.1}$$

$$\varepsilon u_{tt}(t,0) - u_x(t,0) + b_{10}\left(u_t(t,0)\right) + b_{20}\left(u(t,1)\right) = f_0(t), \quad t > 0, \tag{1.2}$$

$$\delta u_{tt}(t,1) + u_x(t,1) + b_{11}\left(u_t(t,1)\right) + b_{21}\left(u(t,1)\right) = f_1(t), \quad t > 0, \tag{1.3}$$

$$u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x), \tag{1.4}$$

where B_1, B_2, b_{10}, b_{11}, b_{20}, b_{21}, f, f_0 and f_1 are given real valued functions. The functions B_1, b_{10} and b_{11} have the form $B_1(s) = \mu|s|^{q-1}s$, $b_{10}(s) = \mu_0|s|^{q_0-1}s$ and $b_{11}(s) = \mu_1|s|^{q_1-1}s$, respectively, where μ, μ_0, μ_1 and q, q_0, q_1 are real constants such that

$$\mu \geq 0,\ \mu_0 \geq 0,\ \mu_1 \geq 0 \text{ and } q > 1,\ q_0 > 1,\ q_1 > 1. \tag{1.5}$$

Let the functions B_2, b_{20} and b_{21} satisfy the local Lipschitz condition, i.e.

$$|B_2(x,s_2) - B_2(x,s_1)| \leq c\left(|s_1|,|s_2|\right)|s_2 - s_1|, \tag{1.6}$$

⋆ Corresponding author

A.B. Aliev
Azerbaijan Technical University, AZ1073, 25 Huseyn Cavid Ave., Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan, AZ 1141, 9 B. Vahabzadeh str., Baku, Azerbaijan
E-mail: alievagil@yahoo.com

S.O. Rustamova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, AZ 1141, 9 B. Vahabzadeh str., Baku, Azerbaijan
E-mail: samira.rustamova.1979@mail.ru

$$|b_{20}(s_2) - b_{20}(s_1)| \le c_0\left(|s_1|, |s_2|\right)|s_2 - s_1|, \tag{1.7}$$

$$|b_{21}(s_2) - b_{21}(s_1)| \le c_1\left(|s_1|, |s_2|\right)|s_2 - s_1|, \tag{1.8}$$

where $0 \le x \le 1,\ s_1, s_2 \in \mathbb{R}, c(\cdot), c_0(\cdot), c_1(\cdot) \in C(\mathbb{R}^2_+; \mathbb{R}_+), \mathbb{R}_+ = (0, \infty)$.

In the present paper, we will first investigate the existence of solutions in the case $\varepsilon > 0$ and $\delta > 0$. Further, in the case $q = q_0 = q_1 = 1$, $\varepsilon = \delta$, we study the existence of the limit $\lim\limits_{\varepsilon\to 0} u_{\varepsilon\,\varepsilon}(t, x)$ and prove that the limit function is a solution of the degenerate problem

$$u_{tt} - u_{xx} + \mu u_t + B_2(t, x, u) = f(t, x),\ \ t > 0,\ \ x \in (0, 1), \tag{1.9}$$

$$-u_x(t, 0) + \mu u_t(t, 0) + b_{20}(t, u(t, 1)) = f_0(t),\ \ t > 0 \tag{1.10}$$

$$u_x(t, 1) + \mu u_t(t, 1) + b_{21}(t, u(t, 1)) = f_1(t),\ \ t > 0, \tag{1.11}$$

$$u(0, x) = \varphi(x),\ u_t(0, x) = \psi(x),\ x \in (0, 1). \tag{1.12}$$

Problem (1.9)-(1.12) are called the mixed problem for the wave equation with a quasistatical boundary condition [1]. We note that some special case of problem (1.9)-(1.12) investigated in [2-6].

2 Preliminaries and technical lemmas

In this paper, problem (1.1)-(1.4) was solved using the semi-group theory.

Let $\|\cdot\|$ be the norm in $L_2(0, 1)$. By H we denote the direct sum of $L_2(0, 1)$ and $\mathbb{R}^2$, i.e.

$$H = L_2(0, 1) \oplus \mathbb{R} \oplus \mathbb{R} = \{w : w = (u, \alpha, \beta),\ u \in L_2(0, 1),\ \alpha, \beta \in \mathbb{R}\}.$$

Moreover, define the scalar product in the space H as follows:

$$< w_1, w_2 >= \int_0^1 u_1(x)\, u_2(x)dx + \varepsilon\alpha_1 \cdot \alpha_2 + \delta\beta_1 \cdot \beta_2,$$

where $w_k = (u_k, \alpha_k, \beta_k),\ u_k \in L_2(0, 1), \alpha_k, \beta_k \in \mathbb{R},\ k = 1,\ 2.$

By H_0 and H_1 we denote the following spaces

$$H_0 = \{\tilde{u} : \tilde{u} = (u, u(0), u(1))\,,\ u \in W_2^1(0, 1)\},$$

$$H_1 = \{\tilde{u} : \tilde{u} = (u, u(0), u(1))\,,\ u \in W_2^2(0, 1)\}.$$

Let us define a linear operator $A_{\varepsilon,\delta}$ in space H as follows

$$\begin{cases} D(A_{\varepsilon,\delta}) = H_1, \\ A_{\varepsilon,\delta}\, \tilde{u} = \left(-u_{xx}(x), -\frac{1}{\varepsilon}u_x(0), \frac{1}{\delta}u_x(1)\right),\ \tilde{u} = (u, u(0), u(1)) \in D(A_{\varepsilon,\delta}). \end{cases}$$

In addition, we define the following nonlinear operators:

$$G_{\varepsilon,\delta}(\tilde{u}) = \left(\mu|u(x)|^{q-1}u(x),\ \frac{\mu_0}{\varepsilon}|u(0)|^{q_0-1}u(0),\ \frac{\mu_1}{\delta}|u(1)|^{q_1-1}u(1)\right)$$

$$\varPhi_{\varepsilon,\delta}(\tilde{u}) = \left(B_2(x, u(x)),\ \frac{1}{\varepsilon}\, b_{20}(u(0)), \frac{1}{\delta}\, b_{21}(u(1))\right),$$

$$\tilde{u} = (u(x), u(0), u(1)) \in H_1.$$

By the definition of the scalar product in the space H and the definition of the operator $A_{\varepsilon,\delta}$ we have

$$< A_{\varepsilon,\delta}\, \tilde{u}, \tilde{v} > = -\int_0^1 u_{xx}(x) \cdot v(x)dx - \frac{1}{\varepsilon}\, \varepsilon u_x(0)v(0) + \frac{1}{\delta}\, \delta u_x(1)v(1)$$

$$= \int_0^1 u_x(x)v_x(x)dx = -\int_0^1 u(x)v_{xx}(x)dx - u(0)v_x(0) + u(1)v_x(1) = < \tilde{u}, A_{\varepsilon,\delta}\tilde{v} > .$$

Thus for any $\varepsilon > 0$ and $\delta > 0$ the linear operator $A_{\varepsilon,\delta}$ is symmetric.

