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On uniform equiconvergence for Dirac type $2m \times 2m$ systems

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Abstract. *The Dirac type $2m \times 2m$ system on $G = (0, \pi)$ is considered. Theorems on componentwise uniform equiconvergence with trigonometric series, and componentwise localization principle are proved.*

Keywords. eigen vector-functions, uniform equiconvergence, localization principle.

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1 Introduction and problem statement

In the paper we study componentwise uniform equiconvergence on a compact with trigonometric series of expansions of $2m$-component vector-functions in orthogonal series in eigen-functions of a Dirac type operator

$$D\psi = B\frac{d\psi}{dx} + P(x)\psi, \quad x \in G = (0, \pi), \tag{1.1}$$

where $B = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ or $B = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$, I_m is a unit operator in $\mathbb{R}^m$, $J_m = (\alpha_{ij})_{i,j=1}^m$, $\alpha_{k,\ m-k+1} = 1$, $k = \overline{1, m}$; $\alpha_{ij} = 0$ for $(i, j) \neq (k,\ m-k+1)$, $k = \overline{1, m}$; $P(x) = diag\ (p_1(x),\ p_2(x), ..., p_{2m}(x))$; $\psi_{(x)} = \left(\psi(x)^1,\ \psi(x)^2, ..., \psi(x)^{2m}\right)^T$; $p_l(x)$, $l = \overline{1, 2m}$ are real-valued functions from $L_p(0, \pi)$, $p > 2$.

A method allowing to establish uniform equiconvergence of spectral expansions responding to differential operators was developed in the papers [3-5]. The given method was modified in the paper [6] and allowed to establish componentwise uniform equiconvergence in the case of a Schrodinger operator with a matrix potential. Later on, a componentwise uniform convergence for an arbitrary order ordinary differential operator was established in the paper [8], and componentwise uniform convergence rate was studied in [9]. For the Dirac operator these issues were studied in the papers [10,1].

In the present paper we study componentwise uniform equiconvergence for a Dirac type $2m \times 2m$ system (1.1), and prove theorems on componentwise uniform equaiconvergence and componentwise localization principle.

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Let $L_p^{2m}(0,\pi)$ be a space of $2m$-component vector-functions $f(x) = (f_1(x),\ f_2(x),$ $..., f_{2m}(x))^T$ with the norm

$$\|f\|_p = \|f\|_{p,2m} = \Big\{ \int_G \Big(\sum_{j=1}^{2m} \Big| f_j(x) \Big|^2 \Big)^{\frac{p}{2}} dx \Big\}^{\frac{1}{p}}.$$

In the case $p = \infty$ the norm of the vector-function $f(x)$ is determined by the equality $\|f\|_\infty = \|f\|_{\infty,2m} = \operatorname*{ess\,sup}_{x\in\overline{G}} |f(x)|$. For $f \in L_2^{2m}(G)$ and $g \in L_2^{2m}(G)$ there exists a scalar product $(f,g) = \int_G \sum_{j=1}^{2m} f_j(x)\ \overline{g_j(x)}\ dx$.

Under the eigen vector-function of the operator (1.1) responding to the eigen-value λ, we will understand any identical non-zero $2m$-component vector-function $\psi(x)$ absolutely continuous on $\overline{G}$ and almost everywhere in G satisfying the equation $D\psi = \lambda\psi$ (see [6]).

Let $\{\psi_k(x)\}_{k=1}^\infty$ be a complete, orthonormed in $L_2^{2m}(G)$ system consisting of eigen vector-functions of the operator D, while $\{\lambda_k\}_{k=1}^\infty$, $\lambda_k \in R$, an appropriate system of eigen-values.

