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The Cech homology theory in the category of soft topological spaces

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Abstract. *In this study, by using the covering of soft topological spaces, an inverse system of simplicial complexes is constituted. Built upon these inverse system of simplicial complexes, Cech homology [cohomology] groups of soft topological spaces are defined. It is proved that Cech homology groups constitue a functor from the category of soft topological spaces to the category of groups. Later, axioms of homology theory are checked for this homology groups.*

Keywords. simplicial complexes, homology theory, homotopy theory.

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1 INTRODUCTION

After the introduction of soft set theory by Russian researcher D. Molodtsov [15], is one of the branches of mathematics, which aims to describe phenomena and concepts of an ambiguous, vague, undefined and imprecise meaning, Shabir and Naz [17] firstly initiated the concept of soft topological spaces which are defined over an initial universe with a fixed set of parameters. Theoretical studies of soft topological spaces have also been researched by some authors in [3,5–8,18]. The theories presented differently from classical methods in studies such as fuzzy set [19], intuitionistic set [10], etc. After the introduction of fuzzy sets by Zadeh [19], Chang [9] first introduced the concept of fuzzy topology. The category of fuzzy topological spaces is an extension of the category of topological spaces. It is known that in the investigation of topological spaces an important place is taken by methods of the algebraic topology. Methods of algebraic topology aren't used on fuzzy topology alot. In our opinion, such a situation was due to the definition of a fuzzy unit interval. In the category of soft topological spaces, methods of algebraic topology were first

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used in [4]. By defining a soft unit interval, Bayramov et al. defined soft singular cubic homology groups of soft topological spaces.

In this study, we define Cech homology [cohomology] groups in the category of pairs of soft topological space for these groups and it is proved that the axioms of dimension and excision of homology groups hold true. Finally, the axiom of homotopy for homology groups is proved.

2 PRELİMİNARY

From the literature, we will give some definitions and results for the development of soft topological spaces which will be needed in this study.

Definition 2.1 *[15] Let X be an initial universe, E be a set of all parameters and $P(X)$ denotes the power set of X. A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \to P(X)$.*

In other words, the soft set is a parameterized family of subsets of the set X. For $e \in E$, $F(e)$ may be conor as the set of $e-$approximate elements of the soft set, i.e.,

$$(F, E) = \{(e, F(e)) : e \in E,\ F : E \to P(X)\}.$$

Definition 2.2 *[14] A soft set (F, E) over X is said to be a null soft set denoted by Φ, if for all $e \in E$, $F(e) = \varnothing$.*

Definition 2.3 *[14] A soft set (F, E) over X is said to be an absolute soft set denoted by $\widetilde{X}$, if for all $e \in E$, $F(e) = X$.*

Definition 2.4 *[1] For two soft sets (F, E) and (G, E) over X, (F, E) is called a soft subset of (G, E), if $\forall e \in E$, $F(e) \subseteq G(e)$. This relationship is denoted by $(F, E)\widetilde{\subseteq}(G, E)$.*

Similarly, (F, E) is called a soft superset of (G, E) if (G, E) is a soft subset of (F, E). This relationship is denoted by $(F, E)\widetilde{\supseteq}(G, E)$. Two soft sets (F, E) and (G, E) over X are called soft equal, if (F, E) is a soft subset of (G, E) and (G, E) is a soft subset of (F, E).

Definition 2.5 *[1] The intersection of two soft sets (F, E) and (G, E) over X is the soft set (H, E), where $\forall e \in E$, $H(e) = F(e) \cap G(e)$. This is denoted by $(F, E)\widetilde{\cap}(G, E) = (H, E)$.*

Definition 2.6 *[1] The union of two soft sets (F, E) and (G, E) over X is the soft set (H, E), where $\forall e \in E$, $H(e) = F(e) \cup G(e)$. This is denoted by $(F, E)\widetilde{\cup}(G, E) = (H, E)$.*

Definition 2.7 *[17] The complement of a soft set (F, E), denoted by $(F, E)^c$, is defined $(F, E)^c = (F^c, E)$, where $F^c : E \to P(X)$ is a mapping given by $F^c(e) = X \backslash F(e)$, $\forall e \in E$.*

Definition 2.8 *([6],[11]) Let (F, E) be a soft set over X. The soft set (F, E) is called a soft point, denoted by (x_e, E), if for the element $e \in E$, $F(e) = \{x\}$ and $F(e') = \varnothing$, for all $e' \in E - \{e\}$ (briefly denoted by x_e).*

It is obvious that each soft set can be expressed as a union of soft points. For this reason, to give the family of all soft sets on X it is sufficient to give only soft points on X.

