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Anisotropic fractional maximal commutators with BMO functions on anisotropic Morrey-type spaces

Ali Akbulut ⋆ · **Victor I. Burenkov** · **Vagif S. Guliyev**

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Abstract. *In the present paper, we shall give necessary and sufficient conditions for the boundedness of anisotropic fractional maximal commutator $M^d_{b,\alpha}$ on anisotropic local Morrey-type spaces, when b belongs to BMO spaces, by which some new characterizations for BMO spaces is obtained. As an application of this results we consider the boundedness of commutators of anisotropic fractional maximal operator $[b, M^d_\alpha]$ on anisotropic global Morrey-type spaces.*

Keywords. Anisotropic local and global Morrey-type spaces; anisotropic fractional maximal function; commutator; BMO

Mathematics Subject Classification (2010): 42B25; 46E30

1 Introduction

Let T be the classical singular integral operator, the commutator $[b, T]$ generated by T and a suitable function b is given by

$$[b,T](f)(x) = T\big((b(x)-b)f\big)(x) = b(x)T(f)(x) - T(bf)(x). \tag{1.1}$$

A well known result due to Coifman, Rochberg and Weiss [21] states that $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ when $b \in BMO(\mathbb{R}^n)$. They also gave a characterization of $BMO(\mathbb{R}^n)$ in virtue of the L^p-boundedness of the above commutator. In 1978, Janson [49] gave some characterizations of Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via commutator $[b, T]$

⋆ Corresponding author

A. Akbulut
Ahi Evran University, Department of Mathematics, Kirsehir, Turkey
E-mail: akbulut72@gmail.com

V.I. Burenkov
S.M. Nikolskii Institute of Mathematics at RUDN University, Moscow, Russia
Cardiff School of Mathematics, Cardiff University, Cardiff, UK
E-mail: burenkov@cardiff.ac.uk

V.S. Guliyev
Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
Dumlupinar University, Department of Mathematics, Kutahya, Turkey
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
E-mail: vagif@guliyev.com

and proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ $(0 < \beta < 1)$ if and only if $[b, T]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ where $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$.

In 1938, Morrey considered regularity of the solution of elliptic differential equations in terms of the solutions themselves and their derivatives. This is a very famous work by Morrey (see [52]). Later many people studied Morrey spaces from a various point of view. After studying Morrey spaces in detail, people are led to considering the local and global counterpart.

Recall that in 1994 the doctoral thesisis [37] (see, also [38]) by third author introduced the new function spaces $\mathit{\Gamma}^*_{p\theta}(\mathbb{G}, w)$ defined on homogeneous Lie groups $\mathbb{G}$ and later called in the paper second author and H.V. Guliyev [10] local Morrey-type space and denote by $LM_{pq,w}(\mathbb{R}^n)$ is given by

$$\|f\|_{LM_{pq,w}(\mathbb{R}^n)} = \big\| w(r) \|f\|_{L_p(B(0,r))} \big\|_{L_\theta(0,\infty)},$$

where w is a positive measurable function defined on $(0;\infty)$.

The main purpose of [37] (see, also [38]) is to give some sufficient conditions for the boundedness of fractional integral operators and singular integral operators defined on homogeneous Lie groups $\mathbb{G}$ in local Morrey-type space $LM_{pq,w}(\mathbb{R}^n)$ (see, [37], pp. 121, Theorem 4.1.6-4.1.11, Theorem 4.2.3-4.2.4.).

In a series of papers (see [10–15]) be given some necessary and sufficient conditions for the boundedness of maximal operators, fractional maximal operators, fractional integral operators and singular integral operators in local Morrey-type space $LM_{pq,w}(\mathbb{R}^n)$.

Let $\mathbb{R}^n$ be the n-dimension Euclidean space with the norm $|x|$ for each $x \in \mathbb{R}^n$, S^{n-1} denotes the unit sphere on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r and ${}^{\complement}B(x,r)$ denote the set $\mathbb{R}^n \backslash B(x,r)$. Let $d = (d_1, \dots, d_n)$, $d_i \geq 1$, $i = 1, \dots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv \big(t^{d_1} x_1, \dots, t^{d_n} x_n\big)$. By [7,23], the function $F(x,\rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points x, $y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([7,9,23]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids

$$\mathcal{E}_d(x,r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \dots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

with the Lebesgue measure $|\mathcal{E}_d(x,r)| = v_n r^{|d|}$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Let also $\mathit{\Pi}_d(x,r) = \{y \in \mathbb{R}^n : \max_{1 \le i \le n} |x_i - y_i|^{1/d_i} < r\}$ denote the parallelopiped, ${}^{\complement}\mathcal{E}_d(x,r) = \mathbb{R}^n \setminus \mathcal{E}_d(x,r)$ be the complement of $\mathcal{E}_d(0,r)$. If $d = \mathbf{1} \equiv (1, \dots, 1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_{\mathbf{1}}(x,r) = B(x,r)$. Note that in the standard parabolic case $d = (1, \dots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The anisotropic fractional maximal operator M^d_α and the anisotropic Riesz potential I^d_α are given by

$$M^d_\alpha f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1+\frac{\alpha}{|d|}} \int_{\mathcal{E}(x,t)} |f(y)| dy, \;\; 0 \le \alpha < |d|,$$

$$I^d_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\rho(x-y)^{|d|-\alpha}} dy, \;\; 0 < \alpha < |d|$$

and the anisotropic maximal commutator of M^d with a locally integrable function b is defined by

$$M^d_{b,\alpha} f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1+\frac{\alpha}{|d|}} \int_{\mathcal{E}(x,t)} |b(x)-b(y)||f(y)|dy, \;\; 0 \le \alpha < |d|,$$

where $|\mathcal{E}(x,t)|$ is the Lebesgue measure of the ellipsoid $\mathcal{E}(x,t)$. If $\alpha = 0$, then $M^d \equiv M^d_0$ is the anisotropic Hardy-Littlewood maximal operator. If $d = \mathbf{1}$, then $M \equiv M^d$ and $M_b \equiv M^d_b$ are the classical Hardy-Littlewood maximal operator and maximal commutator, respectively. If $d = \mathbf{1}$, then $I_\alpha \equiv I^{\mathbf{1}}_\alpha$ is the Riesz potential. The operators M^d_α and I^d_α play an important role in real and harmonic analysis (see, for example [55] and [56]).

On the other hand, we can define the (nonlinear) commutator of the anisotropic fractional maximal operator M^d with a locally integrable function b by

$$[b, M^d_\alpha] f(x) f = b(x) M^d_\alpha f(x) - M^d_\alpha(bf)(x).$$

Obviously, operators $M^d_{b,\alpha}$ and $[b, M^d_\alpha]$ essentially differ from each other since $M^d_{b,\alpha}$ is positive and sublinear and $[b, M^d_\alpha]$ is neither positive nor sublinear.

Our main aim is to characterize the functions involved in the boundedness on anisotropic local Morrey-type spaces $LM_{p\theta,w(\cdot),d}$ of the anisotropic fractional maximal commutator $M^d_{b,\alpha}$ (see, Corollary 4.2). As an application of this result we consider the boundedness of $[b, M^d_\alpha]$ on anisotropic local Morrey-type spaces $LM_{p\theta,w(\cdot),d}$, when b belongs to the BMO space (see, Corollary 4.3).

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Anisotropic fractional maximal operator in anisotropic general Morrey-type spaces

In order to investigate the boundedness properties of the anisotropic fractional maximal commutator $M^d_{b,\alpha}$ it is natural to consider anisotropic local and global Morrey-type spaces.

Definition 2.1 *Let $0 < p, \theta \le \infty$ and let w be a non-negative Lebesgue measurable function on $(0,\infty)$ not equivalent to 0. We denote by $LM_{p\theta,w(\cdot),d}$ and $GM_{p\theta,w(\cdot),d}$, the anisotropic local Morrey-type spaces, the global Morrey-type space respectively, the spaces of all functions f Lebesgue measurable on $\mathbb{R}^n$ with finite quasi-norms*

$$\|f\|_{LM_{p\theta,w(\cdot),d}} \equiv \|f\|_{LM_{p\theta,w(\cdot),d}(\mathbb{R}^n)} = \|w(r)\|f\|_{L_p(\mathcal{E}_d(0,r))}\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GM_{p\theta,w(\cdot),d}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w(\cdot),d}} = \sup_{x\in\mathbb{R}^n} \|w(r)\|f\|_{L_p(\mathcal{E}_d(x,r))}\|_{L_\theta(0,\infty)}$$

respectively.