For arbitrary $f \in L_2^{2m}(G)$ we introduce partial sum of its spectral expansion in the system $\{\psi_k(x)\}_{k=1}^\infty$:

$$\sigma_\nu(x,f) = \left(\sigma_\nu^1(x,f),\ \sigma_\nu^2(x,f), ..., \sigma_\nu^{2m}(x,f)\right)^T,$$

where

$$\sigma_\nu^j(x,f) = \sum_{|\lambda_k|\le\nu} (f,\psi_k)\ \psi_k^j(x), \psi_k(x) = \left(\psi_k^1(x),\ \psi_k^2(x), ..., \psi_k^{2m}(x)\right)^T,$$

$$f(x) = (f_1(x),\ f_2(x),\ ...,\ f_{2m}(x))^T.$$

Along with the partial sum $\sigma_\nu(x,f)$ we determine the vector $S_\nu(x,f) = (S_\nu(x,f_1),$ $S_\nu(x,f_2), ..., S_\nu(x,f_{2m}))^T$, where $S_\nu(x,f_j)$, $j = \overline{1,2m}$, is a modified partial sum of trigonometric series of the function $f_j(x)$, i.e.

$$S_\nu(x,f_j) = \frac{1}{\pi} \int_G \frac{\sin\nu\ (x-y)}{x-y}\ f_j(y)\ dy, \nu > 0.$$

The main results of the present paper are concentrated on the following two theorems.

Theorem 1.1. *Let the functions $p_l(x)$, $l = \overline{1,\ 2m}$, belong to the class $L_p(G)$, $p > 2$. Then for arbitrary vector-function $f \in L_2^{2m}(G)$ on any compact $K \subset G$ the following equality is valid:*

$$\lim_{\nu\to+\infty} \left\|\sigma_\nu^j(\cdot,f) - S_\nu(\cdot,f_j)\right\|_{C(K)} = 0, j = \overline{1,\ 2m}, \tag{1.2}$$

i.e. the j-th component of expansion of the vector-function $f \in L_2^{2m}(G)$ in orthogonal series in the system $\{\psi_k(x)\}_{k=1}^\infty$ uniformly equiconverges on any compact $K \subset G$ with expansion in trigonometric Fourier series of the corresponding j-th component $f_j(x)$ of the vector-function $f(x)$.

Theorem 1.2. *Let the conditions of Theorem 1.1 be fulfilled. Then for orthogonal expansion of arbitrary function $f \in L_2^{2m}(G)$ in the system $\{\psi_k(x)\}_{k=1}^\infty$ the componentwise localization principle is valid: convergence or divergence of the j-th component of the mentioned expansion at the point $x_0 \in G$ depends on behavior in small vicinity of this point x_0 of only appropriate j-th component $f_j(x)$ of the expanded vector-function $f(x)$ (and is independent of behavior of other components).*

2 Some auxiliary facts

In the proof of theorem 1.1 we essentially use the following statements.

Lemma 2.1. *If $p_l \in L_1(G)$, $l = \overline{1,\, 2m}$, and the points $x - t,\ x,\ x + t$ belong to the domain $G = (0, \pi)$, then for the functions $\psi_k(x)$ the following formulas are valid:*

$$\psi_k\,(x \pm t) = (\cos\lambda_k t \cdot I \pm \sin\lambda_k t \cdot B)\ \psi_k(x) \tag{2.1}$$
$$\pm \int_x^{x\pm t} \{\sin\lambda_k(t - |x - \xi|) \cdot I + \mathrm{sign}(\xi - x)\ \cos\lambda_k(t - |x - \xi|) \cdot B\}\ P(\xi)\,\psi_k(\xi)\,d\xi,$$

$$\frac{\psi_k(x - t) + \psi_k(x + t)}{2} = \ \psi_k(x)\ \cos\lambda_k t \tag{2.2}$$
$$+\frac{1}{2}\int_{x-t}^{x+t} \{\sin\lambda_k(t - |x - \xi|) \cdot I + \mathrm{sign}(\xi - x)\ \cos\lambda_k(t - |x - \xi|) \cdot B\}\ P(\xi)\,\psi_k(\xi)\,d\xi,$$

where I is a unit operator in $\mathbb{R}^{2m}$.

