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Non-existence of global solution of a semi-linear parabolic equation with a singular potential

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Abstract. *In the domain* $Q'(R) = \{(x, y); |x| > R, |y| > R\} \times (0; +\infty)$ *the problem of the absence of nonnegative global solutions of a semilinear parabolic equation* $\frac{\partial u}{\partial t} = \operatorname{div}_x (|x|^{\alpha} \nabla_x u) + \Delta_y u + \frac{c_1}{|x|^{2-\alpha}} u + \frac{c_2}{|y|^2} u + |x|^{\sigma_1} |y|^{\sigma_1} |u|^q$ *with an initial condition* $u|_{t=0} = u_0(x, y) \geq 0$ *is investigated. A sufficient condition for the absence of global non-negative solutions is obtained. The proof is based on the test function method.*

Keywords. Semilinear parabolic equation · global solution · singular potential · critical exponent · test function method.

Mathematics Subject Classification (2010): 35K58, 35B09, 35B08, 35B33

1 Introduction

We introduce the following denotation:

$$x = (x_1, x_2, ...x_m), \quad y = (y_1, y_2, ..., y_n), \ |x| = \sqrt{x_1^2 + ... + x_m^2},$$

$$|y| = \sqrt{y_1^2 + ... + y_n^2}, \ B_x(r) = \{x; |x| < r\}, \ \ B_y(r) = \{y; |y| < r\},$$

$$B_x(r_1, r_2) = \{x; \ \ r_1 < |x| < r_2\}, \ \ B_y(r_1, r_2) = \{y; \ \ r_1 < |y| < r_2\},$$

$$B_x^{'}(R) = R^m \backslash B_x(R), \ \ B_y^{'}(R) = \mathbb{R}^n \backslash B_y(R), \quad B'(R) = B_x^{'}(R) \times B_y^{'}(R),$$

$Q'(R) = B'(R) \times (0; +\infty)$ and for $R = 1$ $B_x^{'}(1) = B_x^{'}, B_y^{'}(1) = B_y^{'}, \ B^{'}(1) = B', \ Q'(1) = Q'$.

In $Q'(R)$ we consider the equation

$$\frac{\partial u}{\partial t} = \operatorname{div}_x (|x|^{\alpha} \nabla_x u) + \Delta_y u + \frac{c_1}{|x|^{2-\alpha}} u + \frac{c_2}{|y|^2} u + |x|^{\sigma_1} |y|^{\sigma_2} |u|^q \tag{1.1}$$

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whit the initial condition

$$u|_{t=0} = u_0(x,y) \geq 0, \tag{1.2}$$

where $0 \leq c_1 < \left(\frac{\alpha+m-2}{2}\right)^2$, $0 \leq c_2 < \left(\frac{n-2}{2}\right)^2$, $q > 1$, $\sigma_1, \sigma_2 \in R$, $\alpha < 2$, $\nabla_x = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}\right)$, $\nabla_y = \left(\frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\right)$, $\mathrm{div}_x(f_1, ..., f_m) = \frac{\partial f_1}{\partial x_1} + ... + \frac{\partial f_m}{\partial x_m}$, $\Delta_y = \frac{\partial^2}{\partial y_1^2} + ... + \frac{\partial^2}{\partial y_n^2}$.

We study the existence of global solution of problem (1.1), (1.2).

The solution of problem (1.1),(1.2) is understood in the classic sense. Problems of the existence and nonexistence of global solutions of various class of differential equations and inequalities play an important role in theory and applications therefore attract constant attention of mathematicians and a great number of works were devoted to them. In the classic work of Fujita [1] the following initial problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u^p, & (x,t) \in \mathbb{R}^n \times (0, \infty) \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}^n \end{cases} \tag{1.3}$$

is considered and it is proved that there are non-negative global solutions of problem (1.3) for $1 < p < p^* = 1 + \frac{2}{n}$, and for $p > p^*$ for small (1.3) there exist non-negative global solutions. The case $p = p^*$ was studied in the papers [2],[3] and it was proved that in this case also there no non-negative global solutions. The results of Fujita's papes [1] caused a great interest to the problem of non-existence of global solutions of various class of differential equations and inequalities. One can see the review of such papers in the paper [4], in the monograph [5] and in the book [6].