Note that $LM_{p\theta,w(\cdot)} \equiv LM_{p\theta,w(\cdot),\mathbf{1}}$ and $GM_{p\theta,w(\cdot)} \equiv GM_{p\theta,w(\cdot),\mathbf{1}}$ are the isotropic local and global Morrey-type spaces, respectively. Furthermore, $GM_{p\infty,w(\cdot),d} \equiv M_{p,w(\cdot),d}$ and $WGM_{p\infty,w(\cdot),d} \equiv WM_{p,w(\cdot),d}$ are the anisotropic generalized Morrey and weak generalized Morrey spaces, respectively. Also, $GM_{p\infty,r^{-\lambda/p},d} \equiv M_{p,\lambda,d}$ and $WGM_{p\infty,r^{-\lambda/p},d} \equiv WM_{p,\lambda,d}$ are the anisotropic Morrey and weak Morrey spaces, respectively.

In Definition 2.1 the function w is a very general function. It is natural, first of all, to find conditions ensuring that the spaces $LM_{p\theta,w(\cdot),d}$ and $GM_{p\theta,w(\cdot),d}$ are nontrivial, that is consist not only of functions equivalent to 0 on $\mathbb{R}^n$.

Definition 2.2 *Let $0 < p, \theta \le \infty$. We denote by Ω_θ the set of all functions w which are non-negative and Lebesgue measurable on $(0,\infty)$, not equivalent to 0, and such that for some $t > 0$*

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty. \tag{2.1}$$

Moreover, we denote by $\Omega_{p\theta,d}$, the set of all functions w which are non-negative and Lebesgue measurable on $(0,\infty)$, not equivalent to 0, and such that some $t > 0$

$$\left\|w(r) r^{|d|/p}\right\|_{L_\theta(0,t)} < \infty, \quad \|w(r)\|_{L_\theta(t,\infty)} < \infty. \tag{2.2}$$

Lemma 2.1 *[10, 11] Let $0 < p, \theta \le \infty$ and let w be a non-negative Lebesgue measurable function on $(0,\infty)$, which is not equivalent to 0.*

Then the space $LM_{p\theta,w(\cdot),d}$ is notrivial if and only if $w \in \Omega_\theta$ and the space $GM_{p\theta,w(\cdot),d}$ is notrivial if and only if $w \in \Omega_{p\theta,d}$.

The boundedness of the maximal operator M in Morrey spaces $M_{p,\lambda}$ was studied by Chiarenza and Frasca [18] and the following statement was proved in [47].

Theorem 2.1 (1) *Let $1 < p \le \infty$ and $0 \le \lambda \le \frac{|d|}{p}$. Then the operator M^d is bounded on $M_{p,\lambda,d}$.*

(2) *Let $1 \le p \le \infty$ and $0 \le \lambda \le \frac{|d|}{p}$. Then the operator M^d is bounded from $M_{p,\lambda,d}$ to $WM_{p,\lambda,d}$.*

The boundedness of the maximal operator M from $LM_{p\theta_1,w_1(\cdot)}$ to $LM_{p\theta_2,w_2(\cdot)}$ for general w_1 and w_2 was studied in [10, 11]. In [41, 51, 53] sufficient conditions for the boundedness of M in generalized Morrey spaces where obtained.

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance, [33, 35, 36, 55–57]).

For the first time in the theory of boundedness of operators in general Morrey-type spaces, for a certain range of the numeral parameters $1 < p < \infty$, $0 < \theta_1, \theta_2 \le \infty$ necessary and sufficient conditions on the functions $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$ ensuring the boundedness of M from $LM_{p\theta_1,w(\cdot)}$ to $LM_{p\theta_2,w_2(\cdot)}$ were found by second author and H.V. Guliyev in [10]. Namely, it was assumed that $\theta_1 \le \theta_2$ and $\theta_1 \le p$. Later on in [15] this was proved only under the assumption $\theta_1 \le \theta_2$. We give a complete formulation of these results for the anisotropic case (see [2]).

Theorem 2.2 *[2] Let $1 \le p \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$.*

If $p > 1$, then the condition

$$\left\| \left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{|d|}{p}} \right\|_{L_{\theta_2}(0,\infty)} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \right\|_{L_\infty(0,\infty)} < \infty \tag{2.3}$$

is necessary and sufficient for the boundedness of M^d from $LM_{p\theta_1,w_1(\cdot),d}$ to $LM_{p\theta_2,w_2(\cdot),d}$. Moreover,

$$\|M^d\|_{LM_{p\theta_1,w_1(\cdot),d} \to LM_{p\theta_2,w_2(\cdot),d}} \approx \left\| \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{|d|}{p}} \right\|_{L_{\theta_2}(0,\infty)} \right\|_{L_\infty(0,\infty)} \tag{2.4}$$

uniformly in $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$.

If $p = 1$, then condition (2.3) *with $p = 1$ is necessary and sufficient for the boundedness of M^d from $LM_{1\theta_1,w_1(\cdot),d}$ to $WLM_{1\theta_2,w_2(\cdot),d}$ and*

$$\|M^d\|_{LM_{1\theta_1,w_1(\cdot),d}\to WLM_{1\theta_2,w_2(\cdot),d}} \approx \Big\|\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)}\Big\|w_2(r)\Big(\frac{r}{t+r}\Big)^{|d|}\Big\|_{L_{\theta_2}(0,\infty)}\Big\|_{L_\infty(0,\infty)} \tag{2.5}$$

uniformly in $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$.

In (2.4) *and* (2.5) *it is assumed that* $(+\infty)^{-1} = 0$ *and* $0 \cdot (+\infty) = 0$.

Since

$$\Big\|w_2(r)\Big(\frac{r}{t+r}\Big)^{\frac{|d|}{p}}\Big\|_{L_{\theta_2}(0,\infty)} \approx t^{-\frac{|d|}{p}}\|w_2(r)r^{\frac{|d|}{p}}\|_{L_{\theta_2}(0,t)} + \|w_2\|_{L_{\theta_2}(t,\infty)}$$

uniformly in $t \in (0,\infty)$, condition (2.3) is equivalent to the condition

$$\begin{cases} \Big\|t^{-\frac{|d|}{p}}\|w_2(r)r^{\frac{|d|}{p}}\|_{L_{\theta_2}(0,t)}\,\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)}\Big\|_{L_\infty(0,\infty)} < \infty, \\ \Big\|\|w_2\|_{L_{\theta_2}(t,\infty)}\,\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)}\Big\|_{L_\infty(0,\infty)} < \infty. \end{cases} \tag{2.6}$$

We note the following particular case of Theorem 2.2 related to the boundedness of M^d on $LM_{p\theta,w(\cdot),d}$.

Corollary 2.1 *Let $1 < p < \infty$, $0 < \theta \le \infty$, $w \in \Omega_\theta$.*

Then the condition

$$\big\|t^{-\frac{|d|}{p}}\|w(r)r^{\frac{|d|}{p}}\|_{L_\theta(0,t)}\|w\|^{-1}_{L_\theta(t,\infty)}\big\|_{L_\infty(0,\infty)} < \infty$$

is necessary and sufficient for the boundedness of M^d on $LM_{p\theta,w(\cdot),d}$. Moreover,

$$\|M^d\|_{LM_{p\theta,w(\cdot),d}\to LM_{p\theta,w(\cdot),d}} \approx \Big\|t^{-\frac{|d|}{p}}\|w(r)r^{\frac{|d|}{p}}\|_{L_\theta(0,t)}\|w\|^{-1}_{L_\theta(t,\infty)}\Big\|_{L_\infty(0,\infty)}$$

uniformly in $w \in \Omega_\theta$.

Theorem 2.2 immediately implies sufficient conditions for the boundedness of the operator M^d acting in anisotropic global Morrey-type spaces.