Moreover, the operator $\widetilde{A}_{\varepsilon,\delta}$ is bounded below since

$$< \tilde{u}, A_{\varepsilon,\delta}\tilde{u} > = \int_0^1 |u_x(x)|^2 dx \geq 0.$$

For every $\tilde{h} = (h, \alpha, \beta) \in H$ we consider the equation

$$A_{\varepsilon,\delta}\, \tilde{u} + \tilde{u} = \tilde{h}.$$

Obviously, this equation is equivalent to the following problem for an ordinary differential equation with Robin boundary condition

$$-u_{xx}(x) + u(x) = h(x), \tag{2.1}$$

$$-\frac{1}{\varepsilon}u_x(0) + u(0) = \alpha, \tag{2.2}$$

$$\frac{1}{\delta}u_x(1) + u(1) = \beta. \tag{2.3}$$

Next, we note that for any $h(\cdot) \in L_2(0,1)$ $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\varepsilon > 0$ and $\delta > 0$ problem (2.1)-(2.3) has a solution $u \in W_2^2(0,1)$ (see e.g., [8]).

Thus, by using general theory, we get the following statements: (see [9-10])

Lemma 2.1 *For every $\varepsilon > 0$ and $\delta > 0$ the operator $A_{\varepsilon,\delta}$ is positive self-adjoint in the space $H = L_2(0,1) \oplus \mathbb{R} \oplus \mathbb{R}$.*

Lemma 2.2 *For every $\varepsilon > 0, \delta > 0$, $\mu \geq 0$, $\mu_0 \geq 0$ and $\mu_1 \geq 0$ the operator $G_{\varepsilon,\delta}$ is monotonic, acting from H_0 to H_0', where H_0' is a dual space to H_0.*

Proof. Let $\widetilde{u}_1, \widetilde{u}_2 \in H_0$ then

$$\begin{aligned}
&{}_{H_0}\langle G(\widetilde{u}_2) - G(\widetilde{u}_1), \widetilde{u}_2 - \widetilde{u}_1 \rangle_{H_0^1} \\
&= \int_0^1 \left[\mu\, |u_2(x)|^{q-1}\, u_2(x) - \mu\, |u_1(x)|^{q-1}\, u_1(x)\right] (u_2(x) - u_1(x))\, dx \\
&+ [\mu_0 |u_2(0)|^{q_0-1} u_2(0) - \mu_0 |u_1(0)|^{q_0-1}](u_2(0) - u_1(0)) \\
&+ [\mu_1 |u_2(1)|^{q_1-1} u_2(1) - \mu_1 |u_1(1)|^{q_1-1}](u_2(1) - u_1(1)) \geq 0.
\end{aligned}$$

Lemma 2.3 *The nonlinear operator $\Phi_{\varepsilon,\delta}(\cdot)$ acting from H_0 to H satisfies the local Lipschitz condition, i.e. for every $\|\tilde{w}_1\|_{H_0} \leq r, \|\tilde{w}_2\|_{H_0} \leq r$ the following inequality holds*

$$\|\Phi_{\varepsilon,\delta}(\,\cdot\,,\tilde{w}_1(\,\cdot\,)) - \Phi_{\varepsilon,\delta}(\,\cdot\,,\tilde{w}_2(\,\cdot\,))\|_H \leq c(r)\|\tilde{w}_2(\,\cdot\,) - \tilde{w}_1(\,\cdot\,)\|_{H_0},$$

where $c\,(\,\cdot\,) \in C(\mathbb{R}_+; \mathbb{R}_+)$.

Proof. Let $w_1, w_2 \in H_0$. Then we have

$$\|\Phi_{\varepsilon,\delta}(w_1) - \Phi_{\varepsilon,\delta}(w_2)\|_H = \left(\int_0^1 |B_2(x,u_2(x)) - B_2(x,u_1(x))|^2\, dx\right)^{1/2}$$
$$+ \varepsilon\left|\frac{1}{\varepsilon}[b_{20}(u_2(0)) - b_{20}(u_1(0))]\right| + \delta\left|\frac{1}{\delta}[b_{21}(u_2(1)) - b_{21}(u_1(1))]\right|$$
$$\leq \left(\int_0^1 c^2(|u_1(x)|,|u_2(x)|)\cdot|u_2(x) - u_1(x)|^2 dx\right)^{1/2}$$
$$+ c_0(|u_1(0)|,|u_2(0)|)|u_2(0) - u_1(0)| + c_1(|u_1(1)|,|u_2(1)|)|u_2(1) - u_1(1)|$$
$$\leq \max_{0\leq x\leq 1} c(|u_1(x)|,|u_2(x)|)\left(\int_0^1 |u_2(x) - u_1(x)|^2 dx\right)^{1/2}$$
$$+ c_0\left(|u_1(0)|,|u_2(0)|\right)|u_2(0) - u_1(0)| + c_1\left(|u_1(1)|,|u_2(1)|\right)|u_2(1) - u_1(1)|$$
$$\leq \tilde{c}\left(\|w_1\|_{H_0},\|w_2\|_{H_0}\right)\|w_2 - w_1\|_{H_0}.$$

3 Reducing a mixed problem with a dynamical boundary condition to a operator equation and application of theory of nonlinear semigroups

The problem (1.1)-(1.4) can be rewritten as a Cauchy problem in the space $H = L_2(0,1)\oplus \mathbb{R}\oplus\mathbb{R}$:

$$w''(t) + A_{\varepsilon,\delta}w(t) + G_{\varepsilon,\delta}(w'(t)) + \Phi_{\varepsilon,\delta}(w(t)) = F_{\varepsilon,\delta}(t), \tag{3.1}$$

$$w(0) = w_0, \;\; w'(0) = w_1, \tag{3.2}$$

where

$$F_{\varepsilon,\delta}(t,x) = \begin{pmatrix} f(t,x) \\ \frac{1}{\varepsilon}f_0(t) \\ \frac{1}{\delta}f_1(t)\end{pmatrix}, \;\; w_0 = \begin{pmatrix}\varphi(x)\\ \varphi(0)\\ \varphi(1)\end{pmatrix}, \;\; w_1 = \begin{pmatrix}\psi(x)\\ \psi(0)\\ \psi(1)\end{pmatrix}.$$

For investigating the problem (3.1), (3.2) we define the space $Z = H\left(A_{\varepsilon,\delta^{1/2}}\right)\times H$, where

$$H\left(A_{\varepsilon,\delta}^{1/2}\right) = \left\{\upsilon : \upsilon\in D\left(A_{\varepsilon,\delta^{1/2}}\right)\right\}, \;\; \|\upsilon\|_{H(A_{\varepsilon,\delta}^{1/2})} = \|A_{\varepsilon,\delta}^{1/2}\upsilon\|,$$

and $A_{\varepsilon,\delta}^{1/2}$ is a square root of the positive operator $A_{\varepsilon,\delta}$ defined by spectral decomposition [9].