Lemma 2.1 was proved in [11] for $m = 1$, and for arbitrary $m \geq 1$ in the paper [2]. Formula (2.1) allows to continue continuously the eigen function $\psi_k(x)$ up to the boundary G. Therefore, eigen-functions will be absolutely continuous on $\overline{G}$. Consequently, for $p_l \in L_p(G)\,,\ p \geq 1$, $l = \overline{1,\, 2m}$, from the equation $D\psi_k = \lambda_k\,\psi_k$ we have $\psi_k' \in L_p^{2m}(G)$, i.e. $\psi_k^j \in W_p^1(G)$, $j = \overline{1,\, 2m}$.

Lemma 2.2. *Let $p_l \in L_1(G)$, $l = \overline{1,\, 2m}$. Then there exists a constant $C_0 > 0$ such that*

$$\|\psi_k\|_{\infty,\,2m} \leq C_0\ , \quad k = 1, 2, ... \tag{2.3}$$

Validity of estimation (2.3) follows from the estimation (see [2])

$$\|\psi_k\|_{\infty,\,2m} \leq C\,(1 + |Im\lambda_k|)^{\frac{1}{2}}\ \|\,\psi_k\|_{2,\,2m}$$

with regard to $Im\lambda_k = 0$, $k = 1, 2, ...$, and orthonormality of the system$\{\psi_k(x)\}_{k=1}^{\infty}$.

Lemma 2.3. *Let $p_l \in L_2(G)\,,\, l = \overline{1,\, 2m}$. Then there exists a constant C_1 such that*

$$\sum_{|\,|\lambda_k| - \tau| \leq 1} 1 \leq C_1,\ \ \forall \tau \geq 0. \tag{2.4}$$

Validity of estimation (2.4) was established in the paper [11] for $m = 1$. For arbitrary $m \geq 1$ the given estimation is proved in the same way. This time the shift formula (2.1) that is valid for any $m \geq 1$, is used.

From estimations (2.3) and (2.4) we have

$$\sum_{||\lambda_k| - \tau| \leq 1} |\psi_k(x)|^2 \leq C_2, \quad x \in \overline{G},\ \ \forall \tau \geq 0, \tag{2.5}$$

where $C_2 > 0$ is some constant.

Let us consider the following integrals dependent on the parameters $\lambda_k,\ \nu,\ r,\ R$:

$$T_k^1\,(r, R, \nu) = \int_r^R t^{-1} \sin\nu\,t\,\sin\,\lambda_k\,(t - r)\ \,dt;$$

$$T_k^2\,(r, R, \nu) = \int_r^R t^{-1} \sin\nu\,t\,\cos\,\lambda_k\,(t - r)\ \,dt,$$

$$0 < r < R < \infty\ ,\ \ \nu > 0, \quad k \in N.$$

Lemma 2.4. *(see [10], [1]). For the integrals* $T_k^i\,(r, R, \nu)$*,* $i = 1, 2$*;* $k \in N$*, the following estimation is valid:*

$$\left|T_n^i\right| \le C_3\,(\alpha)\ \begin{cases} |\nu - |\lambda_k||^{-\alpha}\, r^{-\alpha} & for \quad |\nu - |\lambda_k|| \ge 1, \\ \max\ \{|\ln r|\ ,\ |\ln R|\} & for \quad |\nu - |\lambda_k|| \le 1, \end{cases} \tag{2.6}$$

where $\alpha \in (0, 1]$.

Let

$$\delta_k(\nu) = \begin{cases} 1 & for \quad \nu > |\lambda_k|\ , \\ \frac{1}{2} & for \quad \nu = |\lambda_k|\ , \\ 0 & for \quad \nu < |\lambda_k|\ . \end{cases}$$

The following estimation (see [1], §2, lemma 6) is fulfilled for the numbers $\delta_k(\nu)$

$$\left|\frac{2}{\pi}\int_0^R t^{-1}\sin \nu\, t\, \cos \lambda_k t\, dt - \delta_k(\nu)\right| \le \frac{C\,(R)}{1 + |\nu - |\lambda_k||}. \qquad (2.7).$$

3 Proof of theorems 1.1 and 1.2.

Proof of theorem 1.1. Let $f(x) = (f_1(x),\ f_2(x)\,, ...,\ f_{2m}\,(x))^T$ be an arbitrary function from the space $L_2^{2m}(G)$. We fix an arbitrary connected compact $K \subset G$ and the number R, satisfying the condition $0 < 2R < dist\,(K,\ \partial G)$.