Corollary 2.2 *Let $1 \le p \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{p\theta_1,d}$, $w_2 \in \Omega_{p\theta_2,d}$.*

If $p > 1$, then condition (2.3) *is sufficient for the boundedness of M^d from $GM_{p\theta_1,w_1(\cdot),d}$ to $GM_{p\theta_2,w_2(\cdot),d}$. Moreover, the quasi-norm $\|M^d\|_{GM_{p\theta_1,w_1(\cdot),d}\to GM_{p\theta_2,w_2(\cdot),d}}$ is estimated above via the right-hand side of* (2.4) *uniformly in $w_1 \in \Omega_{p\theta_1,d}$ and $w_2 \in \Omega_{p\theta_2,d}$.*

If $p = 1$, then condition (2.3) *is sufficient for the boundedness of M^d from $GM_{1\theta_1,w_1(\cdot),d}$ to $WGM_{1\theta_2,w_2(\cdot),d}$. Moreover, the quasi-norm $\|M^d\|_{GM_{1\theta_1,w_1(\cdot),d}\to GM_{1\theta_2,w_2(\cdot),d}}$ is estimated above via the right-hand side of* (2.5) *uniformly in $w_1 \in \Omega_{1\theta_1,d}$, $w_2 \in \Omega_{1\theta_2,d}$.*

T. Mizuhara [51], E. Nakai [53] and V.S. Guliyev [37] (see also [38–40]) obtained sufficient conditions on weights w_1 and w_2 ensuring boundedness of the maximal operator M and the fractional maximal operator M_α for the limiting case $\alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ from M_{p_1,w_1} to M_{p_2,w_2}.

Theorem 2.3 *[37] Let* $1 \le p_1 \le p_2 < \infty$ *and* $\alpha = |d| \left(\frac{1}{p_1} - \frac{1}{p_2}\right)$. *Moreover, let* w_1, w_2 *be positive measurable functions satisfying the following condition:*

$$\left\| w_1^{-1}(r)\, r^{\alpha - \frac{|d|}{p_1} - 1} \right\|_{L_1(t,\infty)} \lesssim w_2^{-1}(t)\, t^{\alpha - \frac{|d|}{p_1}} \tag{2.7}$$

uniformly in $t > 0$.

Then for $p_1 > 1$ M_α^d *is bounded from* $M_{p_1,w_1(\cdot),d}$ *to* $M_{p_2,w_2(\cdot),d}$ *and for* $p_1 = 1$ M_α^d *is bounded from* $M_{1,w_1(\cdot),d}$ *to* $WM_{p_2,w(\cdot),d}$.

Earlier, in [51], [53] a weaker version of Theorem 2.3 was proved: it was assumed that $w_1 = w_2 = w$ and that w is a positive non-increasing function satisfying the pointwise doubling condition, namely that for some $c > 0$

$$c^{-1}\, w(r) \le w(t) \le c\, w(r)$$

for all t, $r > 0$ such that $0 < r \le t \le 2r$.

For the first time in the theory of boundedness of operators in general Morrey-type spaces, for a certain range of the numeral parameters $1 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $0 < \theta_1, \theta_2 \le \infty$ necessary and sufficient conditions on the functions $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$ ensuring the boundedness of M_α from $LM_{p_1\theta_1,w(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ were found by V.I. Burenkov, H.V. Guliyev, V.S. Guliyev in [11]. Namely, it was assumed that $\theta_1 \le \theta_2$ and $\theta_1 \le p$. Later on in [15] this was proved only under the assumption $\theta_1 \le \theta_2$. We give a complete anisotropic formulation of these results [2] (see also [3,4]).

Theorem 2.4 *[2] Let* $1 \le p_1 \le p_2 \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = |d| \left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$.

If $p_1 > 1$, *then the condition*

$$\left\| w_2(r) \Big(\frac{r}{t+r}\Big)^{\frac{|d|}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} \lesssim \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{2.8}$$

uniformly in $t \in (0,\infty)$ *is necessary and sufficient for the boundedness of* M_α^d *from* $LM_{p_1\theta_1,w_1(\cdot),d}$ *to* $LM_{p_2\theta_2,w_2(\cdot),d}$. *Moreover,*

$$\|M_\alpha^d\|_{LM_{p_1\theta_1,w_1(\cdot),d} \to LM_{p_2\theta_2,w_2(\cdot),d}} \approx \sup_{0<t<\infty} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \left\| w_2(r) \Big(\frac{r}{t+r}\Big)^{\frac{|d|}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} \tag{2.9}$$

uniformly in $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$.

If $p_1 = 1$, *then condition* (2.8) *with* $p = 1$ *is necessary and sufficient for the boundedness of* M_α^d *from* $LM_{1\theta_1,w_1(\cdot),d}$ *to* $WLM_{p_2\theta_2,w_2(\cdot),d}$ *and*

$$\|M_\alpha^d\|_{LM_{1\theta_1,w_1(\cdot),d} \to WLM_{p_2\theta_2,w_2(\cdot),d}} \approx \sup_{0<t<\infty} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \left\| w_2(r) \Big(\frac{r}{t+r}\Big)^{\frac{|d|}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} \tag{2.10}$$

uniformly in $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$.

In (2.9) *and* (2.10) *it is assumed that* $(+\infty)^{-1} = 0$ *and* $0 \cdot (+\infty) = 0$.

Since

$$\Big\| w_2(r)\Big(\frac{r}{t+r}\Big)^{\frac{|d|}{p_2}}\Big\|_{L_{\theta_2}(0,\infty)} \approx t^{-\frac{|d|}{p_2}}\|w_2(r)r^{\frac{|d|}{p_2}}\|_{L_{\theta_2}(0,t)} + \|w_2\|_{L_{\theta_2}(t,\infty)}$$

uniformly in $t \in (0,\infty)$, condition (2.8) is equivalent to the condition

$$\begin{cases} t^{-\frac{|d|}{p_2}}\|w_2(r)r^{\frac{|d|}{p_2}}\|_{L_{\theta_2}(0,t)} \lesssim \|w_1\|_{L_{\theta_1}(t,\infty)} \\ \|w_2\|_{L_{\theta_2}(t,\infty)} \lesssim \|w_1\|_{L_{\theta_1}(t,\infty)} \end{cases} \tag{2.11}$$

uniformly in $t \in (0,\infty)$.

Theorem 2.4 immediately implies sufficient conditions for the boundedness of the operator M^d_α acting in anisotropic global Morrey-type spaces.

Corollary 2.3 *Let* $1 \le p_1 \le p_2 \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{p_1\theta_1,d}$, *and* $w_2 \in \Omega_{p_2\theta_2,d}$.

If $p_1 > 1$, *then condition* (2.8) *is sufficient for the boundedness of* M^d_α *from* $GM_{p_1\theta_1,w_1(\cdot),d}$ *to* $GM_{p_2\theta_2,w_2(\cdot),d}$. *Moreover, the quasi-norm* $\|M^d_\alpha\|_{GM_{p_1\theta_1,w_1(\cdot),d}\to GM_{p_2\theta_2,w_2(\cdot),d}}$ *is estimated above via the right-hand side of* (2.9) *uniformly in* $w_1 \in \Omega_{p\theta_1,d}$ *and* $w_2 \in \Omega_{p\theta_2,d}$.

If $p_1 = 1$, *then condition* (2.8) *is sufficient for the boundedness of* M^d_α *from* $GM_{1\theta_1,w_1(\cdot),d}$ *to* $WGM_{p_2\theta_2,w_2(\cdot),d}$. *Moreover, the quasi-norm* $\|M^d_\alpha\|_{GM_{1\theta_1,w_1(\cdot),d}\to GM_{p_2\theta_2,w_2(\cdot),d}}$ *is estimated above via the right-hand side of* (2.10) *uniformly in* $w_1 \in \Omega_{1\theta_1,d}$, $w_2 \in \Omega_{p_2\theta_2,d}$.

The following lemma was proved in [43].