Definition 2.9 *[6] Two soft points x_e and $y_{e'}$ over a common universe X, we say that the soft points are different if $x \neq y$ or $e \neq e'$.*

Definition 2.10 *[6] The soft point x_e is said to be belonging to the soft set (F, E), denoted by $x_e \widetilde{\in} (F, E)$, if $x_e(e) \in F(e)$, i.e., $\{x\} \subseteq F(e)$.*

Definition 2.11 *[17] Let τ be the collection of soft sets over X, then $\widetilde{\tau}$ is said to be a soft topology on X if*

1) $\Phi, \widetilde{X}$ belong to $\widetilde{\tau}$;

2) the union of any number of soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$;

3) the intersection of any two soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$;

The triplet $(X, \widetilde{\tau}, E)$ is called a soft topological space over X.

Definition 2.12 *[17] Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X, then members of $\widetilde{\tau}$ are said to be a soft open sets in X.*

Definition 2.13 *[17] Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X. A soft set (F, E) over X is said to be a soft closed set in X, if its complement $(F, E)^c$ belongs to $\widetilde{\tau}$.*

3 Cech Homology Theory on Soft Sets

Let $STop$ be a category of soft topological spaces. For each soft topological space (X, τ, E), we show the set of all open coverings of X as $Cov(X)$. $Cov(X)$ is a directed set according to refinement of coverings. Let $\alpha = \{(F_i, E)\}_{i \in I}$ and $\beta = \{(G_j, E)\}_{j \in J}$ be two open coverings of soft topological space (X, τ, E).

Definition 3.1 *The covering β is called a refinement of the covering α, denoted by $\alpha \prec \beta$, if there exists a mapping $p: J \to I$ such that $(G_j, E) \subset (F_{p(j)}, E)$.*

Let $\alpha = \{(F_i, E)\}_{i \in I}$ be an arbitrary soft open covering of soft topological space (X, τ, E) and

$$nerv\alpha = \left\{ (i_1, i_2, ..., i_n) : \bigcap_{k=1}^{n} (F_{i_k}, E) \neq \Phi \right\}$$

be a simplicial complex whose vertices are elements of I. The map $p: J \to I$ extends uniquely to a simplicial mapping $p: nerv\beta \to nerv\alpha$. If $\alpha \prec \beta$ are two coverings of (X, τ, E), then any two mappings $p, p': nerv\beta \to nerv\alpha$ are contiguous simplicial maps [12]. Hence the simplicial mapping $p_\alpha^\beta : nerv\beta \to nerv\alpha$ is uniquely defined in contiguous class of simplicial maps.

Hence, taking these into consideration, we can easily show that

$$nerv(X) = \left(\{nerv\alpha\}_{\alpha \in Cov(X)}, \left\{ p_\alpha^\beta : nerv\beta \to nerv\alpha \right\}_{\alpha \prec \beta} \right) \quad (1)$$

is an inverse system of simplicial complexes.

Remark 3.1 Let $\alpha = \{(F_i, E)\}_{i \in I}$ be a soft open covering of (X, τ, E) and $nerv\alpha$ be a simplicial complex. Then for $e \in E$, $\alpha_e = \{F_i(e)\}_{i \in I}$ is an open covering of topological space (X, τ_e) and $nerv\alpha_e$ is a simplicial complex on this space. It is clear that $nerv\alpha$ is different from $nerv\alpha_e$.