Let $\tilde{S}_\nu(x, f) = \left(\tilde{S}_\nu\,(x, f_1)\ ,\ \tilde{S}_\nu\,(x, f_2)\ , ...,\ \tilde{S}_\nu\,(x, f_{2m})\right)^T$, where $\tilde{S}_\nu\,(x, f_j)\ ,\ j = \overline{1, 2m}$, is a modified partial sum of order ν of trigonometric Fourier series of the function $f_j(x)$, i.e.

$$\tilde{S}_\nu\,(x, f_j) = \frac{1}{\pi}\int_{|x-y| \le R} \frac{\sin\ \nu\,(x-y)}{x-y}\ f_j\,(y)\ dy\,, \quad x \in K\ \ ,\ \ j = \overline{1, 2m}\ .$$

Since the difference $S_\nu\,(x, f_j) - \tilde{S}_\nu\,(x, f_j)$ tends to zero uniformly with respect to $x \in K$ as $\nu \to +\infty$, then to prove theorem 1.1 it suffices to compare the partial sum $\sigma_\nu(x, f)$ with $\tilde{S}_\nu(x, f) = \left(\tilde{S}_\nu\,(x, f_1)\ ,\ \tilde{S}_\nu\,(x, f_2)\ , ...,\ \tilde{S}_\nu\,(x, f_{2m})\right)^T$ and establish estimation (1.2) for $\tilde{S}_\nu(x, f)$.

Since $\{\psi_k(x)\}_{k=1}^\infty$ is a complete orthonormed system in $L_2^{2m}(G)$, then it forms a basis in $L_2^{2m}(G)$. Consequently, any function $f \in L_2^{2m}(G)$ may be represented in the form:

$$f(x) = \sum_{k=1}^{\infty} (f, \psi_k)\ \ \psi_k(x),$$

convergent in the norm of space $L_2^{2m}(G)$. Therefore, for $\tilde{S}_\nu(x, f)$ the following representation is valid:

$$\tilde{S}_\nu(x, f) = \frac{2}{\pi}\ \sum_{k=1}^{\infty} (f, \psi_k)\ \int_0^R \frac{\sin \nu\, t}{t}\ \frac{\psi_k\,(x-t) + \psi_k\,(x+t)}{2}\ dt. \tag{3.1}$$

Applying the mean value formula (2.2), we transform the integral entering into the representation (3.1).

$$\frac{2}{\pi}\int_0^R \frac{\sin\nu t}{t}\ \frac{\psi_k(x-t)+\psi_k(x+t)}{2}\ dt = \frac{2}{\pi}\int_0^R \frac{\sin\nu t}{t}\ \cos\lambda_k t\,dt\ \ \psi_k(x)$$

$$+\frac{1}{\pi}\int_0^R \frac{\sin\nu t}{t}\int_{x-t}^{x+t}\left\{\sin\lambda_k(t-|x-\xi|)\cdot I+\mathrm{sign}(\xi-x)\,\cos\lambda_k(t-|x-\xi|)\cdot B\right\} P(\xi)\,\psi_k(\xi)\,d\xi$$

$$=\frac{2}{\pi}\int_0^R \frac{\sin\nu t}{t}\cos\lambda_k t dt\,\psi_{\mathrm{k}}(x)+\frac{1}{\pi}\int_{x-R}^{x+R}\Big(\int_{|x-\xi|}^R \frac{\sin\,\nu t}{t}\left\{\,\sin\lambda_k(t-|x-\xi|)\cdot I\right.$$