Let us define the scalar product in the space Z as follows:

$$[z_1,z_2] = \left\langle A_{\varepsilon,\delta}^{1/2}u_1, A_{\varepsilon,\delta}^{1/2}u_2\right\rangle + <\upsilon_1,\upsilon_2>,$$

$$z_i = (u_i,\upsilon_i)\in Z, \;\; i = 1,2.$$

Using the substitutions $\upsilon_1 = w, \; \upsilon_2 = w_t$, we reduce problem (3.1), (3.2) to the problem

$$z'(t) + \mathcal{L}_{\varepsilon,\delta}z(t) + \tilde{G}_{\varepsilon,\delta}(z(t)) = \tilde{\Phi}_{\varepsilon,\delta}(z(t)) + \tilde{F}_{\varepsilon,\delta}(t,x), \tag{3.3}$$

$$z(0) = z_0, \tag{3.4}$$

where

$$z = \begin{pmatrix}\upsilon_1\\ \upsilon_2\end{pmatrix}, \; \mathcal{L}_{\varepsilon,\delta} = \begin{pmatrix} 0 & -I \\ \widetilde{A}_{\varepsilon,\delta} & 0\end{pmatrix}, \; D(\mathcal{L}_{\varepsilon,\delta}) = H(A_{\varepsilon,\delta})\times H\left(A_{\varepsilon,\delta}^{1/2}\right),$$

$$\tilde{G}_{\varepsilon,\delta}(z) = \begin{pmatrix} 0 \\ G_{\varepsilon,\delta}(v_2) \end{pmatrix}, \; \tilde{\Phi}_{\varepsilon,\delta}(z) = \begin{pmatrix} 0 \\ -\Phi_{\varepsilon,\delta}(v_1) \end{pmatrix},$$

$$\tilde{F}_{\varepsilon,\delta}(t,x) = \begin{pmatrix} 0 \\ F_{\varepsilon,\delta}(t,x) \end{pmatrix}, \; z_0 = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}.$$

From the definition of the linear operator $\mathcal{L}_{\varepsilon,\delta}$ it follows that $\mathcal{L}_{\varepsilon,\delta}$ is a maximal dissipative operator in the space Z. On the other hand, $\tilde{G}_\varepsilon$ is a monotone operator in Z. Therefore, the following statement is valid [9,10,14].

Lemma 3.1 *Let $\varepsilon > 0, \delta > 0, \mu \geq 0, \mu_0 \geq 0$ and $\mu_1 \geq 0$. Then $M_\varepsilon = \mathcal{L}_\varepsilon + \tilde{G}_\varepsilon$ is a maximal monotone operator in the functional space Z.*

Using Lemmas 1–4 and Theorem 3 of [7], we obtain the following result.

Theorem 3.1 *Let $\varepsilon > 0$, $\delta > 0$, $\mu \geq 0$, $\mu_0 \geq 0$, $\mu_1 \geq 0$ and $\tilde{F}_{\varepsilon,\delta}(\cdot) \in W_1^1(0,T;Z)$. Then for every $z_0 \in D\left(\mathcal{L}_{\varepsilon,\delta} + \tilde{G}_{\varepsilon,\delta}\right)$ the problem*

$$z'(t) + \mathcal{L}_{\varepsilon,\delta} z(t) + \tilde{G}_{\varepsilon,\delta} z(t) = \tilde{F}_{\varepsilon,\delta}(t,x), \tag{3.5}$$

$$z(0) = z_0, \tag{3.6}$$

has a unique solution

$$z = z_{\varepsilon\delta} \in W_\infty^1(0,T;Z)$$

such that $z_{\varepsilon\delta}(t) \in D(\tilde{A}_{\varepsilon,\delta} + \tilde{G}_{\varepsilon,\delta})$, $0 \leq t \leq T$. If $z_0 \in \overline{D(\mathcal{L}_{\varepsilon,\delta} + \tilde{G}_{\varepsilon,\delta})}$ and $\tilde{F}_{\varepsilon,\delta}(\cdot) \in L_1(0,T;Z)$ the the problem (3.5), (3.6) has a generalized solutions $z_{\varepsilon,\delta} \in C([0,T];Z)$.

Using Theorem 3.1, we can conclude results about solvability of the following problem

$$u_{tt} - u_{xx} + B_1(u_t) = f(t,x), \;\; t > 0, \;\; x \in (0,1). \tag{3.7}$$

$$\varepsilon u_{tt}(t,0) - u_x(t,0) + b_{10}(u_t(t,0)) = f_0(t), \;\; t > 0 \tag{3.8}$$

$$\delta u_{tt}(t,1) + u_x(t,1) + b_1(u_t(t,1)) = f_1(t), \;\; t > 0, \tag{3.9}$$

$$u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x), \; x \in (0,1). \tag{3.10}$$

Theorem 3.2 *Let, $\varepsilon > 0, \delta > 0, \mu \geq 0, \mu_0 \geq 0, \mu_1 \geq 0$ and $f(\cdot) \in W_1^1(0,T;W_2^1(0,1),$ $L_2(0,1))$, $f_0(t), f_1(t) \in W_1^1(0,T)$. Then for any $\varphi \in W_2^2(0,1), \psi \in W_2^1\,(0,1)$ and $T > 0$ the problem (3.7)-(3.10) has a unique solution*

$$u_{\varepsilon\delta}(\cdot) \in W_\infty^2(0,T;W_2^2(0,1), W_2^1(0,1), L_2(0,1))$$

such that $u_{\varepsilon\delta}(0,\cdot), u_{\varepsilon\delta}(1,\cdot) \in W_\infty^2(0,T;\mathbb{R})$.