Lemma 2.2 *Let* $1 \le p < \infty$, $0 \le \alpha < \frac{|d|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|d|}$. *Then for any ball* $\mathcal{E} = \mathcal{E}_d(x,r)$ *in* $\mathbb{R}^n$, *the inequality*

$$\|M^d_\alpha f\|_{L_q(\mathcal{E})} \lesssim r^{\frac{|d|}{q}} \sup_{t>2r} t^{\alpha-\frac{|d|}{p}}\|f\|_{L_p(\mathcal{E}_d(x,t))} \tag{2.12}$$

holds for all $f \in L^{\rm loc}_p(\mathbb{R}^n)$.

Moreover, the inequality

$$\|M^d_\alpha f\|_{WL_q(\mathcal{E})} \lesssim r^{\frac{|d|}{q}} \sup_{t>2r} t^{\alpha-|d|}\|f\|_{L_1(\mathcal{E}_d(x,t))} \tag{2.13}$$

holds for all $f \in L^{\rm loc}_1(\mathbb{R}^n)$.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

Let $1 \le p < \infty$. Denote by $\mathcal{G}_p$ the the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto t^{\frac{|d|}{p}}\varphi(t) \in (0,\infty)$ is almost increasing.

Seemingly the requirement $\phi \in \mathcal{G}_p$ is superfluous but it turns out that this condition is natural. Indeed, Nakai established that there exists a function φ such that φ itself is decreasing, that $\varphi(t)t^{|d|/p} \le \varphi(r)r^{|d|/p}$ for all $0 < t \le r < \infty$ and that $M_{p,\phi,d}(\mathbb{R}^n) = M_{p,\varphi,d}(\mathbb{R}^n)$. By elementary calculations, we have the following, which shows particularly that the spaces $M_{p,\varphi,d}(\mathbb{R}^n)$ and $WM_{p,\varphi,d}(\mathbb{R}^n)$ are not trivial, see for example, [31].

Lemma 2.3 *[32] Let $\varphi \in \mathcal{G}_p$, $1 \le p < \infty$, $\mathcal{E}_0 = \mathcal{E}_d(x_0, r_0)$ and $\chi_{\mathcal{E}_0}$ is the characteristic function of the ball $\mathcal{E}_0$, then $\chi_{\mathcal{E}_0} \in M_{p,\varphi,d}(\mathbb{R}^n)$. Moreover, there exists $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \le \|\chi_{\mathcal{E}_0}\|_{WM_{p,\varphi,d}} \le \|\chi_{\mathcal{E}_0}\|_{M_{p,\varphi,d}} \le \frac{C}{\varphi(r_0)}.$$

We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \;\; t \in (0,\infty).$$

The following theorem was proved in [14].

Theorem 2.5 *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on $(0,\infty)$.*

Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{2.14}$$

3 Anisotropic maximal commutators in anisotropic general Morrey-type spaces

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [16,17] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [19,20,27–30,44–46]). Let T be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$.

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 3.1 *Define $BMO(\mathbb{R}^n) = \{b \in L_1^{\text{loc}}(\mathbb{R}^n) : \|b\|_* < \infty\}$, where*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}| dy < \infty,$$

and $b_{\mathcal{E}(x,r)} = |\mathcal{E}(x,r)|^{-1} \int_{\mathcal{E}(x,r)} b(y)dy$.

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

Remark 3.1 [24,56]

1 Let $b \in BMO(\mathbb{R}^n)$. Then

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}|^p dy \right)^{\frac{1}{p}} \tag{3.1}$$

for $1 < p < \infty$.

2 Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left| b_{\mathcal{E}(x,r)} - b_{\mathcal{E}(x,\tau)} \right| \le C\|b\|_* \log \frac{\tau}{r} \quad \text{for} \quad 0 < 2r < \tau, \tag{3.2}$$

where C is independent of f, x, r and τ.

The mapping properties of M_b and $[b, M]$ have been studied extensively by many authors. See, for instance, [1,5,6,22,34,49]. The operator M_b plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance, [34,48,54]). The nonlinear commutator of Hardy-Littlewood maximal function, $[b, M]$, can be used in studying the product of a function in H_1 and a function in BMO (see [8] for instance).

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) = \begin{cases} 0\,, & \text{if } b(x) \ge 0, \\ |b(x)|, & \text{if } b(x) < 0, \end{cases}$$

and $b^+(x) = |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M^d_\alpha]$ and $M^d_{b,\alpha}$ are valid. Let b be any non-negative locally integrable function. Then

$$|[b, M^d_\alpha]f(x)| \le M^d_{b,\alpha} f(x), \quad x \in \mathbb{R}^n$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$.

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M^d_\alpha]f(x)| \le M^d_{b,\alpha} f(x) + 2b^-(x) M^d_\alpha f(x), \quad x \in \mathbb{R}^n \tag{3.3}$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ (see, for example, [1]).

Note that, the boundedness of the operator M_b on L^p spaces was proved by Garcia-Cuerva et al. [34]. In 2000, Bastero et al. [6] studied the necessary and sufficient condition for the boundedness of $[b, M]$ on L^p spaces.

The maximal commutator operator M_b plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance, [34]). The operator M_b has been studied intensively and there exist plenty of results about it. Garcia-Cuerva et al. [34] proved the following statement for the isotropic case.

Theorem 3.1 *Let $1 < p < \infty$. M^d_b is bounded on $L_p(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

The operator $[M, b]$ was studied by Milman et al. in [50] and [6]. This operator arises, for example, when one tries to give a meaning to the product of a function in H^1 and a function in BMO (which may not be a locally integrable function, see, for instance, [8]). Using real interpolation techniques, in [50], Milman and Schonbek proved the L_p-boundedness of the operator $[M, b]$. Bastero, Milman and Ruiz [6] proved the next theorem for the isotropic case.

Theorem 3.2 *Let $1 < p < \infty$. Then the following assertions are equivalent:*
(i) *$[M^d, b]$ is bounded on $L_p(\mathbb{R}^n)$;*
(ii) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L_\infty(\mathbb{R}^n)$;*
The opertors M^d_b and $[M^d, b]$ enjoy weak-type $L(1+\log^+ L)$ estimate.

The proof of previous theorem is based on the following inequalities.

Theorem 3.3 *([1, Corollary 1.11 and 1.12]) Let $b \in BMO(\mathbb{R}^n)$. Then, there exists a positive constant C such that*

$$M^d_b(f)(x) \le C\,\|b\|_* \left(M^d\right)^2 f(x), \quad x \in \mathbb{R}^n \tag{3.4}$$

for all $f \in L^{loc}_1(\mathbb{R}^n)$. Here $\left(M^d\right)^2 \equiv M^d\big(M^d\big)$.

Moreover, if $b^- \in L_\infty(\mathbb{R}^n)$, then, there exists a positive constant C such that

$$|[M^d, b]f(x)| \le C\left(\|b^+\|_* + \|b^-\|_\infty\right)\left(M^d\right)^2 f(x) \tag{3.5}$$

for all $f \in L^{loc}_1(\mathbb{R}^n)$.

In this section we investigate boundedness of anisotropic maximal commutator and commutator of anisotropic maximal operator in Morrey spaces. The following theorem is true.

Theorem 3.4 *Let $1 < p < \infty$, $0 \le \lambda \le \frac{|d|}{p}$. The following assertions are equivalent:*
(i) *$b \in BMO(\mathbb{R}^n)$.*
(ii) *The operator M^d_b is bounded on $M_{p,\lambda,d}$.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. By Theorems 2.1 and 3.3 it follows that M^d_b is bounded in Morrey space $M_{p,\lambda,d}$ and the following inequality holds:

$$\|M^d_b f\|_{M_{p,\lambda,d}} \lesssim \|b\|_* \|f\|_{M_{p,\lambda,d}}.$$

$(ii) \Rightarrow (i)$. Assume that there exists $C > 0$ such that

$$\|M^d_b(f)\|_{M_{p,\lambda,d}} \le C\|f\|_{M_{p,\lambda,d}}$$

for all $f \in M_{p,\lambda,d}$. Obviously,

$$\|f\|_{M_{p,\lambda,d}} \approx \sup_{\Pi'} |\Pi'|^{-\frac{\lambda}{p}} \|f\|_{L_p(\Pi')},$$

where the supremum is taken with respect to all the parallelopipeds Π'.