Example 1 Let $X = \{x_1, x_2, x_3\}$ be an universe set, $E = \{e_1, e_2\}$ be a parameter set and $\tau = \left\{\Phi,\ \widetilde{X},\ (F_1, E),\ (F_2, E), ..., (F_{11}, E)\right\}$ be a soft topology over X. Here, the soft sets $(F_1, E), (F_2, E), ..., (F_{11}, E)$ over X are defined as following:

$$\begin{aligned}
F_1(e_1) &= \{x_1\},\ F_1(e_2) = \{x_1, x_3\},\\
F_2(e_1) &= \{x_2\},\ F_2(e_2) = \{x_2, x_3\},\\
F_3(e_1) &= \{x_3\},\ F_3(e_2) = X,\\
F_4(e_1) &= \varnothing,\ F_4(e_2) = \{x_3\},\\
F_5(e_1) &= \varnothing,\ F_5(e_2) = \{x_1, x_3\},\\
F_6(e_1) &= \{x_1, x_2\},\ F_6(e_2) = X,\\
F_7(e_1) &= \{x_1, x_3\},\ F_7(e_2) = X,\\
F_8(e_1) &= \{x_2, x_3\},\ F_8(e_2) = X,\\
F_9(e_1) &= \varnothing,\ F_9(e_2) = \{x_2, x_3\},\\
F_{10}(e_1) &= \{x_1\},\ F_{10}(e_2) = X,\\
F_{11}(e_1) &= \{x_2\},\ F_{11}(e_2) = X.
\end{aligned}$$

The family $\alpha = \{(F_1, E), (F_2, E), (F_3, E)\}$ is an soft open covering of (X, τ, E) and $nerv\alpha = \{(1), (2), (3), (1,2), (1,3), (2,3), (1,2,3)\}$. But, we take $e_1 \in E$,

$$\alpha_{e_1} = \{F_1(e), F_2(e), F_3(e)\}$$

is an open covering of (X, τ_{e_1}) and

$$nerv\alpha_{e_1} = \{(1), (2), (3)\}.$$

Let G be an arbitrary group. If we apply the homology [cohomology] functor $H_q(H^q)$ for the inverse system (1), then we obtain the following inverse (direct) system of groups

$$\begin{aligned}
&H_q(nerv(X); G)\\
&\quad = \Big(\{H_q(nerv\alpha, G)\}_{\alpha\in Cov(X)}, \{H_q(p_\alpha^\beta) : H_q(nerv\beta, G) \to H_q(nerv\alpha, G)\}_{\alpha\prec\beta}\Big),
\end{aligned}$$

$$\begin{aligned}
&\Big[H^q(nerv(X); G)\\
&\quad = \Big(\{H^q(nerv\alpha, G)\}_{\alpha\in Cov(X)}, \{H^q(p_\alpha^\beta) : H^q(nerv\alpha, G) \to H^q(nerv\beta, G)\}_{\alpha\prec\beta}\Big)\Big].
\end{aligned}$$

Definition 3.2 *The group* $H_q(X; G) = \varprojlim_\alpha H_q(nerv\alpha; G)$ $\left[H^q(X; G) = \varinjlim_\alpha H^q(nerv\alpha; G)\right]$ *is said to be $q-$level homology [cohomology] group of soft topological space* (X, τ, E).

Let (X, τ, E) and (Y, τ', E') be two soft topological spaces and $(f, \varphi) : (X, \tau, E) \to (Y, \tau', E')$ be a soft continuous mapping of soft topological spaces. For every soft open covering $\alpha = \{(G_j, E')\}_{j\in J}$ of (Y, τ', E'), the family $(f, \varphi)^{-1}(\alpha) = \left\{(f, \varphi)^{-1}(G_j, E')\right\}_{j\in J'}$ is an soft open covering of (X, τ, E) and $J' \subset J$. It is clear that if $\beta \succ \alpha$, then $(f, \varphi)^{-1}(\beta) \succ (f, \varphi)^{-1}(\alpha)$. If

$$(f, \varphi)^{-1}\left(G_{j_1}, E'\right) \cap ... \cap (f, \varphi)^{-1}\left(G_{j_k}, E'\right) \neq \Phi,$$

then $(G_{j_1}, E') \cap ... \cap (G_{j_k}, E') \neq \Phi$. Hence simplicial complex $nerv\,(f,\varphi)^{-1}(\alpha)$ is a subcomplex of the simplicial complex $nerv\alpha$.