Using Lemma 4 and applying theorem 7.2 of [8], we get the following statement:

Theorem 3.3 *Let $\varepsilon > 0, \delta > 0$, $\mu \geq 0$, $\mu_0 \geq 0$, $\mu_1 \geq 0$, $\tilde{F}_{\varepsilon\delta}(\cdot) \in W_1^1(0,T;Z))$, and conditions (1.6)-(1.8) are satisfied. Then for every $z_0 \in D\left(L_{\varepsilon\delta} + \tilde{G}_{\varepsilon\delta}\right)$ there exists $T' > 0$ such that problem (3.3)- (3.4) has unique solution $z = z_{\varepsilon\delta} \in W_\infty^1(0,T';Z)$ with $z_{\varepsilon\delta}(t) \in D\left(\tilde{A}_{\varepsilon\delta} + \tilde{G}_{\varepsilon\delta}\right)$. If $z_0 \in \overline{D(\mathcal{L}_{\varepsilon\delta} + \tilde{G}_{\varepsilon\delta})}$ and $\tilde{F}_{\varepsilon\delta}(\cdot) \in L_1(0,T';Z)$, then the problem (3.3), (3.4) has a generalized solution $z_{\varepsilon\delta} \in C([0,T];Z)$. Moreover, if for the existence of the local solution $T_{\max} > 0$ is a maximum interval then one of the following alternatives is true*

1. $\lim\limits_{t \to T_{\max}} \|z_{\varepsilon\delta}(t)\|^2 = +\infty,$
2. $T_{\max} = +\infty.$

Using Theorem 3.3 for the existence of the local solutions of (1.1)-(1.4) we obtain the following result.

Theorem 3.4 *Let* $\varepsilon > 0$, $\delta > 0$, $\mu \geq 0$, $\mu_0 \geq 0$, $\mu_1 \geq 0$, $f(\cdot) \in W_1^1(0,T; W_2^1(0,1), L_2(0,1))$, $f_0(t), f_1(t) \in W_1^1(0,T)$, *and conditions (1.6)-(1.8) are satisfied. Then for any* $\varphi \in W_2^2(0,1), \psi \in W_2^1(0,1)$ *there exists* $T' > 0$ *such that problem (3.7)-(3.10) has a unique solution* $u_{\varepsilon\delta} \in W_\infty^2(0,T'; W_2^2(0,1), W_2^1(0,1), L_2(0,1))$ *with* $u_{\varepsilon\delta}(0,t), u_{\varepsilon\delta}(1,t) \in W_\infty^2(0,T;\mathbb{R})$. *Moreover, if* $T_{\max} > 0$ *is the length of maximal interval in which the local solution exists, then one of the following alternatives is true*

1. $\lim\limits_{t\to T_{\max}} \left\{ \|u_{\varepsilon\delta t}(t,\cdot)\|^2 + \|u_{\varepsilon\delta x}(t,\cdot)\|^2 + |u_{\varepsilon\delta t}(t,0)| + |u_{t\varepsilon\delta}(t,1)| \right\} = +\infty$,
2. $T_{\max} = +\infty$.

4 Global solvability of the mixed problem for a one-dimensional nonlinear wave equation with dynamic boundary conditions

In this section we will study the problem (1.1)-(1.4) in the case when the following conditions are satisfied:

$$B_2(x,u) = \eta|u|^{p-1}u,\ \eta \geq 0,\ p > 0, \tag{4.1}$$

$$b_{20}(\xi) = \eta_0|\xi|^{p_0-1}\xi,\ \eta_0 \geq 0,\ p_0 > 0, \tag{4.2}$$

$$b_{21}(\xi) = \eta_1|\xi|^{p_1-1}\xi,\ \eta_1 \geq 0,\ p_1 > 0. \tag{4.3}$$

In this case we can prove the existence and uniqueness of global solution of problem (1.1)-(1.4).

Theorem 4.1 *Assume that,* $\varepsilon > 0$, $\delta > 0$, $\mu \geq 0$, $\mu_0 \geq 0, \mu_1 \geq 0$, $f(\cdot) \in W_1^1(0,T; W_2^1(0,1), L_2(0,1))$, $f_0(t), f_1(t) \in W_1^1(0,T)$ *and conditions (4.1)-(4.3) are satisfied. Then for any* $\varphi \in W_2^2(0,1)$ *and* $\psi \in W_2^1(0,1)$ *the problem(1.1)-(1.4) has a unique solution* $u_{\varepsilon\delta} \in W_\infty^2(0,T; W_2^2(0,1), W_2^1(0,1), L_2(0,1))$ *so that* $u_{\varepsilon\delta}(0,t), u_{\varepsilon\delta}(1,t) \in W_\infty^2(0,T;\mathbb{R})$.

Proof. By virtue of Theorem 3.4, in order to prove the theorem it is sufficient for $u_{\varepsilon\delta}(\,\cdot\,)$ to obtain an a priori estimate

$$\|u_{\varepsilon\delta t}(t,\cdot)\|^2 + \|u_{\varepsilon\delta x}(t,\cdot)\|^2 + |u_{\varepsilon\delta t}(t,0)|^2 + |u_{\varepsilon\delta t}(t,1)|^2 \leq c. \tag{4.4}$$

First, we multiply both sides of (3.1) by $u_{\varepsilon\delta t}(t,x)$ and integrate in $[0,t] \times (0,1)$. Then, we multiply both sides of (3.3) by $u_{\varepsilon\delta t}(t,0)$ and $u_{\varepsilon\delta t}(t,1)$, respectively, and integrate the resulting equalities in $[0,t]$. Next, by applying the integration by parts and using the Hölder inequality one can get

$$\frac{1}{2}\int_0^1 |u_{\varepsilon\delta t}(t,x)|^2 dx + \frac{1}{2}\int_0^1 |u_{\varepsilon\delta x}(t,x)|^2 dx$$
$$+\,\mu\int_0^1 |u_{\varepsilon\delta s}(s,x)|^{q_0+1} dxds + \mu_0\int_0^t |u_{\varepsilon\delta s}(s,0)|^{q_0+1} ds$$
$$+\,\mu_1\int_0^t |u_{\varepsilon\delta s}(s,1)|^{q_1+1} ds + \frac{\eta}{p+1}\int_0^1 |u_{\varepsilon\delta}(t,x)|^{p+1} dx$$
$$+\,\frac{\eta_0}{p_0+1}|u_{\varepsilon\delta}(t,0)|^{p_0+1} + \frac{\eta_1}{p_1+1}|u_{\varepsilon\delta}(t,1)|^{p_1+1}$$
$$\le \frac{1}{2}|\psi(x)|^2 dx + \frac{1}{2}\int_0^1 |\varphi_x(x)|^2 dx$$
$$+\,\frac{\eta}{p+1}\int_0^1 |\varphi(x)|^{p+1} dx + \frac{\eta_0}{p_0+1}|\varphi(0)|^{p_0+1}$$
$$+\,\frac{\eta_1}{p_1+1}|\varphi(1)|^{p_1+1} + \frac{\varepsilon}{2}|\psi(0)|^2 + \frac{\varepsilon}{2}|\psi(1)|^2$$
$$+\,\frac{1}{2}\int_0^1 |f(t,x)|^2 dxds + \frac{1}{2}|f_0(t)|^2 dt + \frac{1}{2}\int_0^T |f_1(t)|^2 dt$$
$$+\,\frac{1}{2}\int_0^T\int_0^1 |u_{\varepsilon_s}(s,x)|^2 dxds + \frac{1}{2}\int_0^T |u_\varepsilon(s,0)|^2 ds + \frac{1}{2}\int_0^T u_\varepsilon(s,1)|^2 ds.$$

Using the Gronwalls inequality from this relation we get a priori estimation (4.9).