Let Π be a fixed parallelopiped. We consider $f = \chi_\Pi$. It is easy to compute that

$$\begin{aligned}\|\chi_\Pi\|_{M_{p,\lambda,d}} &\approx \sup_{\Pi'} |\Pi'|^{-\frac{\lambda}{p}} \|\chi_\Pi\|_{L_p(\Pi')} = \sup_{\Pi'} |\Pi'|^{-\frac{\lambda}{p}}\, |\Pi' \cap \Pi|^{\frac{1}{p}} \\ &= \sup_{\Pi' \subseteq \Pi} |\Pi'|^{-\frac{\lambda}{p}}\, |\Pi' \cap \Pi|^{\frac{1}{p}} = |\Pi|^{\frac{1}{p}-\frac{\lambda}{p}}.\end{aligned} \tag{3.6}$$

On the other hand, since

$$M^d_b(\chi_\Pi)(x) \gtrsim \frac{1}{|\Pi|}\int_\Pi |b(y) - b_\Pi|\,dy \quad \text{for all} \quad x \in \Pi,$$

then

$$\|M_b^d(\chi_\Pi)\|_{M_{p,\lambda,d}} \approx \sup_{\Pi'} |\Pi'|^{-\frac{\lambda}{p}} \|M_b^d(\chi_\Pi)\|_{L_p(\Pi')}$$
$$\gtrsim |\Pi|^{\frac{1}{p}-\frac{\lambda}{p}} \frac{1}{|\Pi|} \int_Q |b(y) - b_\Pi| dy. \quad (3.7)$$

Since by assumption

$$\|M_b^d(\chi_\Pi)\|_{M_{p,\lambda,d}} \lesssim \|\chi_\Pi\|_{M_{p,\lambda,d}},$$

by (3.6) and (3.7), we get that

$$\frac{1}{|\Pi|} \int_\Pi |b(y) - b_\Pi| dy \lesssim 1.$$

The following theorem was proved in [58].

Theorem 3.5 *Let* $1 < p < \infty$, $0 \le \lambda \le \frac{|d|}{p}$. *Suppose that* b *be a real valued, locally integrable function in* $\mathbb{R}^n$. *The following assertions are equivalent:*
(i) *b is in* $BMO(\mathbb{R}^n)$ *such that* $b^- \in L_\infty(\mathbb{R}^n)$.
(ii) *The commutator* $[M^d, b]$ *is bounded in* $M_{p,\lambda,d}$.

Remark 3.2 $(i) \Rightarrow (ii)$. Assume that b is in $BMO(\mathbb{R}^n)$ such that $b^- \in L_\infty(\mathbb{R}^n)$. By Theorems 3.5 and 3.3 it follows that $[M^d, b]$ is bounded in Morrey space $M_{p,\lambda,d}$ and the following inequality holds:

$$\|[M^d, b]f\|_{M_{p,\lambda,d}} \lesssim \left(\|b^+\|_* + \|b^-\|_\infty\right) \|f\|_{M_{p,\lambda,d}}.$$

From Theorem 2.2 we obtain the following result.

Theorem 3.6 *Let* $1 < p \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$.
Then the conditions (2.3) *and*

$$\Big\| t^{-\frac{|d|}{p}} \|w_1(r) r^{\frac{|d|}{p}}\|_{L_{\theta_1}(0,t)} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \Big\|_{L_\infty(0,\infty)} < \infty \quad (3.8)$$

is necessary and sufficient for the boundedness of $\left(M^d\right)^2$ *from* $LM_{p\theta_1,w_1(\cdot),d}$ *to* $LM_{p\theta_2,w_2(\cdot),d}$. *Moreover,*

$$\|\left(M^d\right)^2\|_{LM_{p\theta_1,w_1(\cdot),d}\to LM_{p\theta_2,w_2(\cdot),d}} \approx \Big\| t^{-\frac{|d|}{p}} \|w_1(r) r^{\frac{|d|}{p}}\|_{L_{\theta_1}(0,t)} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \Big\|_{L_\infty(0,\infty)}$$
$$+ \Big\| \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \Big\| w_2(r) \Big(\frac{r}{t+r}\Big)^{\frac{|d|}{p}} \Big\|_{L_{\theta_2}(0,\infty)} \Big\|_{L_\infty(0,\infty)} \quad (3.9)$$

uniformly in $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$.

From Theorem 2.2 we obtain the following result.

Corollary 3.1 *Let* $b \in BMO(\mathbb{R}^n)$, $1 < p \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$.
Then the conditions (2.3) *and* (3.8) *is sufficient for the boundedness of* M_b^d *from* $LM_{p\theta_1,w_1(\cdot),d}$ *to* $LM_{p\theta_2,w_2(\cdot),d}$. *Moreover,*

$$\|M_b^d\|_{LM_{p\theta_1,w_1(\cdot),d}\to LM_{p\theta_2,w_2(\cdot),d}} \lesssim \|b\|_* \left(\Big\| t^{-\frac{|d|}{p}} \|w_1(r) r^{\frac{|d|}{p}}\|_{L_{\theta_1}(0,t)} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \Big\|_{L_\infty(0,\infty)}\right.$$
$$\left. + \Big\| \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \Big\| w_2(r) \Big(\frac{r}{t+r}\Big)^{\frac{|d|}{p}} \Big\|_{L_{\theta_2}(0,\infty)} \Big\|_{L_\infty(0,\infty)} \right) \quad (3.10)$$

uniformly in $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$.

From Theorem 3.6 we obtain the following result.

Theorem 3.7 *Let* $1 < p \leq \infty$, $0 < \theta \leq \infty$, $w \in \Omega_\theta$.
Then the condition

$$\Big\|t^{-\frac{|d|}{p}}\|w(r)r^{\frac{|d|}{p}}\|_{L_\theta(0,t)}\|w\|^{-1}_{L_\theta(t,\infty)}\Big\|_{L_\infty(0,\infty)} < \infty \tag{3.11}$$

is necessary and sufficient for the boundedness of $\big(M^d\big)^2$ *on* $LM_{p\theta,w(\cdot),d}$. *Moreover,*

$$\|\big(M^d\big)^2\|_{LM_{p\theta,w(\cdot),d}\to LM_{p\theta,w(\cdot),d}} \approx \Big\|t^{-\frac{|d|}{p}}\|w(r)r^{\frac{|d|}{p}}\|_{L_\theta(0,t)}\|w\|^{-1}_{L_\theta(t,\infty)}\Big\|_{L_\infty(0,\infty)}$$

uniformly in $w \in \Omega_\theta$.

Theorems 3.3 and 3.7 immediately implies sufficient conditions for the boundedness of the operator M^d_b acting in anisotropic global Morrey-type spaces.

Corollary 3.2 *Let* $b \in BMO(\mathbb{R}^n)$, $1 < p \leq \infty$, $0 < \theta \leq \infty$, $w \in \Omega_\theta$.
Then the condition (3.11) *is sufficient for the boundedness of* M^d_b *on* $LM_{p\theta,w(\cdot),d}$. *Moreover,*

$$\|M^d_b\|_{LM_{p\theta,w(\cdot),d}\to LM_{p\theta,w(\cdot),d}} \lesssim \|b\|_* \Big\|t^{-\frac{|d|}{p}}\|w(r)r^{\frac{|d|}{p}}\|_{L_\theta(0,t)}\|w\|^{-1}_{L_\theta(t,\infty)}\Big\|_{L_\infty(0,\infty)}$$

uniformly in $w \in \Omega_\theta$.

4 Anisotropic fractional maximal commutators in anisotropic general Morrey-type spaces

In this section we investigate boundedness of anisotropic fractional maximal commutator and commutator of anisotropic maximal operators in anisotropic general Morrey-type spaces.

For proving our main results, we need the following estimate.