Let $i_{\alpha,(f,\varphi)} : nerv\,(f,\varphi)^{-1}(\alpha) \to nerv\alpha$ be an embedding map. Then the family

$$\underline{f} = \left(\left\{(f,\varphi)^{-1} : Cov(Y) \to Cov(X)\right\}, \left\{i_{\alpha,(f,\varphi)} : nerv(f,\varphi)^{-1}(\alpha) \to nerv\alpha\right\}_{\alpha\in Cov(Y)}\right) \quad (2)$$

is a morphism from the inverse system $nerv\,(X)$ to the inverse system $nerv\,(Y)$. By using the morphism $\underline{f}$, we define the following homomorphism of homology [cohomology] groups

$$f_* = \varprojlim_{\alpha} H_q\left(\underline{f}\right) : H_q\left(X;G\right) \to H_q\left(Y;G\right) \left[f^* = \varinjlim_{\alpha} H^q\left(\overline{f}\right) : H^q\left(Y;G\right) \to H^q\left(X;G\right)\right].$$

Theorem 3.1 *The corresponding*

$$(X,\tau,E) \longmapsto H_q\left(X;G\right) \quad \left[(X,\tau,E) \longmapsto H^q\left(X;G\right)\right]$$

is a covariant [contravariant] functor from the category STop to the category of groups.

Proof. *The proof is clear.*

Let (X,τ,E) be a soft topological space, $A \subset X$ and (A,τ_A,E) be a soft subtopological space.

Definition 3.3 $(f,\varphi) : (X,A,\tau,E) \to (Y,B,\tau',E')$ *is called a morphism of pairs of soft topological spaces, if* $f : X \to Y$ *such that* $f(A) \subset B$.

Let the family $\alpha = \{(F_i,E)\}_{i\in I}$ be a soft open covering of (X,τ,E). Then the family $\alpha \cap \widetilde{A} = \left\{(F_i,E) \cap \widetilde{A}\right\}_{i\in I}$ is a soft open covering of (A,τ_A,E). We denote family of this coverings as $Cov\,(X,A)$. It is clear that the simplicial map $p_\alpha^\beta : nerv\beta \to nerv\alpha$ induces the following map of pairs of simplicial complexes

$$p_\alpha^\beta : \left(nerv\beta, nerv\left(\beta \cap \widetilde{A}\right)\right) \to \left(nerv\alpha, nerv\left(\alpha \cap \widetilde{A}\right)\right).$$

Hence

$$\left(\left\{\left(nerv\alpha, nerv\left(\alpha \cap \widetilde{A}\right)\right)\right\}_{\alpha\in Cov(X,A)}, \left\{p_\alpha^\beta\right\}_{\alpha\prec\beta}\right)$$

is an inverse system of pairs of simplicial complexes.

Definition 3.4 *The group*

$$H_q\left(X,A;G\right) = \varprojlim_{\alpha} H_q\left(\left(nerv\alpha, nerv\left(\alpha \cap \widetilde{A}\right)\right); G\right),$$

$$\left[H^q\left(X,A;G\right) = \varinjlim_{\alpha} H^q\left(\left(nerv\alpha, nerv\left(\alpha \cap \widetilde{A}\right)\right); G\right)\right]$$

is said to be homology [cohomology] group of the pair (X,A,τ,E).

It is obvious that the $(f,\varphi):(X,A,\tau,E)\to(Y,B,\tau',E')$ induces the homomorphism of homology [cohomology] groups as:

$$(f,\varphi)_{*q}:H_q(X,A;G)\to H_q(Y,B;G),$$

$$[(f,\varphi)^{*q}:H^q(Y,B;G)\to H^q(X,A;G)].$$

Then the corresponding

$$(X,A,\tau,E)\mapsto H_q(X,A;G)\quad[(X,A,\tau,E)\mapsto H^q(X,A;G)]$$

is a covariant [contravariant] functor from the category of pairs of soft topological spaces to the category of groups.

Now we show dimension axiom.

Let x_e be an arbitrary soft point in (X,τ,E). If the family $\alpha=\{(F_i,E)\}_{i\in I}$ is a soft open covering of x_e, then we choose a soft open set (F_{i_1},E) such that $x_e\in(F_{i_1},E)$ and $\alpha_1=\{(F_{i_1},E)\}$ which consist of single soft element is an open covering the soft point x_e. Then every soft open covering of x_e has a refinement of single element. We denote this covering as $Cov_1(x_e)$. This means that $Cov_1(x_e)$ consisting of the single element is cofinal in $Cov(x_e)$ and $Cov_1(x_e)\subset Cov(x_e)$. Thus for $q>0$ and $\alpha_1\in Cov_1(x_e)$,

$$H_q(x_e;G)=\lim_{\substack{\leftarrow\\ \alpha_1}}H_q(nerv\alpha;G)=0,$$

$$H_0(x_e;G)=G.$$

We prove the following theorem.