5 Mixed problem in the linear case of dissipative term

In this section we will consider problem (1.1)-(1.4) in linear dissipation case, more precisely, we consider the mixed problem

$$u_{tt} - u_{xx} + u_t + B_2(u) = f(t,x),\ x\in(0,1),\ t>0,\ 0\le t\le T, \tag{5.1}$$

$$\varepsilon u_{tt}(t,0) - u_x(t,0) + u_t(t,0) + b_{20}(u(t,0)) = f_0(t),\ 0\le t\le T, \tag{5.2}$$

$$\varepsilon u_{tt}(t,1) + u_x(t,1) + u_t(t,1) + b_{21}(u(t,1)) = f_1(t),\ 0\le t\le T, \tag{5.3}$$

$$u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),\ x\in(0,1), \tag{5.4}$$

where $B_2(s) = \eta|s|^{p-1}s, b_{20}(s) = \eta_0|s|^{p_0-1}s,\ b_{21}(s) = \eta_1|s|^{p_1-1}s, f(t,x)\in W_2^1((0,T)\times(0,1)), f_0(t), f_1(t)\in W_2^1(0,T)$.

Acting as in the previous section, we reduce problem (5.1)-(5.4) in the space $H = L_2(0,1)\oplus\mathbb{R}\oplus\mathbb{R}$ to the following Cauchy problem:

$$\tilde{u''}(t) + \Im_\varepsilon\tilde{u}(t) + \tilde{u}'(t) + \Re_\varepsilon(\tilde{u}(t)) = F_\varepsilon(t) \tag{5.5}$$

$$\tilde{u}(0) = \tilde{u}_0,\ \tilde{u}'(0) = \tilde{u}_1, \tag{5.6}$$

where

$$\tilde{u} = (u(x), u(0), u(1)) \in D(\Im_\varepsilon) = H_1,$$

$$\Im_\varepsilon\tilde{u} = A_{\varepsilon\varepsilon} = \left(-u_{xx}(x), \frac{1}{\varepsilon}u_x(0) - \frac{1}{\varepsilon}u_x(1)\right),$$

$$\Re_\varepsilon(\tilde{u}) = \Phi_{\varepsilon\varepsilon}(\tilde{u}) = \left(\eta|u(x)|^{p-1}u(x), \frac{\eta_0}{\varepsilon}|u(0)|^{p_0-1}u(0), \frac{\eta_1}{\varepsilon}|u(1)|^{p_1-1}u(1)\right),$$

$$F_\varepsilon(t) = \left(f(t,\cdot), \frac{1}{\varepsilon} f_0(t), \frac{1}{\varepsilon} f_1(t) \right),$$

$$\tilde{u}_0 = (\varphi(x), \varphi(0), \varphi(1)),\ \tilde{u}_1 = (\psi(x), \psi(0), \psi(1)).$$

Using Lemmas 3.1-3.4 and [7] for problem (5.1)-(5.4) we get the following result.

Theorem 5.1 *Assume $\varepsilon > 0,\ f(t,x) \in W_2^1((0,T) \times (0,1)),\ f_0(\cdot),\ f_1(\cdot) \in W_2^1(0,T)$. Then for any $\varphi \in W_2^2(0,1)$, $\psi \in W_2^1(0,1)$ there exists $T' > 0$ such that problem (5.1)-(5.4) has a unique solution $u_\varepsilon(\cdot) \in C^2([0,T']; W_2^2(0,1), W_2^1(0,1), L_2(0,1))$ such that $u_\varepsilon(0,t), u_\varepsilon(1,t) \in C^2([0,T'];\mathbb{R})$. If $T_{\max}$ is the length of the maximum interval for the existence of the solution, then the one of the following alternatives is true:*

1. $\lim\limits_{t\to T_{\max}} \left[\|\tilde{u}_{\varepsilon t}(t)\|_H^2 + \|\tilde{u}_\varepsilon(t)\|_{H_1} \right] = +\infty$,
2. $T_{\max} = +\infty$.

Theorem 5.2 *Assume $\varepsilon > 0, \eta_1 \geq 0, \eta_2 \geq 0,\ f(t,x) \in W_2^1((0,T) \times (0,1))$, $f_0(\cdot), f_1(\cdot) \in W_2^1(0,T)$. Then for any $\varphi \in W_2^2(0,1)$, $\psi \in W_2^1(0,1)$ the problem (5.1)-(5.4) has a unique solution $u_\varepsilon(\cdot) \in C^2([0,T]; W_2^2(0,1), W_2^1(0,1), L_2(0,1))$ such that $u_\varepsilon(0,t), u_\varepsilon(1,t) \in C^2([0,T],\mathbb{R})$.*

6 Mixed problem of nonlinear wave equation with quasi-statical boundary condition

In this section, we will investigate the solvability of the problem (5.1)-(5.4) in the case when the dynamic boundary conditions degenerate to quasi-statical boundary conditions.

We consider the following mixed problem

$$u_{tt} - u_{xx} + u_t + B_2(u) = f(t,x),\ x \in (0,1), t \in [0,T], \tag{6.1}$$

$$-u_x(t,0) + u_t(t,0) + b_{20}(u(t,0)) = f_0(t),\ t \in [0,T], \tag{6.2}$$

$$u_x(t,1) + u_t(t,1) + b_{21}(u(t,1)) = f_1(t),\ t \in [0,T], \tag{6.3}$$

$$u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),\ x \in (0,1), \tag{6.4}$$

where $B_2(s) = \eta|s|^{p-1}s, b_{20}(s) = \eta_0|s|^{p_0-1}s, b_{21}(s) = \eta_1|s|^{p_1-1}s$.