$$\left.+\mathrm{sign}(\xi-x)\,\cos\lambda_k(t-|x-\xi|)\cdot B\ dt\right\}\Big)\times P(\xi)\ \psi_k(\xi)\,d\xi$$

$$=\delta_k(\nu)\,\psi_k(x)+\left[\frac{2}{\pi}\int_0^R \frac{\sin\nu t}{t}\ \cos\lambda_k t\,dt-\delta_k(\nu)\right]\psi_k(x)$$

$$+\frac{1}{\pi}\int_{x-R}^{x+R}\left\{T_k^1(|x-\xi|,R,\nu)\cdot I+\mathrm{sign}(\xi-x)\,T_k^2(|x-\xi|,R,\nu)\cdot B\right\}\ P(\xi)\ \psi_k(\xi)\,d\xi\,.$$

Taking this into account in equality (3.1), applying estimation (2.7) and equality

$$\sum_{k=1}^{\infty}(f,\psi_k)\ \delta_k(\nu)\ \psi_k(x)=\sigma_\nu(x,f)-\frac{1}{2}\sum_{|\lambda_k|=\nu}(f,\psi_k)\ \psi_k(x)\quad,$$

we get

$$\left|\sigma_\nu(x,f)-\tilde{S}_\nu(x,f)\right|\le\frac{1}{2}\sum_{|\lambda_k|=\nu}|(f,\psi_k)|\ |\psi_k(x)|+C\,(R)\sum_{k=1}^{\infty}|(f,\psi_k)|\ \frac{|\psi_k(x)|}{1+||\lambda_k|-\nu|}$$

$$+C\sum_{k=1}^{\infty}|(f,\psi_k)|\ \left|\int_0^R\{P(x-r)\,\psi_k(x-r)+P(x+r)\,\psi_k(x+r)\}\ T_k^1\,(r,R,\nu)\ dr\right|$$

$$+C\sum_{k=1}^{\infty}\left|(f,\psi_k)\right|\,\Big|\int_0^R\big\{P(x+r)\,\psi_k(x+r)$$

$$-P(x-r)\,\psi_k(x-r)\big\}\ T_k^2(r,R,\nu)\,dr\Big|=\sum_{l=1}^{4}A_l\,, \tag{3.2}$$

where $C=\frac{1}{\pi}$.

We estimate the expressions $A_l\,,\ \ l=\overline{1,4}$. To estimate the expressions A_1 we use the Bessel inequality and estimation (2.5).

$$A_1=\frac{1}{2}\sum_{|\lambda_k|=\nu}|(f,\psi_k)|\ |\psi_k(x)|\le\frac{1}{2}\Big(\sum_{|\lambda_k|=\nu}|(f,\psi_k)|^2\Big)^{\frac{1}{2}}\cdot\Big(\sum_{|\lambda_k|=\nu}|\psi_k|^2\Big)^{\frac{1}{2}}\le C_4\|f\|_{2,2m}.$$

From estimations (2.3), (2.4) and the Bessel inequality we have

$$A_2\le C\,(R)\left(\sum_{k=1}^{\infty}|(f,\psi_k)|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\frac{|\psi_k(x)|^2}{(1+|\nu-|\lambda_k|\,|)^2}\right)^{\frac{1}{2}}$$

$$\le C_5\|f\|_{2,2m}\left(\sum_{k=1}^{\infty}\frac{1}{(1+|\nu-|\lambda_k||)^2}\right)^{\frac{1}{2}}\le C_5\|f\|_{2,2m}\left(\sum_{n=1}^{\infty}(1+n)^{-2}\sum_{n\le|\nu-|\lambda_k||\le n+1}1\right)^{\frac{1}{2}}$$

$$\le C_6\|f\|_{2,2m}\left(\sum_{l=1}^{\infty}l^{-2}\right)^{\frac{1}{2}}\le C_7\|f\|_{2,2m}.$$

We estimate series A_3 (series A_4 is estimated in the same way). We prove that the series

$$B_3=\sum_{k=1}^{\infty}\left(\int_0^R L(x,r)\left|T_k^1(r,R,\nu)\right|dr\right)^2, \tag{3.3}$$

where

$$L(x,r)=\sum_{j=1}^{2m}\left(|p_j(x+r)|+|p_j(x-r)|\right)$$

uniformly converges with respect to $x\in K$.