Theorem 3.2 *For every soft point $x_e\in(X,\tau,E)$,*

$$H_q(x_e;G)=\begin{cases}0, & q>0,\\ G, & q=0.\end{cases}.$$

Let (X,A,τ,E) be a pair of soft topological spaces, $i:A\to X$ and $j:X\to(X,A)$ be embedding maps. For every $\alpha\in Cov(X,A)$, the mapping i,j induces simplicial mappings

$$i_\alpha:nerv\left(\alpha\cap\widetilde{A}\right)\to nerv\alpha,$$

$$j_\alpha:nerv\alpha\to\left(nerv\alpha,nerv\left(\alpha\cap\widetilde{A}\right)\right).$$

Hence for every $\alpha\in Cov(X,A)$, we obtain the following exact sequence of homology groups

$$\begin{aligned}H(nerv\alpha)&=...\leftarrow H_q(nerv\alpha)\leftarrow H_q\left(nerv\left(\alpha\cap\widetilde{A}\right)\right)\leftarrow H_{q+1}\left(nerv\alpha,nerv\left(\alpha\cap\widetilde{A}\right)\right)\\&\leftarrow H_{q+1}(nerv\alpha)\leftarrow ...\end{aligned}\tag{3}$$

If we take limit of (3), then we obtain a sequence of homology groups of (X,A,τ,E) as follows:

$$...\leftarrow H_q(X;G)\leftarrow H_q(A;G)\leftarrow H_{q+1}(X,A;G)\leftarrow H_{q+1}(X)\leftarrow ...\tag{4}$$

Since the inverse limit of exact sequence is not exact, the sequence (4) is not exact, but cohomology sequence is exact.

Theorem 3.3 *(Excision axiom) Let (X, A, τ, E) be a soft topological space, $(U, E) \in \tau$ and its soft closure $\overline{(U, E)}$ is contained in the soft interior of $\widetilde{A}$, i.e, $\overline{(U, E)} \subset \widetilde{A}^o$. Then the inclusion mapping $j : (X - U, A - U) \to (X, A)$ induces an isomorphism*

$$j_{*q} : H_q (X - U, A - U; G) \to H_q (X, A; G),$$
$$[j^{*q} : H^q (X, A; G) \to H^q (X - U, A - U; G)].$$

Proof. For the covering $\alpha = \{(F_i, E)\}_{i \in I} \in Cov(X)$, let us consider the covering

$$\left(\alpha, \alpha \cap \widetilde{A}\right) = \left(\{(F_i, E)\}_{i \in I}, \left\{(F_i, E) \cap \widetilde{A}\right\}_{i \in I'}\right)$$

of (X, A). Let D be a subset of $Cov(X, A)$ such that $(F_i, E) \cap (U, E) \neq \Phi$. Then $i \in I'$ and $(F_i, E) \subset \widetilde{A}$. Now we show that D is a cofinal subset in $Cov(X, A)$.

Let us consider any covering γ of (X, A) with indexing pair $(I \cup J; I' \cup J)$. Hence $I \cap J = \varnothing$ and there exists one to one correspondence between I and I'. Consider the covering γ as follows:

$$(\gamma_i, E) = (F_i, E) \cap (U, E), \ \text{ for } j \in J,$$
$$(\gamma_j, E) = (F_{i'}, E) \cap \widetilde{A}^o, \ \text{ for } i' \in J.$$

It is clear that the family γ is a soft open covering of (X, A) and $\gamma \succ \alpha$, $\gamma \in D$, i.e, D is a cofinal subset in $Cov(X, A)$.