Theorem 6.1 *Assume $\eta \geq 0,\ \eta_0 \geq 0,\ \eta_1 \geq 0,\ f(t,x) \in W_2^1((0,T) \times (0,1)),\ f_0(\cdot)$, $f_1(\cdot) \in W_2^1(0,T)$. Then for any $\varphi \in W_2^2(0,1), \psi \in W_2^1(0,1)$ problem (6.1)-(6.4) has a unique solution $u(\cdot)$ such that*
$u(\cdot) \in L_\infty(0,T; W_2^2(0,1))$, $u_t(\cdot) \in L_\infty(0,T; W_2^1(0,1))$,
$u_{tt}(\cdot) \in L_\infty(0,T; L_2(0,1))$, $u_t(0,\cdot), u_t(1,\cdot), u_x(0,\cdot), u_t(1,\cdot) \in L_2(0,T)$.

Proof. Assume that $\varepsilon > 0$. Then by Theorem 5.2 problem (5.1)-(5.4) has a unique solution $u_\varepsilon(\cdot) \in C^2([0,T]; W_2^2(0,1), W_2^1(0,1), L_2(0,1))$ such that $u_\varepsilon(0,\cdot)$, $u_\varepsilon(1,\cdot) \in C^2([0,T],\mathbb{R})$.

Multiply both sides of (5.1) by $u_{\varepsilon t}(t,x)$ and integrate in $[0,t] \times [0,1]$. In the similar way, multiply both sides of (5.2) by $u_{\varepsilon t}(0,t)$ and both sides of (5.3) by $u_{\varepsilon t}(1,t)$, and integrate the resulting relations in $[0,t]$. Then by applying the integration by parts we get

$$\frac{1}{2}\int_0^1 |u_{\varepsilon t}(t,x)|^2 dx + \frac{1}{2}\int_0^1 |u_{\varepsilon x}(t,x)|^2 dx$$

$$+\int_0^t\int_0^1 |u_{\varepsilon s}(s,x)|^2dxds+\frac{\eta}{p+1}\int_0^1 |u_\varepsilon(s,x)|^{p+1}dx$$
$$+\int_0^t |u_{\varepsilon s}(s,0)|^2ds+\frac{\varepsilon}{2}|u_{\varepsilon t}(t,0)|^2$$
$$+\int_0^t |u_{\varepsilon s}(s,1)|^2ds+\frac{\varepsilon}{2}|u_{\varepsilon t}(t,1)|^2+\frac{\eta_0}{p_0+1}|u_\varepsilon(t,0)|^{p_0+1}$$
$$+\frac{\eta_1}{p_1+1}|u_\varepsilon(t,1)|^{p_1+1}=\frac{1}{2}\int_0^1 |\varphi'_x(x)|^2dx+\frac{1}{2}\int_0^1 |\psi(x)|^2dx$$
$$+\ \frac{\eta}{p+1}\int_0^1 |\varphi(x)|^{p+1}dx\ +\ \frac{\varepsilon}{2}|\psi(1)|^2+\frac{\eta_0}{p_0+1}|\varphi(0)|^{p_0+1}+\frac{\eta_1}{p_1+1}|\varphi(1)|^{p_1+1}.$$

Hence we have the following a priori estimation

$$\int_0^1 |u_{\varepsilon t}(t,x)|^2dx\le c,\ 0\le t\le T, \tag{6.5}$$

$$\int_0^1 |u_{\varepsilon x}(t,x)|^2dx\le c,\ 0\le t\le T, \tag{6.6}$$

$$\int_0^1 |u_\varepsilon(t,x)|^{p+1}dx\le c,\ 0\le t\le T, \tag{6.7}$$

$$\int_0^t\int_0^1 |u_{\varepsilon_s}(s,x)|^2dxds\le c,\ 0\le t\le T, \tag{6.8}$$

$$\int_0^t |u_{\varepsilon s}(s,0)|^2ds\le c,\ 0\le t\le T, \tag{6.9}$$

$$\int_0^t |u_{\varepsilon s}(s,1)|^2ds\le c,\ 0\le t\le T, \tag{6.10}$$

$$\varepsilon\,|u_{\varepsilon t}(t,0)|^2\le c,\ \varepsilon\,|u_{\varepsilon t}(t,1)|^2\le c,\ 0\le t\le T, \tag{6.11}$$

$$|u_\varepsilon(t,0)|\le c,\ |u_\varepsilon(t,1)|\le c,\ 0\le t\le T, \tag{6.12}$$

where the constant $c>0$ is independent of $\varepsilon\in(0,1)$.

Since $u_\varepsilon\in C^2([0,T];W_2^2(0,1),W_2^1(0,1),L_2(0,1))$ it follows from (5.1) that

$$\int_0^1 |u_{\varepsilon_{tt}}(0,x)|^2dx\le c\left[\int_0^1 |\varphi_{xx}(x)|^2dx\right.$$
$$\left.+\int_0^1 |\varphi(x)|^{2p}dx+\int_0^1 |\psi(x)|^2dx+\int_0^1 |f(0,x)|^2dx\right].$$

Hence we get

$$\int_0^1 |u_{\varepsilon_{tt}}(0,x)|^2dx\le c_1, \tag{6.13}$$

where $c_1>0$ is independent of $\varepsilon\in(0,1)$.

By (5.2) we have

$$\varepsilon\,|u_{\varepsilon_{tt}}(0,0)|\le|\varphi_x(0)|+|\psi(0)|+\eta_0|\varphi(0)|^{p_0}+|f_0(0)|\le c_2. \tag{6.14}$$

In the same way, from (5.3) we get

$$\varepsilon \left|u_{\varepsilon tt}(0,1)\right| \leq |\varphi_x(1)| + |\psi(1)| + \eta_1|\varphi(1)|^{p_1} + |f_1(0)| \leq c_2, \tag{6.15}$$

so, $c_2 > 0$ is also independent of $\varepsilon \in (0,1)$.

Denoting $y_h(t,x) = \frac{1}{h}[u_\varepsilon(t+h,x) - u_\varepsilon(t,x)]$ from (5.1)-(5.4) we obtain

$$\frac{1}{2}\int_0^1 |y_{h_t}(t,x)|^2dx + \frac{1}{2}\int_0^1 |y_{h_x}(t,x)|^2dx$$

$$+\int_0^t\int_0^1 |y_{h_s}(s,x)|^2dxds + \int_0^t y_{h_x}(s,0)y_{h_s}(s,0)ds - \int_0^t y_{h_x}(s,1)y_{h_s}(s,1)ds$$

$$+\eta_0\int_0^t\int_0^1 \frac{1}{h}\left[B(u(s+h,x)) - B(u(s,x))\right] y_{h_s}(s,x)dxds$$

$$= \frac{1}{2}\int_0^1 |y_{h_t}(0,x)|^2dx + \frac{1}{2}\int_0^1 |y_{h_x}(0,x)|^2dx + \int_0^t\int_0^1 f_h(s,x)y_{h_s}(s,x)dxds. \tag{6.16}$$