Lemma 4.1 *If* $b \in L^1_{\rm loc}(\mathbb{R}^n)$ *and* $\mathcal{E}_0 := \mathcal{E}_d(x_0, r_0)$, *then*

$$r_0^\alpha |b(x) - b_{\mathcal{E}_0}| \leq C M^d_{b,\alpha}\chi_{\mathcal{E}_0}(x) \ \textit{for every} \ x \in \mathcal{E}_0.$$

Proof. It is well-known that

$$M^d_{b,\alpha}f(x) \leq \mathrm{M}^d_{b,\alpha}f(x) \leq 2^{|d|-\alpha}\, M^d_{b,\alpha}f(x). \tag{4.1}$$

Now let $x \in \mathcal{E}_0$. By using (4.1), we get

$$\begin{aligned} M^d_{b,\alpha}\chi_{\mathcal{E}_0}(x) &\geq 2^{-|d|+\alpha}\,\mathrm{M}^d_{b,\alpha}f(x) = 2^{-|d|+\alpha}\sup_{\mathcal{E}\ni x}|\mathcal{E}|^{-1}\int_{\mathcal{E}\cap\mathcal{E}_0}|b(x)-b(y)|dy \\ &\geq 2^{-|d|+\alpha}\,\big||\mathcal{E}_0|^{-1}\int_{\mathcal{E}_0}(b(x)-b(y))dy\big| = 2^{-|d|+\alpha}\,|b(x)-b_{\mathcal{E}_0}|. \end{aligned}$$

Lemma 4.2 *[22] Let $0 < \alpha < |d|$. Then*

$$M^d_\alpha\big(M^d f\big)(x) \approx \sup_{\mathcal{E}\ni x} |\mathcal{E}|^{\frac{\alpha}{|d|}} \, \|f\|_{L(1+\log^+ L),\mathcal{E}}, \tag{4.2}$$

where $\|\cdot\|_{L(1+\log^+ L),\mathcal{E}}$ is the Luxemburg type average defined by

$$\|f\|_{L(1+\log^+ L),\mathcal{E}} = \inf\Big\{\lambda > 0 : \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} \frac{|f(x)|}{\lambda}\Big(1+\log^+\frac{|f(x)|}{\lambda}\Big)dx \le 1\Big\},$$

$\log^+ t = 0$ *if* $0 < t \le 1$, *and* $\log^+ t = \log t$ *if* $t > 1$.

Lemma 4.3 *[42] Let $0 < \alpha < |d|$ and $b \in BMO(\mathbb{R}^n)$. Then there exists a positive constant C such that*

$$M^d_{b,\alpha} f(x) \le C\|b\|_* \Big(M^d\big(M^d_\alpha f\big)(x) + M^d_\alpha\big(M^d f\big)(x)\Big) \tag{4.3}$$

for almost every $x \in \mathbb{R}^n$ and for all functions from $f \in L^{\rm loc}_1(\mathbb{R}^n)$.

Proof. Let $x \in \mathbb{R}^n$, $r > 0$, $\mathcal{E} = \mathcal{E}_d(x,r)$ and $\lambda\mathcal{E} = \mathcal{E}_d(x,\lambda r)$. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{2\mathcal{E}}(y)$, $f_2(y) = f(y)\chi_{\complement_{(2\mathcal{E})}}(y)$, and $\chi_{2\mathcal{E}}$ denotes the characteristic function of $2\mathcal{E}$. Then for any $y \in \mathbb{R}^n$

$$\begin{aligned} M^d_{b,\alpha} f(y) &= M^d_\alpha\big((b-b(y))f\big)(y) = M^d_\alpha\big((b-b_{2\mathcal{E}}+b_{2\mathcal{E}}-b(y))f\big)(y)\\ &\le M^d_\alpha\big((b-b_{2\mathcal{E}})f\big)(y) + M^d_\alpha\big((b_{2\mathcal{E}}-b(y))f\big)(y)\\ &\le M^d_\alpha\big((b-b_{2\mathcal{E}})f_1\big)(y) + M^d_\alpha\big((b-b_{2\mathcal{E}})f_2\big)(y) + \big|b_{2\mathcal{E}}-b(y)\big| M^d_\alpha f(y). \end{aligned}$$

For $0 < \delta < 1$ we have

$$\begin{aligned} &\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} \big(M^d_{b,\alpha} f(y)\big)^\delta dy\Big)^{\frac{1}{\delta}} \le \Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} \big(M^d_\alpha\big((b-b_{2\mathcal{E}})f_1\big)(y)\big)^\delta dy\Big)^{\frac{1}{\delta}}\\ &+ \Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} \big(M^d_\alpha\big((b-b_{2\mathcal{E}})f_2\big)(y)\big)^\delta dy\Big)^{\frac{1}{\delta}} + \Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} \big|b(y)-b_{2\mathcal{E}}\big|\big(M^d_\alpha f\big)(y)\big)^\delta dy\Big)^{\frac{1}{\delta}}\\ &= I_1 + I_2 + I_3. \end{aligned}$$

Applying Hölder's inequality, by (3.2) and Lemma 4.2 we get

$$\begin{aligned} I_1 &\le \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} M^d_\alpha\big((b-b_{2\mathcal{E}})f_1\big)(y)dy \lesssim \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} I^d_\alpha\big|(b-b_{2\mathcal{E}})f_1\big|(y)dy\\ &= \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\int_{\mathbb{R}^n} \frac{\big|b(z)-b_{2\mathcal{E}}\big|\,|f_1(z)|}{\rho(y-z)^{|d|-\alpha}}dzdy\\ &= \frac{1}{|\mathcal{E}|}\int_{\mathbb{R}^n} \big|b(z)-b_{2\mathcal{E}}\big|\,|f_1(z)|\,\Big(\int_{\mathcal{E}} \rho(y-z)^{\alpha-|d|}dy\Big)dz\\ &\lesssim \frac{|\mathcal{E}|^{\frac{\alpha}{|d|}}}{|\mathcal{E}|}\int_{2\mathcal{E}} \big|b(z)-b_{2\mathcal{E}}\big|\,|f(z)|dz \approx \frac{|2\mathcal{E}|^{\frac{\alpha}{|d|}}}{|2\mathcal{E}|}\int_{2\mathcal{E}} \big|b(z)-b_{2\mathcal{E}}\big|\,|f(z)|dz\\ &\lesssim |2\mathcal{E}|^{\frac{\alpha}{|d|}} \big\|b(z)-b_{2\mathcal{E}}\big\|_{\exp L,2\mathcal{E}}\, \|f\|_{L(1+\log^+ L),2\mathcal{E}}\\ &\lesssim \|b\|_* |2\mathcal{E}|^{\frac{\alpha}{|d|}}\, \|f\|_{L(1+\log^+ L),2\mathcal{E}} \lesssim \|b\|_*\, M^d_\alpha\big(M^d f\big)(x). \end{aligned}$$

Let us estimate I_2.

$$
\begin{aligned}
I_2 &\leq \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} M^d_\alpha\big((b-b_{2\mathcal{E}})f_2\big)(y)dy \lesssim \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} I^d_\alpha\big|(b-b_{2\mathcal{E}})f_2\big|(y)dy \\
&= \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\sum_{k=1}^{\infty}\int_{2^{k+1}\mathcal{E}\setminus 2^k\mathcal{E}} \frac{\big|b(z)-b_{2\mathcal{E}}\big|\,|f(z)|}{\rho(y-z)^{|d|-\alpha}}dzdy \\
&\approx \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\sum_{k=1}^{\infty}\frac{1}{\big|2^{k+1}\mathcal{E}\big|^{1-\frac{\alpha}{|d|}}}\int_{2^{k+1}\mathcal{E}\setminus 2^k\mathcal{E}}\big|b(z)-b_{2\mathcal{E}}\big|\,|f(z)|dzdy \\
&\approx \sum_{k=1}^{\infty}\frac{2^{-k}}{\big|2^{k+1}\mathcal{E}\big|^{1-\frac{\alpha}{|d|}}}\int_{2^{k+1}\mathcal{E}\setminus 2^k\mathcal{E}}\big|b(z)-b_{2\mathcal{E}}\big|\,|f(z)|dz \\
&\approx \sum_{k=1}^{\infty}\frac{2^{-k}}{\big|2^{k+1}\mathcal{E}\big|^{1-\frac{\alpha}{|d|}}}\int_{2^{k+1}\mathcal{E}\setminus 2^k\mathcal{E}}\big|b(z)-b_{2^{k+1}\mathcal{E}}\big|\,|f(z)|dz \\
&+ \sum_{k=1}^{\infty}\frac{2^{-k}}{\big|2^{k+1}\mathcal{E}\big|^{1-\frac{\alpha}{|d|}}}\int_{2^{k+1}\mathcal{E}\setminus 2^k\mathcal{E}}\big|b_{2\mathcal{E}}-b_{2^{k+1}\mathcal{E}}\big|\,|f(z)|dz = A+B.
\end{aligned}
$$