Obviously,

$$\left(\int_0^R L^p(x,r)\,dr\right)^{\frac{1}{p}}\le C(m)\sum_{j=1}^{2m}\|p_j\|_p\,, \tag{3.4}$$

where $\|\cdot\|_p=\|\cdot\|_{L_p(G)}$, $p>2$.

Apply lemma (2.4) to series (3.3) for $\alpha\in\left(\frac{1}{2},\ \frac{p-1}{p}\right)$, $p>2$. As a result we get

$$B_3=\sum_{|\nu-|\lambda_k||<1}\left(\int_0^R L(x,r)\left|T_k^1(r,R,\nu)\right|dr\right)^2+\sum_{|\nu-|\lambda_k||\ge1}\left(\int_0^R L(x,r)\left|T_k^1(r,R,\nu)\right|dr\right)^2$$

$$\le\sum_{|\nu-|\lambda_k||<1}\left(\int_0^R L(x,r)\max\{|\ln R|\,,\,|\ln r|\}\,dr\right)^2+\sum_{|\nu-|\lambda_k||\ge1}\left(\int_0^R L(x,r)\,r^{-\alpha}\,dr\right)^2|\nu-|\lambda_k||^{2\alpha}.$$

For $p>2$ apply the Holder inequality in each integral, take into account estimations (2.4), (3.4) and the inequality $\frac{1}{2}<\alpha<\frac{p-1}{p}=\frac{1}{q}$:

$$B_3\le C_1(m)\sum_{j=1}^{2m}\|p_j\|_p\left\{\left(\int_0^R(\max\{|\ln R|\,,\,|\ln r|\})^q\,dr\right)^{\frac{2}{q}}\cdot\sum_{|\nu-|\lambda_k||\le1}1\right.$$

$$\left.+\left(\int_0^R r^{-\alpha q}\,dr\right)^{\frac{2}{q}}\sum_{|\nu-|\lambda_k||\ge1}|\nu-|\lambda_k||^{-2\alpha}\right\}$$

$$\le C_2(m)\Big\{\sum_{|\nu-|\lambda_k||\le1}1+\sum_{n=1}^{\infty}\sum_{n\le|\nu-|\lambda_k||\le n+1}|\nu-|\lambda_k||^{-2\alpha}\Big\}$$

$$\le C_3(m)\Big\{1+\sum_{n=1}^{\infty}n^{-2\alpha}\Big(\sum_{n\le|\nu-|\lambda_k||\le n+1}1\Big)\Big\}\le C_4(m)\Big\{1+\sum_{n=1}^{\infty}n^{-2\alpha}\Big\}\le C_5(m).$$

From the Bessel inequality, estimation (2.3) and convergence of series B_3, we get

$$A_3 \leq C \sum_{k=1}^{\infty} |(f, \psi_k)| \left| \int_0^R \{P(x-r)\,\psi_k(x-r) + P(x+r)\,\psi_k(x-r)\} \; T_k^1\,(r, R, \nu)\; dr \right|$$

$$\leq C_8 \sum_{k=1}^{\infty} |(f, \psi_k)| \int_0^R L(x,r) \; \left| T_k^1\,(r, R, \nu) \right| \; dr$$

$$\leq \; C_8 \left(\sum_{k=1}^{\infty} |(f, \psi_k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \left(\int_0^R L(x,r) \; \left| T_k^1\,(r, R, \nu) \right| \; dr \right)^2 \right)^{\frac{1}{2}} \leq C_9 \|f\|_{2,2m}.$$

The same estimation is fulfilled for A_4 as well.