Using the Lagrange's mean value theorem and a priori estimation (6.5), (6.12) we have

$$\left|\int_0^t\int_0^1 \frac{1}{h}[B(u(s+h)) - B(u(s,x))]\, y_{h_s}(s,x)dxds\right|$$

$$\leq c\int_0^t\int_0^1 \left(|u(s+h,x)|^p + |u(s,x)|^p\right)|y_h(s,x)|\,|y_{h_s}(s,x)|\,dxds$$

$$\leq c_1\int_0^t\int_0^1 \left[|y_h(s,x)|^2\cdot|y_{h_s}(s,x)|^2\right]dxds. \tag{6.17}$$

From (6.16) and (6.17) we get

$$\frac{1}{2}\int_0^1 |y_{h_s}(s,x)|^2dx + \frac{1}{2}\int_0^1 |y_{h_x}(s,x)|^2dx + \int_0^t\int_0^1 |y_{h_s}(s,x)|^2dxds$$

$$+\int_0^t y_{h_x}(s,0)y_{h_s}(s,0)ds - \int_0^t y_{h_x}(s,1)y_{h_s}(s,1)ds \leq \frac{1}{2}\int_0^1 |y_{h_t}(0,x)|^2dx$$

$$+\frac{1}{2}\int_0^1 |y_{h_x}(0,x)|^2dx + c_2\int_0^t\int_0^1 |y_h(s,x)|^2dxds$$

$$+c_3\int_0^t\int_0^1 |y_{h_s}(s,x)|^2dxds + \int_0^t\int_0^1 |f_h(s,x)|^2dxds. \tag{6.18}$$

In the similar way from (5.2) and (5.3) we obtain

$$\frac{\varepsilon}{2}|y_{h_t}(t,0)|^2 + \int_0^t |y_{h_s}(s,0)|^2ds - \int_0^t y_{h_x}(s,0)y_{h_s}(s,0)ds \leq \frac{\varepsilon}{2}\,|y_{h_t}(0,0)|^2$$

$$+\alpha\int_0^t |y_h(s,0)|^2ds + \frac{1}{\alpha}\int_0^t |f_{0h}(s)|^2ds; \tag{6.19}$$

$$\frac{\varepsilon}{2}|y_{h_t}(t,1)|^2 + \int_0^t |y_{h_s}(s,1)|^2ds + \int_0^t y_{h_x}(s,1)y_{h_s}(s,1)ds \leq \frac{\varepsilon}{2}\,|y_{h_t}(0,1)|^2$$

$$+\alpha\int_0^t |y_{h_s}(s,1)|^2 ds + \frac{1}{\alpha}\int_0^t |f_{1h}(s)|^2 ds\ , \tag{6.20}$$

where $\alpha > 0$.

Summing (6.16) and (6.20) we get

$$\frac{1}{2}\int_0^1 |y_{h_t}(t,x)|^2 dx + \frac{1}{2}\int_0^1 |y_{h_x}(t,x)|^2 dx$$
$$+\int_0^t\int_0^1 |y_{h_s}(s,x)|^2 dxds + \frac{\varepsilon}{2}|y_h(t,0)|^2 + \frac{\varepsilon}{2}|y_h(t,1)|^2$$
$$+\int_0^t |y_{h_s}(s,0)|^2 ds + \int_0^t |y_{h_s}(s,1)|^2 ds \le \frac{1}{2}\int_s^1 |y_{h_t}(0,x)|^2 dx$$
$$+\frac{1}{2}\int_s^1 |y_{h_x}(0,x)|^2 dx + c_2\int_0^t\int_0^1 |y_h(s,x)|^2 dxds$$
$$+c_3\int_0^t\int_0^1 |y_{h_s}(s,x)|^2 dxds + \frac{\varepsilon}{2}|y_{h_t}(0,1)|^2 + 2\alpha\int_0^t |y_h(s,0)|^2 ds$$
$$+2\alpha\int_0^t |y_h(s,1)|^2 ds + \frac{1}{\alpha}\int_0^t |f_{0h}(s)|^2 ds + \frac{1}{\alpha}\int_0^t |f_{1h}(s)|^2 ds. \tag{6.21}$$

On the other hand, we have

$$\int_0^1 |y_h(0,x)|^2 dx = \frac{1}{h^2}\int_0^1 |u(h,x)-u(0,x)|^2 dx = \int_0^1 |u_t'(\theta h,x)|^2 dx, \tag{6.22}$$

where $0 < \theta < 1$.

By a priori estimation (6.5) it follows from (6.22) that

$$\int_0^1 |y_h(0,x)|^2 dx \le c, \quad h > 0, \tag{6.23}$$

where the constant c is independent of h. According (5.2), (5.3) and (6.12) we have the estimations

$$\varepsilon\,|y_{h_t}(0,1)|^2 \le c, \quad \varepsilon\,|y_{h_t}(1,1)|^2 \le c, \tag{6.24}$$

where the constant $c > 0$ is independent of $h > 0$ and $\varepsilon \in (0,1)$.

On the other hand $u_{\varepsilon xt}(\cdot) \in C([0,T], L_2(0,1))$, and therefore

$$\lim_{h\to 0}\int_0^1 |y_{h_x}(0;x)|^2 dx = \lim_{h\to 0}\int_0^1 \left|\int_0^1 u_{\varepsilon xt}(\tau h,x)d\tau\right|^2 dx$$
$$\le c\lim_{h\to 0}\int_0^1\int_0^1 |u_{\varepsilon xt}(\tau h,x)|^2 dxd\tau = c\int_0^1 |\psi_x'(x)|^2 dx \le c. \tag{6.25}$$

According to a priori estimation (6.8) we have

$$\lim_{h\to 0}\int_0^t\int_0^1 |y_h(s;x)|^2 dxds \le \int_0^t\int_0^1 |u_{\varepsilon_s}(s;x)|^2 dxds \le c. \tag{6.26}$$

In the same way we get the following equalities:

$$\lim_{h\to 0}\int_0^1 |y_{h_t}(t;x)|^2 dx = \int_0^1 |u_{\varepsilon tt}(t;x)|^2 dx, \tag{6.27}$$

$$\lim_{h\to 0}\int_0^1 |y_{h_x}(t;x)|^2 dx = \int_0^1 |u_{\varepsilon xt}(t;x)|^2 dx, \tag{6.28}$$

$$\lim_{h\to 0}\int_0^t\int_0^1 |y_{h_s}(s;x)|^2 dxds = \int_0^t\int_0^1 |u_{\varepsilon ss}(s;x)|^2 dxds. \tag{6.29}$$