In the represented work, in the equation there is a minior term with a singular potential, and the principal part of the equation is a Baouendi-Grushin type operator. In fact, if we take substitution $u = |x|^{-\frac{\alpha}{2}} v$ in equation (1), then it is reduced to the

$$|x|^{-\alpha} \frac{\partial v}{\partial t} = \Delta_x v + |x|^{-\alpha} \Delta_y v + \left(\frac{\alpha}{2}\left(2 - n - \frac{\alpha}{2}\right) + c_1\right) \frac{1}{|x|^2} v$$
$$+ \frac{|x|^{-\alpha}}{|y|^2} v + |x|^{\sigma_1 - \frac{\alpha}{2}(q+1)} |y|^{\sigma_2} |v|^q ,$$

where principal part is a Baoendi-Grushin operator.

Linear and semilinear equations with Baouendi-Grushin operator in the principal part were also widely studied in [7]-[12].

The issues considered in this paper were earlier studied in the papers [13]-[19].

2 The basic theorem and its proof

Denote

$$D_1 = \left(\frac{\alpha + m - 2}{2}\right)^2 - c_1, \quad D_2 = \left(\frac{n-2}{2}\right)^2 - c_2,$$
$$\lambda_+ = -\frac{\alpha + m - 2}{2} + \sqrt{D_1}, \quad \lambda_- = -\frac{\alpha + m - 2}{2} - \sqrt{D_1},$$
$$\mu_+ = -\frac{n-2}{2} + \sqrt{D_2}, \quad \mu_- = -\frac{n-2}{2} - \sqrt{D_2}.$$

We consider the following functions

$$\xi = |x|^{\lambda_+} - |x|^{\lambda_-}, \quad \eta = |y|^{\mu_+} - |y|^{\mu_-} .$$

Obviously

$$\operatorname{div}_{\mathrm{x}}\left(|\mathrm{x}|^{\alpha}\nabla_{\mathrm{x}}\xi\right)+\frac{\mathrm{c}_1}{|\mathrm{x}|^{2-\alpha}}\xi=0 \tag{2.1}$$

in $R^m\setminus\{0\}$, $\xi|_{|x|=1}=0$ and

$$\Delta_y\eta+\frac{c_2}{|y|^2}\eta=0 \text{ in } \mathbb{R}^n\setminus\{0\}, \quad \eta|_{|y|=1}=0. \tag{2.2}$$

We consider the following functions

$$\varphi(x)=\begin{cases}1, & R\le|x|\le\rho\\ \left(\frac{1}{2}\left(\cos\left(\left(\frac{|x|}{\rho}-1\right)\pi\right)+1\right)\right)^{k_1}, & \rho\le|x|\le2\rho\\ 0, & |x|\ge2\rho,\end{cases}$$

$$\psi(y)=\begin{cases}1, & R\le|y|\le\rho^{\varepsilon}\\ \left(\frac{1}{2}\left(\cos\left(\left(\frac{|y|}{\rho^{\varepsilon}}-1\right)\pi\right)+1\right)\right)^{k_2}, & \rho^{\varepsilon}\le|y|\le2\rho^{\varepsilon}\\ 0, & |y|\ge2\rho^{\varepsilon},\end{cases}$$

$$T(t)=\begin{cases}1, & 0\le t\le\rho^{\varkappa}\\ \left(\frac{1}{2}\left(\cos\left(\left(\frac{|y|}{\rho^{\varkappa}}-1\right)\pi\right)+1\right)\right)^{k_2}, & \rho^{\varkappa}\le t\le2\rho^{\varkappa}\\ 0, & t\ge2\rho^{\varkappa},\end{cases}$$

where $\varepsilon,\varkappa$ are constants and their concrete values will be determined later, k_1,k_2,k_3 are large positive numbers.