Consequently, by virtue of the obtained estimations for the sum $A_l\,,\;\; l = \overline{1,4}$, from (3.2) it follows that for any function $f \in L_2^{2m}(G)$ the estimation

$$\left| \sigma_\nu(x,f) - \tilde{S}_\nu(x,f) \right| \leq C_{10}(K) \|f\|_{2,2m}. \tag{3.5}$$

uniform with respect to $x \in K$ is fulfilled.

Now from estimation (3.5) we derive the relation

$$\lim_{\nu \to +\infty} \left\| \sigma_\nu\,(\cdot, f) - \tilde{S}_\nu\,(\cdot, f) \right\|_{C(K)} = 0. \tag{3.6}$$

From the completeness of the system $\{\psi_k(x)\}_{k=1}^{\infty}$ in the space $L_2^{2m}(G)$ it follows that for any $\varepsilon > 0$ there exist the constants α_k, $k = \overline{1,\, n(\varepsilon, f)}$ such that

$$\left\| f - \sum_{k=1}^{n(\varepsilon,f)} \alpha_k \psi_k \right\|_{2,2m} < \frac{\varepsilon}{4\,C_{10}(K)}.$$

Denote $\varphi(x) = \sum\limits_{k=1}^{n(\varepsilon,f)} \alpha_k \psi_k(x)$. Then

$$\left\| \tilde{S}_\nu\,(\cdot,\; f) - \sigma_\nu\,(\cdot,\; f) \right\|_{C(K)} = \left\| \tilde{S}_\nu\,(\cdot, f-\varphi) + \tilde{S}_\nu\,(\cdot,\; \varphi) - \sigma_\nu\,(\cdot,\; f-\varphi) - \sigma_\nu\,(\cdot,\; \varphi) \right\|_{C(K)}$$

$$\leq \left\| \sigma_\nu\,(\cdot,\; f-\varphi) - \tilde{S}_\nu\,(\cdot, f-\varphi) \right\|_{C(K)} + \left\| \tilde{S}_\nu\,(\cdot,\; \varphi) - \sigma_\nu\,(\cdot,\; \varphi) \right\|_{C(K)}.$$

From estimation (3.5) and equality $\sigma_\nu\,(x, \varphi) = \varphi(x)$ for rather large ν we get

$$\left\| \sigma_\nu(x,f) - \tilde{S}_\nu(x,f) \right\|_{C(K)} \leq C_{10}(K)\,\|f - \varphi\|_{2,2m} + \left\| \tilde{S}_\nu\,(\cdot,\; \varphi) - \varphi \right\|_{C(K)},$$

$$\varphi(x) = (\varphi_1(x),\, \varphi_2(x),\, ...,\, \varphi_{2m}(x))^T.$$

Since $\varphi_j(x) \in W_2^1(G)\,,\;\; j = \overline{1,2m}$, then the difference $\tilde{S}_\nu\,(x, \varphi_j) - \varphi_j(x)$ tends to zero uniformly with respect to $x \in K$ as $\nu \to +\infty$ for each fixed j. Consequently, $\nu > \nu_0 > 0$ (ν_0 is a rather large number)

$$\left\| \sigma_\nu\,(\cdot, f) - \tilde{S}_\nu\,(\cdot, f) \right\|_{C(K)} \leq \frac{C_{10}(K)\,\varepsilon}{(4\,C_{10}(K))} + \frac{\varepsilon}{2} < \varepsilon$$

is fulfilled, i.e.

$$\lim_{\nu \to +\infty} \left\| \sigma_\nu^j \left(\cdot, f\right) - \tilde{S}_\nu \left(\cdot, f_j\right) \right\|_{C(K)} = 0 \ , \quad j = \overline{1,\, 2m},$$

is fulfilled.
Theorem 1.1 is completely proved.

The statement of Theorem 1.2 follows from the proved theorem 1.1 with regard to the localization principle for Fourier trigonometric series.

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