On the other hand,

$$\lim_{h\to 0}\int_0^t |f_{0h}(s)|^2 ds = \int_0^t |f_0'(s)|^2 ds \le c, \tag{6.30}$$

$$\lim_{h\to 0}\int_0^t |f_{1h}(s)|^2 ds = \int_0^t |f_1'(s)|^2 ds \le c, \tag{6.31}$$

$$\lim_{h\to 0}\int_0^t\int_0^1 |f_h(s,x)|^2 dxds = \int_0^t\int_0^1 |f_s'(s,x)|^2 dxds \le c. \tag{6.32}$$

Taking into account (6.23)-(6.32) in (6.21), and passing to the limit as $h\to 0$, we obtain

$$\int_0^1 |u_{\varepsilon tt}(t,x)|^2 dx + \int_0^1 |u_{\varepsilon xt}(t,x)|^2 dx + \varepsilon|u_{\varepsilon tt}(t,0)|^2$$

$$\le +\varepsilon\,|u_{\varepsilon tt}(t,1)|^2 \le c_{10} + c_{11}\int_0^t\int_0^1 |u_{\varepsilon_{ss}}(s,x)|^2 dxds.$$

Using Gronwalls inequality, we get the following a priori estimation

$$\int_0^1 |u_{\varepsilon_{tt}}(t,x)|^2 dx + \int_0^1 |u_{\varepsilon xt}(t,x)|^2 dx$$

$$+\varepsilon|u_{\varepsilon tt}(t,0)|^2 + \varepsilon|u_{\varepsilon tt}(t,1)|^2 \le c, \tag{6.33}$$

where the constant $c > 0$ is independent of $\varepsilon \in (0,1)$.

It follows from (5.1) and (6.33) the following a priori estimation

$$\int_0^1 |u_{\varepsilon xx}(t,x)|^2 dx \le c\left[\int_0^1 |u_{\varepsilon tt}(t,x)|^2 dx + \int_0^1 |u_{\varepsilon t}(t,x)|^2 dx\right.$$

$$\left. + \int_0^1 |B_2(u_\varepsilon)|^2 dx + \int_0^1 |f(t,x)|^2 dx\right] \le c. \tag{6.34}$$

By virtue of a priori estimations (6.33) and (6.34) from $\{u_\varepsilon(t,x)\}$ we can choose a subsequence $\{u_{\varepsilon_k}(t,x)\}$ such that as $\varepsilon_k \to 0$

$$u_{\varepsilon_k} \to u \quad * - \text{weakly in} \quad L_\infty(0,T;W_2^2(0,1)), \tag{6.35}$$

$$u_{\varepsilon_k t} \to u_t \quad * - \text{weakly in} \quad L_\infty(0,T;W_2^1(0,1)), \tag{6.36}$$

$$u_{\varepsilon_k tt} \to u_{tt} \quad * - \text{weakly in} \quad L_\infty(0,T;L_2((0,1))) \tag{6.37}$$

(see [11]).

From (6.36), (6.37) we have:

$$u_{\varepsilon_{kt}} \to u_t \text{ in } C([0,1]:L_2(0,T)) \tag{6.38}$$

(see [11]). In addition, we also have

$$u_{\varepsilon t}(t,0) \to u_t(t,0) \quad \text{in} \quad L_2(0,T), \tag{6.39}$$

$$u_{\varepsilon t}(t,1) \to u_t(t,1) \ \text{ in } \ L_2(0,T). \tag{6.40}$$

In the similar way, we get

$$u_{\varepsilon_k x}(t,1) \to u_x(t,1) \ \text{ in } \ L_2(0,T), \tag{6.41}$$

$$u_{\varepsilon_k x}(t,1) \to u_x(t,1) \ \text{ in } \ L_2(0,T) \tag{6.42}$$

It follows from (6.33) that

$$\varepsilon_k u_{\varepsilon_k tt}(t,0) \to 0 * -\text{ weakly } \text{ in } \ L_\infty(0,T), \tag{6.43}$$

$$\varepsilon_k u_{\varepsilon_k tt}(t,1) \to 0 \ * -\text{ weakly } \text{ in } \ L_\infty(0,T). \tag{6.44}$$

Now, having written problem (5.1)-(5.4) for $\varepsilon = \varepsilon_k$, we pass to the limit as $\varepsilon_k \to 0$. Then allowing for (6.35)-(6.44) we get that the limit function $u(t,x)$ is a solution of problem (6.1)-(6.4). The proof of this theorem is complete.

References

1. Durant, G., Lions, J.L.: *Inequalities in Mechanics and Physics*, Berlin, Springer-Verlag, (1976).
2. Long, N.T., Ut, L.V., Truc, N.T.: *On a shock problem involving a linear viscoelastic bar*, Nonl. Anal. **63**, 198–224 (2015).
3. Long, N.T., Diem, T.N.: *On the nonlinear wave equation* $u_{tt} - u_{xx} = f(x,t,u,u_x,u_t)$ *associated with the mixed homogeneous conditions*, Nonl. Anal. **29**, 1217-1230 (1997).
4. Long, N.T., Dinh, A.P.N., Diem, T.N.: *Linear recursive schemes and asymptotic expansion associated with the Kirchoff-Carrier operator*, J. Math. Anal.Appl. **267**, 116–134 (2002).
5. Nguyen, Thanh, Long: *On the nonlinear wave equation* $u_{tt} - B(t, |u_x|^2 u_{xx}) = f(x,t,u,u_x,u_t)$ *associated with the mixed homogeneous conditions*, J. Math. Anal.Appl. **274** (1), 102–123 (2002).
6. Aliev, A.B., Rzayeva, N.A.: *The mixed problem for nonlinear Tymoshenko systems*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics **38** (1), 13–21 (2018).
7. Showalter, R.E.: *Monotone operators in Banach spaces and nonlinear partial differential equations*, Mathematical Surveys and Monographs, AMS, **49** (1997).
8. Chueshov, I., Eller, M., Lasiecka, I.: *On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation*, Comm. Part. Diff. Equat. **27**, 1901–1951 (2002).
9. Lumer G., Phillips R.: *Dissipative operators in a Banach space*, Pacific J. Math. **11** (2), 679–698 (1961).
10. Kato, T.: *Perturbation Theory for linear operators*, Springer-Verlag, Berlin Heidelberg (1966).
11. Krein, S.Q.: *Linear differential equation in Banach space*, Springer, Boston (1982).
12. Lions, J.L., Majenes, E.: *Non-homogeneous Boundary value Pronlems*, Springer, Berlin, Heidelberg, New York (1972).
13. Barbu, V.: *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff, Leyden (1976).
14. Brezis, H.: *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam (1973).