The following theorem is the basic result of this paper.

Theorem 2.1 *Let* $m,n>2,\quad \alpha<2, q>1, 0\le c_1<\left(\frac{\alpha+m-2}{2}\right)^2,\quad 0\le c_2<\left(\frac{n-2}{2}\right)^2,\ 2-\alpha+\sigma_1+\frac{2-\alpha}{2}\sigma_2>0,\quad q\le1+\frac{2-\alpha+\sigma_1+\frac{2-\alpha}{2}\sigma_2}{\lambda_++m+\frac{2-\alpha}{2}(\mu_++n)}$. *Then, if* $u(x,y,t)$ *is the solution of problem (1.1), (1.2), then* $u(x,y,t)\equiv0$.

Proof. For simplicity of notation we take $R=1$. Mulitiply equation (1.1) by the function $f(x,y,t)=\xi(x)\eta(y)\varphi(x)\psi(y)T(t)$ and make integration parts. Then we get are following relation

$$G\equiv\int\limits_{Q'}|x|^{\sigma_1}|y|^{\sigma_2}|u|^q\xi(x)\eta(y)\varphi(x)\psi(y)T(t)dxdydt$$

$$=-\int\limits_{Q'}u\xi\eta\varphi\psi\frac{\partial T}{\partial t}dxdydt-\int\limits_{B'}u_0(x,y)\xi\eta\varphi\psi dxdy$$

$$-\int\limits_{Q'}u\eta\psi T\operatorname{div}_{\mathrm{x}}\left(|\mathrm{x}|^{\alpha}\left(\nabla_{\mathrm{x}}\xi\varphi+\xi\nabla_{\mathrm{x}}\varphi\right)\right)\mathrm{dxdydt}$$

$$-\int\limits_{Q'}u\xi\varphi T\Delta_y(\psi\eta)dxdtdt-\int\limits_{Q'}\frac{c_1}{|x|^{2-\alpha}}u\xi\eta\varphi\psi Tdxdydt$$

$$-\int\limits_{Q'}\frac{c_2}{|y|^2}u\xi\eta\varphi\psi Tdxdydt=-\int\limits_{Q'}u\xi\eta\varphi\psi\frac{\partial T}{\partial t}dxdydt-\int\limits_{B'}u_0(x,y)\xi\eta\varphi\psi dxdy$$

$$-\int\limits_{Q'} u\eta\psi T\varphi\left[\mathrm{div_x}\left(|\mathrm{x}|^{\alpha}\nabla_{\mathrm{x}}\xi\right)+\frac{\mathrm{c}_1}{|\mathrm{x}|^{2-\alpha}}\xi\right]dxdydt$$

$$-\int\limits_{Q'} u\eta\psi TH(x,\xi,\varphi)dxdydt-\int\limits_{Q'} u\xi\varphi T\psi\left[\Delta_y\eta+\frac{c_2}{|y|^2}\eta\right]dxdydt$$

$$-\int\limits_{Q'} u\xi\varphi T\left[2(\nabla_y\psi,\nabla_y\eta)+\eta\nabla_y\psi\right]dxdydt, \tag{2.3}$$

where

$$H(x,\xi,\varphi)=|x|^{\alpha}\left(\nabla_x\xi,\nabla_x\varphi\right)+|x|^{\alpha}\xi\Delta_x\varphi+\left(\nabla_x\left(|x|^{\alpha}\xi\right),\nabla_x\varphi\right)$$

$$=|x|^{\alpha}\left(\nabla_x\xi,\nabla_x\varphi\right)+|x|^{\alpha}\xi\Delta_x\varphi+\alpha|x|^{\alpha-2}\xi\left(x,\nabla_x\varphi\right)+|x|^{\alpha}\left(\nabla_x\xi,\nabla_x\varphi\right)$$

$$=\alpha|x|^{\alpha-2}\xi\left(x,\nabla_x\varphi\right)+2|x|^{\alpha}\left(\nabla_x\xi,\nabla_x\varphi\right)+|x|^{\alpha}\xi\Delta_x\varphi.$$