Now we show that $f^{-1}(D)$ is a cofinal subset in $Cov(X - U, A - U)$. For every $\beta = \{(V_j, E)\}_{j \in J} \in Cov(X - U, A - U)$, the family $\alpha = \{(U, E) \cup (V_j, E)\}_{j \in J}$ is a soft open covering of (X, A) and $\beta = f^{-1}(\alpha)$. If $\gamma \in D$ is a refinement of α, then $\beta = f^{-1}(\alpha) \prec f^{-1}(\gamma)$. Thus $f^{-1}(D)$ is a cofinal subset in $Cov(X - U, A - U)$. Since $D \subset Cov(X, A)$ and $f^{-1}(D) \subset Cov(X - U, A - U)$ are cofinal subsets, we get this sets to define homology groups.

For every $\alpha \in D$ and $\beta = f^{-1}(\alpha)$, we prove that

$$i_{\alpha, f_*} : H_q\left(nerv\beta, nerv\left(\beta \cap \widetilde{A}\right)\right) \to H_q\left(nerv\alpha, nerv\left(\alpha \cap \widetilde{A}\right)\right)$$

is an isomorphism. For this, it sufficies to prove that

$$nerv\alpha = nerv\beta \cup nerv\left(\alpha \cap \widetilde{A}\right),$$

$$nerv\beta \cap \widetilde{A} = nerv\beta \cap nerv\left(\alpha \cap \widetilde{A}\right).$$

Since $\left(nerv\beta, nerv\beta \cap \widetilde{A}\right) \subset \left(nerv\alpha, nerv\alpha \cap \widetilde{A}\right)$, we have the inclusions

$$nerv\beta \cup nerv\left(\alpha \cap \widetilde{A}\right) \subset nerv\alpha,$$

$$nerv\beta \cap \widetilde{A} \subset nerv\beta \cap nerv\left(\alpha \cap \widetilde{A}\right).$$

Thus it remains to prove

$$nerv\alpha \subset nerv\beta \cup nerv\left(\alpha \cap \widetilde{A}\right),$$

$$nerv\beta \cap nerv\left(\alpha \cap \widetilde{A}\right) \subset nerv\beta \cap \widetilde{A}.$$

Let $S=(i_1,i_2,...,i_n)$ be a simplex of $nerv\alpha$ which is not in $nerv\beta$. Then,

$$(F_{i_1},E)\cap...\cap(F_{i_n},E)\neq\Phi \text{ and } (F_{i_1},E)\cap...\cap(F_{i_n},E)\cap(U,E)^c=\Phi.$$

Hence, $(F_{i_1},E)\cap...\cap(F_{i_n},E)\subset(U,E)$ is satisfied and $(F_{i_k},E)\cap(U,E)\neq\Phi$, for $1\leq k\leq n$. Consequently, the simplex S belongs to $nerv\left(\alpha\cap\widetilde{A}\right)$, i.e,

$$nerv\alpha=nerv\beta\cup nerv\left(\alpha\cap\widetilde{A}\right)$$

is obtained. Now for $S=(i_1,i_2,...,i_n)\in nerv\beta\cap nerv\left(\alpha\cap\widetilde{A}\right)$,

$$(F_{i_1},E)\cap...\cap(F_{i_n},E)\cap(U,E)^c\neq\Phi \text{ and } (F_{i_1},E)\cap...\cap(F_{i_n},E)\cap\widetilde{A}\neq\Phi.$$

If $(F_{i_1},E)\cap...\cap(F_{i_n},E)\subset(U,E)^c$, then

$$(F_{i_1},E)\cap...\cap(F_{i_n},E)\cap\widetilde{A}\cap(U,E)^c=(F_{i_1},E)\cap...\cap(F_{i_n},E)\cap\widetilde{A}\neq\Phi,$$

i.e, $S\in nerv\beta\cap\widetilde{A}$. If $(F_{i_1},E)\cap...\cap(F_{i_n},E)\neq\Phi$, for $\alpha\in D$, then $(F_{i_k},E)\subset\widetilde{A}$ for every $1\leq k\leq n$. Thus since $f^{-1}(F_{i_1},E)\cap...\cap f^{-1}(F_{i_n},E)\cap\widetilde{A}\cap(U,E)^c=f^{-1}(F_{i_1},E)\cap...\cap f^{-1}(F_{i_n},E)\cap(U,E)^c\neq\Phi$, so $S\in nerv\left(\beta\cap\widetilde{A}\right)$. Hence

$$nerv\beta\cap nerv\left(\alpha\cap\widetilde{A}\right)=nerv\beta\cap\widetilde{A}$$

is obtained.