We estimate each integral in turn. For A, by Hölder's inequality and Lemma 4.2 we obtain

$$
\begin{aligned}
A &\lesssim \sum_{k=1}^{\infty} 2^{-k}\big|2^{k+1}\mathcal{E}\big|^{\frac{\alpha}{|d|}}\big\|b(z)-b_{2^{k+1}\mathcal{E}}\big\|_{\exp L,2^{k+1}\mathcal{E}}\,\big\|f\big\|_{L(1+\log^+ L),2^{k+1}\mathcal{E}} \\
&\lesssim \|b\|_*\sum_{k=1}^{\infty} 2^{-k}|2^{k+1}\mathcal{E}|^{\frac{\alpha}{|d|}}\,\|f\|_{L(1+\log^+ L),2^{k+1}\mathcal{E}} \\
&\lesssim \|b\|_*\, M^d_\alpha\big(M^d f\big)(x)\sum_{k=1}^{\infty}2^{-k} = \|b\|_*\, M^d_\alpha\big(M^d f\big)(x).
\end{aligned}
$$

For B, by Hölder's inequality and Lemma 4.2 we get

$$
\begin{aligned}
B &= \sum_{k=1}^{\infty}\frac{2^{-k}\,\big|b_{2\mathcal{E}}-b_{2^{k+1}\mathcal{E}}\big|}{\big|2^{k+1}\mathcal{E}\big|^{1-\frac{\alpha}{|d|}}}\int_{2^{k+1}\mathcal{E}\setminus 2^k\mathcal{E}}|f(z)|dz \\
&\lesssim \|b\|_*\sum_{k=1}^{\infty} k\,2^{-k}|2^{k+1}\mathcal{E}|^{\frac{\alpha}{|d|}}\,\|f\|_{L(1+\log^+ L),2^{k+1}\mathcal{E}} \\
&\lesssim \|b\|_*\, M^d_\alpha\big(M^d f\big)(x)\sum_{k=1}^{\infty}k\,2^{-k} \approx \|b\|_*\, M^d_\alpha\big(M^d f\big)(x).
\end{aligned}
$$

Therefore we get

$$
I_2 \lesssim \|b\|_*\, M^d_\alpha\big(M^d f\big)(x).
$$

Finally, for estimate I_3, applying Hölder's inequality with exponent $1/\delta$, by Lemmas by (3.2) we get

$$
I_3 \leq \Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\big|b(y)-b_{2\mathcal{E}}\big|^t dy\Big)^{\frac{1}{t}}\frac{1}{|\mathcal{E}|}\int_B M^d_\alpha f(y)dy \lesssim \|b\|_*\, M^d\big(M^d_\alpha f\big)(x).
$$

Lemma 4.3 is proved by the estimate of I_1, I_2, I_3 and the Lebesgue differentiation theorem.

The following lemma is true.

Lemma 4.4 *Let* $1 < p < \infty$, $0 < \alpha < \frac{|d|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|d|}$, $b \in BMO(\mathbb{R}^n)$. *Then for any ball* $\mathcal{E} = \mathcal{E}_d(x, r)$ *in* $\mathbb{R}^n$, *the inequality*

$$\|M^d_{b,\alpha} f\|_{L_q(\mathcal{E})} \lesssim \|b\|_* \, r^{\frac{|d|}{q}} \sup_{t>2r} t^{-\frac{|d|}{q}} \|f\|_{L_p(\mathcal{E}_d(x,t))} \tag{4.4}$$

holds for all $f \in L^{\rm loc}_p(\mathbb{R}^n)$.

Proof. Let $0 < \alpha < |d|$ and $b \in BMO(\mathbb{R}^n)$. Then from Lemmas 4.3 and 2.2 we get

$$\begin{aligned}
&\|M^d_{b,\alpha} f\|_{L_q(\mathcal{E})} \lesssim \|b\|_* \|M^d\big(M^d_\alpha f\big) + M^d_\alpha\big(M^d f\big)\|_{L_q(\mathcal{E})} \\
&\lesssim \|b\|_* \Big(\|M\big(M_\alpha f\big)\|_{L_q(\mathcal{E})} + \|M^d_\alpha\big(M^d f\big)\|_{L_q(\mathcal{E})} \Big) \\
&\lesssim \|b\|_* \Big(r^{\frac{|d|}{q}} \sup_{t>r} t^{-\frac{|d|}{q}} \|M^d_\alpha f\|_{L_q(\mathcal{E}_d(x,t))} + r^{\frac{|d|}{q}} \sup_{t>r} t^{-\frac{|d|}{q}} \|M^d f\|_{L_p(\mathcal{E}_d(x,t))} \Big) \\
&\lesssim \|b\|_* \Big(r^{\frac{|d|}{q}} \sup_{t>r} \sup_{\tau>t} t^{-\frac{|d|}{q}} \|f\|_{L_p(\mathcal{E}_d(x,\tau))} + r^{\frac{|d|}{q}} \sup_{t>r} t^{\frac{|d|}{p} - \frac{|d|}{q}} \sup_{\tau>t} t^{-\frac{|d|}{p}} \|f\|_{L_p(\mathcal{E}_d(x,\tau))} \Big) \\
&= \|b\|_* \Big(r^{\frac{|d|}{q}} \sup_{t>r} t^{-\frac{|d|}{q}} \|f\|_{L_p(\mathcal{E}_d(x,t))} + r^{\frac{|d|}{q}} \sup_{\tau>r} \tau^{-\frac{|d|}{p}} \|f\|_{L_p(\mathcal{E}_d(x,\tau))} \sup_{r<t<\tau} t^{\frac{|d|}{p} - \frac{|d|}{q}} \Big) \\
&= \|b\|_* \, r^{\frac{|d|}{q}} \sup_{t>r} t^{-\frac{|d|}{q}} \|f\|_{L_p(\mathcal{E}_d(x,t))}.
\end{aligned}$$

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $M^d_{b,\alpha}$ from $M_{p,\varphi_1,d}(\mathbb{R}^n)$ to $M_{q,\varphi_2,d}(\mathbb{R}^n)$, when b belongs to the $BMO(\mathbb{R}^n)$ space.

Theorem 4.1 (Spanne type boundedness characterization)
Let $b \in BMO(\mathbb{R}^n) \setminus \{const\}$, $p, q \in [1, \infty)$, $0 \le \alpha < |d|$, $\varphi_1 \in \Omega_p$ *and* $\varphi_2 \in \Omega_q$.

i) Let $1 < p < \frac{|d|}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|d|}$, *then the condition*

$$\sup_{t<r<\infty} r^{-\frac{Q}{q}} \operatorname*{ess\,inf}_{r<s<\infty} \varphi_1(x, s) s^{\frac{Q}{p}} \le C \, \varphi_2(x, t), \tag{4.5}$$

where C *does not depend on* x *and* r, *is sufficient for the boundedness of* $M^d_{b,\alpha}$ *from* $M_{p,\varphi_1,d}(\mathbb{R}^n)$ *to* $M_{q,\varphi_2,d}(\mathbb{R}^n)$.

ii) If $\varphi_1 \in \mathcal{G}_p$, *then the condition*

$$t^\alpha \varphi_1(t) \lesssim \varphi_2(t) \tag{4.6}$$

is necessary for the boundedness of $M^d_{b,\alpha}$ *from* $M_{p,\varphi_1,d}(\mathbb{R}^n)$ *to* $M_{q,\varphi_2,d}(\mathbb{R}^n)$.

iii) Let $1 < p < \frac{|d|}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|d|}$. *If* $\varphi_1 \in \mathcal{G}_p$, *then the condition* (4.6) *is necessary and sufficient for the boundedness of* $M^d_{b,\alpha}$ *from* $M_{p,\varphi_1,d}(\mathbb{R}^n)$ *to* $M_{q,\varphi_2,d}(\mathbb{R}^n)$.