Taking into account (2.1), (2.2) and $\int\limits_{B'} u_0(x,y)\xi\eta\varphi\psi dxdy\geq 0$, from (2.3) we get

$$G\leq-\int\limits_{Q'} u\xi\eta\varphi\psi\frac{\partial T}{\partial t}dxdydt-\int\limits_{Q'} u\eta\psi TH(x,\xi,\varphi)dxdydt$$

$$-\int\limits_{Q'} u\xi\varphi T\left[2(\nabla_y\psi,\nabla_y\eta)+\eta\Delta_y\psi\right]dxdydt. \tag{2.4}$$

Using the Holder inequality from (2.4), we can write

$$G\leq\left(\int\limits_{\rho^{\varkappa}}^{2\rho^{\varkappa}}\int\limits_{B'}|x|^{\sigma_1}|y|^{\sigma_2}|u|^q\xi\eta\varphi\psi Tdxdydt\right)^{\frac{1}{q}}$$

$$\times\left(\int\limits_{\rho^{\varkappa}}^{2\rho^{\varkappa}}\int\limits_{B'}\frac{\left|\frac{\partial T}{\partial t}\right|^{q'}\xi\eta\varphi\psi}{T^{q'-1}|x|^{\sigma_1(q'-1)}|y|^{\sigma_2(q'-1)}}dxdydt\right)^{\frac{1}{q'}}$$

$$+\left(\int\limits_0^{2\rho^{\varkappa}}\int\limits_{B'_y}\int\limits_{B_x(\rho,2\rho)}|x|^{\sigma_1}|y|^{\sigma_2}|u|^q\xi\eta\varphi\psi Tdxdydt\right)^{\frac{1}{q}}$$

$$\times\left(\int\limits_0^{2\rho^{\varkappa}}\int\limits_{B'_y}\int\limits_{B_x(\rho,2\rho)}\frac{|H(x,\xi,\varphi)|^{q'}\eta\psi T}{|x|^{\sigma_1(q'-1)}|y|^{\sigma_2(q'-1)}\xi^{q'-1}\varphi^{q'-1}}dxdydt\right)^{\frac{1}{q'}}$$

$$+\left(\int\limits_0^{2\rho^{\varkappa}}\int\limits_{B_y(\rho^{\varepsilon},2\rho^{\varepsilon})}\int\limits_{B'_x}|x|^{\sigma_1}|y|^{\sigma_2}|u|^q\xi\eta\varphi\psi Tdxdydt\right)^{\frac{1}{q}}$$

$$\times\left(\int\limits_0^{2\rho^{\varkappa}}\int\limits_{B_y(\rho^{\varepsilon},2\rho^{\varepsilon})}\int\limits_{B'_x}\frac{\left|2\left(\nabla_y\eta,\nabla_y\psi\right)+\eta\Delta_y\psi\right|^{q'}\xi\varphi T}{|x|^{\sigma_1(q'-1)}|y|^{\sigma_2(q'-1)}\eta^{q'-1}\psi^{q'-1}}dxdydt\right)^{\frac{1}{q'}}, \quad (2.5)$$

where $\frac{1}{q}+\frac{1}{q'}=1$.