To prove homotopy axiom in the category of soft toplogical spaces, we consider the unit interval $I=[0,1]$ depending on one parameter as soft topological space. For the parameter set $E=\{*\}$, soft topological space $(I,\tau_e,*)$ has soft point as τ_*.

Definition 3.5 *Let (X,τ,E) and (Y,τ',E') be two soft topological spaces and (f,φ), $(g,\varphi):(X,\tau,E)\to(Y,\tau',E')$ be two soft continuous mappings. If there exists a soft continuous mapping $(F,\varphi):(X\times I,\tau\times\tau',E\times\{*\})\to(Y,\tau',E')$ such that*

$$(F,\varphi)(x_e,0_*)=(f,\varphi)(x_e)=(f(x))_{\varphi(e)},$$

$$(F,\varphi)(x_e,1_*)=(g,\varphi)(x_e)=(g(x))_{\varphi(e)},$$

then (F,φ) is called soft homotopy and the mappings $(f,\varphi),(g,\varphi)$ are said to be soft homotopic maps.

It is clear that soft homotopy relation is an equivalence relation and this relation is invariant according to composition operation.

Theorem 3.4 *(Homotopy Axiom) If $(f,\varphi),(g,\varphi):(X,\tau,E)\to(Y,\tau',E')$ are soft homotopic maps, then*

$$(f,\varphi)_*=(g,\varphi)_*:H_q(X;G)\to H_q(Y;G).$$

It is enough to prove the theorem, we firstly give necessary lemmas. Since the coverings of $(I,\tau_e,*)$ is the covering of (I,τ_e), we get regular coverings.

Let the family $\alpha=\{(F_j,E)\}_{j\in J}\in Cov(X)$ be a covering. For every $i\in I$, we consider regular covering $\beta^j=\left\{V_i^j\right\}_{i=1,N^i}$ of $(I,\tau_e,*)$. For the set $W=\left\{(j,i):j\in J,\ i\in N^i\right\}$, the family $\gamma=\{\gamma_{j,i}\}=\left\{(F_j,E)\times V_i^j\right\}_{(j,i)\in W}$ is a soft open covering of $X\times I$. The covering $\gamma\in Cov(X\times I)$ indexed by W is called a stacked covering over α.

Lemma 3.1 *Stacked coverings form a cofinal subset of* $Cov\left(X \times I\right).$

Proof. Let $\gamma = \{\gamma_{j,i}\} = \left\{(F_j, E) \times V_i^j\right\}_{(j,i)\in W}$ be any covering of the space $X \times I$. For each (x_e, t_*) choose soft open sets $F_j\left(x_e\right) \subset X$, $V_i^j\left(t_*\right) \subset I$ such that $(F_j, E) \times V_i^j \subset \gamma_{j,i}$. For each fixed soft point x_e, the family $\left\{V_i^j\left(t_*\right)\right\}$ constitute soft open covering of the unit interval $(I, \tau_e, *)$. This family has a finite regular refinement β^j. For each soft open set V_i^j, there is a soft open set (F_j, E) such that $(F_j, E)\times V_i^j \subset \gamma_{j,i}$. If $(F, E) = \underset{j\in J}{\cap} (F_j, E)$, then the family $\left\{(F_j, E) \times V_i^j\right\}$ is a stacked covering of $X\times I$ and refinement of covering γ.

Lemma 3.2 *Let the family γ be a stacked covering of $X \times I$ over base α. If $nerv\alpha$ is a simplex, then $nerv\alpha$ is a cyclic.*

Proof. Without loss of generality we may assume that none of the soft sets of the covering of α is null soft set. Let $\alpha = \{(F_j, E)\}_{j\in J}$ and $\gamma = \left\{(F_j, E) \times V_i^j\right\}_{(j,i)\in W}$. We define a covering δ of the space $(I, \tau_e, *)$ as $\delta = \left\{\delta_{i,j} = V_i^j\right\}_{(j,i)\in W}$. If S is any simplex with vertices $(i_0, j_0), ..., (i_n, j_n) \in W$, then

$$\begin{aligned}\overset{n}{\underset{k=0}{\cap}} \left[(F_{j_k}, E) \times V_{i_k}^{j_k}\right] &= \left(\overset{n}{\underset{k=0}{\cap}} (F_{j_k}, E)\right) \times \left(\overset{n}{\underset{k=0}{\cap}} V_{i_k}^{j_k}\right) \\ &= \left(\overset{n}{\underset{k=0}{\cap}} (F_{j_k}, E)\right) \cap \left(\overset{n}{\underset{k=0}{\cap}} \delta_{i_k,j_k}\right).\end{aligned}$$