Hence

$$G\le G^{\frac{1}{q}}\left[\left(\int\limits_{Q'}\frac{\left|\frac{\partial T}{\partial t}\right|^{q'}\xi\eta\varphi\psi}{T^{q'-1}|x|^{\sigma_1(q'-1)}|y|^{\sigma_2(q'-1)}}dxdydt\right)^{\frac{1}{q'}}\right.$$

$$+\left(\int\limits_{Q'}\frac{|H(x,\xi,\varphi)|^{q'}\eta\psi T}{|x|^{\sigma_1(q'-1)}|y|^{\sigma_2(q'-1)}\xi^{q'-1}\varphi^{q'-1}}dxdydt\right)^{\frac{1}{q'}}$$

$$\left.+\left(\int\limits_{Q'}\frac{\left|2\left(\nabla_y\eta,\nabla_y\psi\right)+\eta\Delta_y\psi\right|^{q'}\xi\varphi T}{|x|^{\sigma_1(q'-1)}|y|^{\sigma_2(q'-1)}\eta^{q'-1}\psi^{q'-1}}dxdydt\right)^{\frac{1}{q'}}\right]=G^{\frac{1}{q}}\left[I_1^{\frac{1}{q'}}+I_2^{\frac{1}{q'}}+I_3^{\frac{1}{q'}}\right],$$

where I_1, I_2, I_3 denote the first, second, third integral respectively in the brackets.

Hence

$$G^{1-\frac{1}{q}}=G^{\frac{1}{q'}}\le I_1^{\frac{1}{q'}}+I_2^{\frac{1}{q'}}+I_3^{\frac{1}{q'}},$$
$$G\le C(I_1+I_2+I_3). \quad (2.6)$$

Now using the substitution

$$\widetilde{x}=\frac{x}{\rho},\widetilde{y}=\frac{y}{\rho^{\varepsilon}},\ \tau=\frac{t}{\rho^{\varkappa}},\widetilde{\varphi}(\widetilde{x})=\varphi(x)=\varphi(\widetilde{x}\rho),$$

$$\widetilde{\psi}(\widetilde{y})=\psi(y)=\psi(\widetilde{y}\rho^{\varepsilon}),\ \widetilde{T}(\tau)=T(t)=T(\tau\rho^{\varkappa})$$

we estimate each integral I_1, I_2, I_3 separately.

$$I_1=\int\limits_{Q'}\frac{\left|\frac{\partial T}{\partial t}\right|^{q'}\xi\eta\varphi\psi}{T^{q'-1}|x|^{\sigma_1(q'-1)}|y|^{\sigma_2(q'-1)}}dxdydt\le\int\limits_{\rho^{\varkappa}}^{2\rho^{\varkappa}}\frac{\left|\frac{\partial T}{\partial t}\right|^{q'}}{T^{q'-1}}dt$$

$$\times\int\limits_{B_y(1,2\rho^{\varepsilon})}\frac{\eta}{|y|^{\sigma_2(q'-1)}}dy\int\limits_{B_x(1,2\rho)}\frac{\xi}{|x|^{\sigma_1(q'-1)}}dx\le\rho^{-\varkappa q'+\varkappa}\int\limits_1^2\frac{\left|\frac{\partial\widetilde{T}}{\partial\tau}\right|^{q'}}{\widetilde{T}^{q'-1}}d\tau$$

$$\times\int\limits_{B_y(1,2\rho^{\varepsilon})}|y|^{\mu_+-\sigma_2(q'-1)}dy\int\limits_{B_x(1,2\rho)}|x|^{\lambda_+-\sigma_1(q'-1)}dx\le c\rho^{-\varkappa q'+\varkappa}\widetilde{I}_1$$

$$\times\int\limits_1^{2\rho^{\varepsilon}}r^{\mu_+-\sigma_2(q'-1)+n-1}dr\int\limits_1^{2\rho}r^{\lambda_+-\sigma_1(q'-1)+m-1}dr$$

$$\le c\rho^{-\varkappa(q'-1)+\lambda_++\varepsilon\mu_++m+\varepsilon n-\sigma_1(q'-1)-\varepsilon\sigma_2(q'-1)}\widetilde{I}_1, \quad (2.7)$$

where $\widetilde{I}_1 = \int\limits_1^2 \frac{\left|\frac{\partial \widetilde{T}}{\partial \tau}\right|^{q'}}{\widetilde{T}^{q'-1}} d\tau$.

$$I_2 = \int\limits_{Q'} \frac{|H(x,\xi,\varphi)|^{q'} \eta\psi T}{|x|^{\sigma_1(q'-1)}|y|^{\sigma_2(q'-1)}\xi^{q'-1}\varphi^{q'-1}} dxdydt$$

$$\leq \int\limits_0^{2\rho^{\varkappa}} \int\limits_{B_y(1,2\rho^{\varepsilon})} \int\limits_{B_x(\rho,2\rho)} \frac{|H(x,\xi,\varphi)|^{q'} \eta}{|x|^{\sigma_1(q'-1)}|y|^{\sigma_2(q'-1)}\xi^{q'-1}\varphi^{q'-1}} dxdydt$$

$$= \int\limits_0^{2\rho^{\varkappa}} dt \int\limits_{B_y(1,2\rho^{\varepsilon})} \frac{\eta}{|y|^{\sigma_2(q'-1)}} dy \int\limits_{B_x(\rho,2\rho)} \frac{|H(x,\xi,\varphi)|^{q'}}{|x|^{\sigma_1(q'-1)}\xi^{q'-1}\varphi^{q'-1}} dx$$

$$\leq c\rho^{\varkappa}\rho^{\varepsilon(\mu_+ - \sigma_2(q'-1)+n)}$$

$$\times \int\limits_{B_x(\rho,2\rho)} \frac{\left|\alpha\,|x|^{\alpha-2}\,\xi(x,\nabla_x\varphi) + 2\,|x|^{\alpha}\,(\nabla_x\xi,\nabla_x\varphi) + |x|^{\alpha}\,\xi\Delta_x\varphi\right|^{q'}}{|x|^{\sigma_1(q'-1)}\xi^{q'-1}\varphi^{q'-1}} dx$$

$$\leq c\rho^{\varkappa+\varepsilon\mu_+ +\varepsilon n-\varepsilon\sigma_2(q'-1)}$$

$$\times \int\limits_{B_x(\rho,2\rho)} \frac{\left|\alpha\,|x|^{\alpha-2+\lambda_+}\,|(x,\nabla_x\varphi)| + 2\lambda_+\,|x|^{\alpha-2+\lambda_+}\,|(x,\nabla_x\varphi)| + |x|^{\alpha+\lambda_+}\,\Delta_x\varphi\right|^{q'}}{|x|^{\sigma_1(q'-1)}|x|^{\lambda_+(q'-1)}\varphi^{q'-1}} dx$$

$$\leq c\rho^{\varkappa+\varepsilon\mu_+ +\varepsilon n-\varepsilon\sigma_2(q'-1)}\rho^{(\alpha-2+\lambda_+)q'-\sigma_1(q'-1)-\lambda_+(q'-1)+m}$$

$$\times \int\limits_{B_{\widetilde{x}}(1,2)} \frac{\left|(\alpha+2\lambda_+)|\widetilde{x}|^{\alpha-2+\lambda_+}|(\widetilde{x},\nabla_{\widetilde{x}}\widetilde{\varphi})| + |\widetilde{x}|^{\alpha+\lambda_+}|\Delta_{\widetilde{x}}\widetilde{\varphi}|\right|^{q'}}{|\widetilde{x}|^{(\sigma_1+\lambda_+)(q'-1)}\widetilde{\varphi}^{q'-1}} d\widetilde{x}$$

$$\leq c\rho^{\varkappa+\lambda_+ +\varepsilon\mu_+ +m+\varepsilon n-\sigma_1(q'-1)-\varepsilon\sigma_2(q'-1)+(\alpha-2)q'}\widetilde{I}_2, \tag{2.8}$$

where $\widetilde{I}_2$ denotes the last integral in (2.8).