Since $nerv\alpha$ is a simplex, then $\overset{n}{\underset{k=0}{\cap}} \delta_{i_k,j_k} \neq \varnothing$. It follows that $\overset{n}{\underset{k=0}{\cap}} \left[(F_{j_k}, E) \times V_{i_k}^{j_k}\right] = \varPhi \Leftrightarrow \overset{n}{\underset{k=0}{\cap}} \delta_{i_k,j_k} \neq \varnothing$. Thus $nerv\alpha$ is a cyclic.

Proposition 3.1 *Let γ be a stacked covering of soft topological space $X \times I$ over base α. Consider the simplicial maps*

$$l, u : nerv\alpha \to nerv\gamma$$

which is defined by

$$l\left(j\right) = (j, 0)\,,\ u\left(j\right) = \left(j, N^i\right),$$

induces

$$l_{*_q} = u_{*_q} : H_q\left(nerv\alpha\right) \to H_q\left(nerv\gamma\right).$$

Proof. The proof is clear.

Now we prove of the theorem.

Proof. We define soft mappings

$$(f_0, \varphi)\,, (g_0, \varphi) : (X, \tau, E) \to (X \times I, \tau \times \tau_e, E \times \{e\})$$

for every soft point $x_e \in X$ as

$$(f_0, \varphi)\left(x_e\right) = \left(f_0\left(x\right)\right)_{\varphi(e)} = (x_e, 0_*)\,,$$

$$(g_0, \varphi)(x_e) = (g_0(x))_{\varphi(e)} = (x_e, 1_*).$$

Therefore if $(F, \varphi) : (X \times I, \tau \times \tau', E \times \{*\}) \to (Y, \tau', E')$ is a soft homotopy between soft mappings (f, φ) and (g, φ), then

$$(f, \varphi) = (F, \varphi) \circ (f_0, \varphi), \ \ (g, \varphi) = (F, \varphi) \circ (g_0, \varphi).$$

Hence to prove the theorem, it is sufficient to show that

$$(f_0, \varphi)_{*_q} = (g_0, \varphi)_{*_q}.$$

Since stacked coverings is a cofinal subset, we can use the stacked coverings to define homology groups of the soft topological space $X \times I$. Let the family γ be a stacked covering of $X \times I$ over base α. Consider the coverings $\gamma_0 = (f_0, \varphi)^{-1}(\gamma)$, $\gamma_1 = (g_0, , \varphi)^{-1}(\gamma)$ and simplicial mappings

$$i_{(f_0,\varphi),\gamma} : nerv\,(f_0, \varphi)^{-1}(\gamma) \to nerv\gamma,$$

$$i_{(g_0,\varphi),\gamma} : nerv\,(g_0, , \varphi)^{-1}(\gamma) \to nerv\gamma.$$

If maps $u' : nerv\alpha \to nerv\gamma$, $\pi : nerv\gamma \to nerv\,(f_0, \varphi)^{-1}(\gamma)$ are defined by

$$u'(i) = (i, n^i), \ \pi(i, j) = (i, 0),$$

then

$$u = i_{(g_0,\varphi),\gamma} \circ u', \ \ l = i_{(f_0,\varphi),\gamma} \circ \pi \circ u'$$

are obtained. Further it can be observed that

$$(i_{(g_0,\varphi),\gamma})_{*_q} = (i_{(f_0,\varphi),\gamma})_{*_q}$$

and consequently

$$(f_0, \varphi)_{*_q} = (g_0, \varphi)_{*_q}.$$

4 CONCLUSION

In this paper, Cech homology [cohomology] groups of soft topological spaces are defined. It is proved that Cech homology groups constitue a functor from the category of soft topological spaces to the category of groups. Later, axioms of homology theory are checked for this homology groups. We hope that, the results of this study may help in the investigation of Cech homology groups and in many researches.

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