$$I_3 = \int\limits_{Q'} \frac{|2\,(\nabla_y\eta,\nabla_y\psi) + \eta\Delta_y\psi|^{q'}\,\xi\varphi T}{|x|^{\sigma_1(q'-1)}|y|^{\sigma_2(q'-1)}\psi^{q'-1}\eta^{q'-1}} dxdydt$$

$$\leq \int\limits_0^{2\rho^{\varkappa}} dt \int\limits_{B_x(1,2\rho)} \frac{\xi}{|x|^{\sigma_1(q'-1)}} dx \int\limits_{B_y(\rho^{\varepsilon},2\rho^{\varepsilon})} \frac{|2\,(\nabla_y\eta,\nabla_y\psi) + \eta\Delta_y\psi|^{q'}}{|y|^{\sigma_2(q'-1)}\psi^{q'-1}\eta^{q'-1}} dy$$

$$\leq c\rho^{\varkappa} \int\limits_{B_x(1,2\rho)} |x|^{\lambda_+ - \sigma_1(q'-1)} dx \int\limits_{B_y(\rho^{\varepsilon},2\rho^{\varepsilon})} \frac{\left|2|y|^{\mu_+ -2}|\,(y,\nabla_y\psi)\,| + |y|^{\mu_+}|\Delta_y\psi|\right|^{q'}}{|y|^{\sigma_2(q'-1)}|y|^{\mu_+(q'-1)}\psi^{q'-1}} dy$$

$$\leq c\rho^{\varkappa}\rho^{\lambda_+ - \sigma_1(q'-1)+m}\rho^{\varepsilon((\mu_+ -2)q'-\sigma_2(q'-1)-\mu_+(q'-1)+n)}$$

$$\times \int\limits_{B_{\widetilde{y}}(1,2)} \frac{\left|2|\widetilde{y}|^{\mu_+-2}| \left(\widetilde{y}, \nabla_{\widetilde{y}}\widetilde{\psi}\right) | + |\widetilde{y}|^{\mu_+} |\Delta_{\widetilde{y}}\widetilde{\psi}|\right|^{q'}}{|\widetilde{y}|^{(\sigma_2+\mu_+)(q'-1)}\widetilde{\psi}^{q'-1}} d\widetilde{y}$$

$$= c\rho^{\varkappa+\lambda_+ +\varepsilon\mu_+ +m+\varepsilon n-\sigma_1(q'-1)-\varepsilon\sigma_2(q'-1)-2\varepsilon q'} \widetilde{I}_3, \tag{2.9}$$

where $\widetilde{I}_3$ denotes the last integral in (2.9).

Now we take $\varepsilon, \varkappa$ such that

$$-\varkappa q' = (\alpha - 2)q' = -2\varepsilon q'.$$

Hence,

$$\varkappa = 2 - \alpha, \quad \varepsilon = \frac{2-\alpha}{2}.$$

Taking this and inequalities (2.7),(2.8), (2.9) into account, from (2.6) we get

$$G \leq c\rho^{-(2-\alpha)(q'-1)+\lambda_+ +m+\frac{2-\alpha}{2}(\mu_+ +n)-\left(\sigma_1+\frac{2-\alpha}{2}\sigma_2\right)(q'-1)} \left(\widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3\right)$$

$$= c\rho^{-\left((2-\alpha)+\sigma_1+\frac{2-\alpha}{2}\sigma_2\right)\frac{1}{q-1}+\lambda_+ +m+\frac{2-\alpha}{2}(\mu_+ +n)} \left(\widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3\right).$$

It is easy to show that for large k_1, k_2, k_3 the integrals $\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3$ are bounded (see [5]).

Then from (2.10) it follows

$$G \leq c\rho^{-\frac{2-\alpha+\sigma_1+\frac{2-\alpha}{2}\sigma_2-\left(\lambda_+ +m+\frac{2-\alpha}{2}(\mu_+ +n)\right)(q-1)}{q-1}} \tag{2.10}$$

If $2-\alpha+\sigma_1+\frac{2-\alpha}{2}\sigma_2 - \left(\lambda_+ + m + \frac{2-\alpha}{2}(\mu_+ + n)\right)(q-1) > 0$, i.e.

$$q < 1 + \frac{2-\alpha+\sigma_1+\frac{2-\alpha}{2}\sigma_2}{\lambda_+ + m + \frac{2-\alpha}{2}(\mu_+ + n)},$$

then passing to limit as $\rho \to +\infty$ from (2.11) we get,

$$\int\limits_{Q'} |x|^{\sigma_1}|y|^{\sigma_2}|u|^q \xi\eta dxdydt \leq 0.$$

Hence it follows that in this case $u(x,y,t) \equiv 0$.

Now let

$$q = 1 + \frac{2-\alpha+\sigma_1+\frac{2-\alpha}{2}\sigma_2}{\lambda_+ + m + \frac{2-\alpha}{2}(\mu_+ + n)}. \tag{2.11}$$

Then from (2.11) we get

$$\int\limits_{Q'} |x|^{\sigma_1}|y|^{\sigma_2}|u|^q \xi\eta dxdydt \leq C. \tag{2.12}$$

Hence and from the properties of the integral it follows $\rho \to \infty$ as

$$G_t(\rho) = \int\limits_{\rho^\varkappa}^{2\rho^\varkappa}\int\limits_{B'} |x|^{\sigma_1}|y|^{\sigma_2}|u|^q \xi\eta\varphi\psi T dxdydt$$

$$\leq \int_{\rho^{\varkappa}}^{2\rho^{\varkappa}} \int_{B'} |x|^{\sigma_1}|y|^{\sigma_2}|u|^q \xi\eta dxdydt \to 0, \tag{2.13}$$

$$G_x(\rho) = \int_0^{2\rho^{\varkappa}} \int_{B_y(1,2\rho^{\varepsilon})} \int_{B_x(\rho,2\rho)} |x|^{\sigma_1}|y|^{\sigma_2}|u|^q \xi\eta\varphi\psi T dxdydt$$

$$\leq \int_0^{2\rho^{\varkappa}} \int_{B_y(1,2\rho^{\varepsilon})} \int_{B_x(\rho,2\rho)} |x|^{\sigma_1}|y|^{\sigma_2}|u|^q \xi\eta dxdydt \to 0, \tag{2.14}$$

$$G_y(\rho) = \int_0^{2\rho^{\varkappa}} \int_{B_y(\rho^{\varepsilon},2\rho^{\varepsilon})} \int_{B_x(1,2\rho)} |x|^{\sigma_1}|y|^{\sigma_2}|u|^q \xi\eta\varphi\psi T dxdydt$$

$$\leq \int_0^{2\rho^{\varkappa}} \int_{B_y(\rho^{\varepsilon},2\rho^{\varepsilon})} \int_{B_x(1,2\rho)} |x|^{\sigma_1}|y|^{\sigma_2}|u|^q \xi\eta dxdydt \to 0. \tag{2.15}$$

From (2.7),(1.8), (2.10) it follows that for (2.11) the integrals I_1, I_2, I_3 are bounded. Then using (2.13),(2.14), (2.15) from (2.5) we have

$$G^{q'} \leq C\left(G_t^{\frac{q'}{q}}(\rho)I_1 + G_y^{\frac{q'}{q}}(\rho)I_2 + G_x^{\frac{q'}{q}}(\rho)I_3\right) \leq C\left(G_t^{\frac{q'}{q}}(\rho) + G_y^{\frac{q'}{q}}(\rho) + G_x^{\frac{q'}{q}}(\rho)\right).$$

Hence,

$$G \leq C\left(G_t^{\frac{q'}{q}}(\rho) + G_y^{\frac{q'}{q}}(\rho) + G_x^{\frac{q'}{q}}(\rho)\right) \to 0 \quad as \quad \rho \to +\infty.$$

So, in this case also

$$\int_{Q'} |x|^{\sigma_1}|y|^{\sigma_2}|u|^q \xi\eta dxdydt \leq 0.$$

Hence we again obtain $u(x,y,t) \equiv 0$.
The theorem is completely proved.

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