Proof. 1. Let $0 < \alpha < |d|$ and $b \in BMO(\mathbb{R}^n)$. By Theorem 2.5 and Lemma 4.4 we get

$$\begin{aligned}
\|M^d_{b,\alpha} f\|_{M_{q,\varphi_2,d}(\mathbb{R}^n)} &\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x, r)^{-1} \sup_{\tau>r} \tau^{-\frac{|d|}{q}} \|f\|_{L_p(\mathcal{E}_d(x,\tau))} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x, r)^{-1} r^{-\frac{|d|}{p}} \|f\|_{L_p(\mathcal{E}_d(x,r))} = \|b\|_* \|f\|_{M_{p,\varphi_1,d}(\mathbb{R}^n)},
\end{aligned}$$

2. We shall now prove the second part. Let $\mathcal{E}_0 = \mathcal{E}_d(x_0, r_0)$ and $x \in \mathcal{E}_0$. By Lemma 4.1, we have $r_0^\alpha |b(x) - b_{\mathcal{E}_0}| \le C M^d_{b,\alpha} \chi_{\mathcal{E}_0}(x)$. Therefore, by Lemma 2.3 and Remark 3.1

$$r_0^\alpha \lesssim \frac{\|M^d_{b,\alpha}\chi_{\mathcal{E}_0}\|_{L_q(\mathcal{E}_0)}}{\|b(\cdot) - b_{\mathcal{E}_0}\|_{L_q(\mathcal{E}_0)}} \lesssim |\mathcal{E}_0|^{-\frac{1}{q}} \|M^d_{b,\alpha}\chi_{\mathcal{E}_0}\|_{L_q(\mathcal{E}_0)}$$
$$\lesssim \varphi_2(r_0)\|M^d_{b,\alpha}\chi_{\mathcal{E}_0}\|_{M_{q,\varphi_2}(\mathbb{R}^n)} \lesssim \varphi_2(r_0)\|\chi_{\mathcal{E}_0}\|_{M_{p,\varphi_1,d}(\mathbb{R}^n)} \lesssim \frac{\varphi_2(r_0)}{\varphi_1(r_0)}.$$

Since this is true for every $r_0 > 0$, we are done.

3. The third statement of the theorem follows from the first and second parts of the theorem.

Corollary 4.1 *Let* $1 < p < q < \infty$, $0 < \alpha < |d|$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|d|}$, $b \in BMO(\mathbb{R}^n)$, $b^- \in L^\infty(\mathbb{R}^n)$, $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ *and the pair* (φ_1, φ_2) *satisfy the condition* (4.5). *Then the operator* $[b, M^d_\alpha]$ *is bounded from* $M_{p,\varphi_1,d}(\mathbb{R}^n)$ *to* $M_{q,\varphi_2,d}(\mathbb{R}^n)$.

From Theorem 2.4 we obtain the following result.

Theorem 4.2 *Let* $1 < p_1 \le p_2 \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$.

Then the conditions (2.8) *and*

$$\Big\|t^{-\frac{|d|}{p}} \|w_2(r) r^{\frac{|d|}{p}}\|_{L_{\theta_2}(0,t)} \, \|w_2\|^{-1}_{L_{\theta_2}(t,\infty)}\Big\|_{L_\infty(0,\infty)} < \infty \tag{4.7}$$

uniformly in $t \in (0,\infty)$ *is necessary and sufficient for the boundedness of* $M^d\big(M^d_\alpha\big)$ *from* $LM_{p_1\theta_1,w_1(\cdot),d}$ *to* $LM_{p_2\theta_2,w_2(\cdot),d}$. *Moreover,*

$$\|M^d\big(M^d_\alpha\big)\|_{LM_{p_1\theta_1,w_1(\cdot),d}\to LM_{p_2\theta_2,w_2(\cdot),d}} \approx \Big\|t^{-\frac{|d|}{p}} \|w_2(r) r^{\frac{|d|}{p}}\|_{L_{\theta_2}(0,t)} \, \|w_2\|^{-1}_{L_{\theta_2}(t,\infty)}\Big\|_{L_\infty(0,\infty)}$$
$$+ \sup_{0<t<\infty} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \Big\| w_2(r) \Big(\frac{r}{t+r}\Big)^{\frac{|d|}{p_2}} \Big\|_{L_{\theta_2}(0,\infty)} \tag{4.8}$$

uniformly in $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$.

Theorem 4.3 *Let* $1 < p_1 \le p_2 \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$.

Then the conditions (2.8) *and* (3.8) *uniformly in* $t \in (0,\infty)$ *is necessary and sufficient for the boundedness of* $M^d_\alpha\big(M^d\big)$ *from* $LM_{p_1\theta_1,w_1(\cdot),d}$ *to* $LM_{p_2\theta_2,w_2(\cdot),d}$. *Moreover,*

$$\|M^d_\alpha\big(M^d\big)\|_{LM_{p_1\theta_1,w_1(\cdot),d}\to LM_{p_2\theta_2,w_2(\cdot),d}} \approx \Big\|t^{-\frac{|d|}{p}} \|w_1(r) r^{\frac{|d|}{p}}\|_{L_{\theta_1}(0,t)} \, \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)}\Big\|_{L_\infty(0,\infty)}$$
$$+ \sup_{0<t<\infty} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \Big\| w_2(r) \Big(\frac{r}{t+r}\Big)^{\frac{|d|}{p_2}} \Big\|_{L_{\theta_2}(0,\infty)} \tag{4.9}$$

uniformly in $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$.

From Lemma 4.3 and Theorems 4.2, 4.3 we obtain the following result.

Corollary 4.2 *Let $b \in BMO(\mathbb{R}^n)$, $1 < p_1 \le p_2 \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = |d| \left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$.*

Then the conditions (2.8) *and* (3.8) *uniformly in $t \in (0,\infty)$ is sufficient for the boundedness of $M^d_{b,\alpha}$ from $LM_{p_1\theta_1,w_1(\cdot),d}$ to $LM_{p_2\theta_2,w_2(\cdot),d}$. Moreover,*

$$\|M^d_{b,\alpha}\|_{LM_{p_1\theta_1,w_1(\cdot),d}\to LM_{p_2\theta_2,w_2(\cdot),d}} \lesssim \|b\|_* \Big(\big\|t^{-\frac{|d|}{p}} \|w_1(r) r^{\frac{|d|}{p}}\|_{L_{\theta_1}(0,t)} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)}\big\|_{L_\infty(0,\infty)}$$
$$+ \big\|t^{-\frac{|d|}{p}} \|w_2(r) r^{\frac{|d|}{p}}\|_{L_{\theta_2}(0,t)} \|w_2\|^{-1}_{L_{\theta_2}(t,\infty)}\big\|_{L_\infty(0,\infty)}$$
$$+ \sup_{0<t<\infty} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \Big\| w_2(r) \Big(\frac{r}{t+r}\Big)^{\frac{|d|}{p_2}} \Big\|_{L_{\theta_2}(0,\infty)} \Big) \tag{4.10}$$

uniformly in $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$.

Corollary 4.3 *Assume that b is in $BMO(\mathbb{R}^n)$ such that $b^- \in L_\infty(\mathbb{R}^n)$. Let $1 < p_1 \le p_2 \le \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = |d| \left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$.*

Then the conditions (2.8) *and* (3.8) *uniformly in $t \in (0,\infty)$ is sufficient for the boundedness of $[b, M^d_\alpha]$ from $LM_{p_1\theta_1,w_1(\cdot),d}$ to $LM_{p_2\theta_2,w_2(\cdot),d}$. Moreover,*

$$\|[b, M^d_\alpha]\|_{LM_{p_1\theta_1,w_1(\cdot),d}\to LM_{p_2\theta_2,w_2(\cdot),d}} \lesssim \big(\|b^+\|_* + \|b^-\|_\infty\big)$$
$$\times \Big(\big\|t^{-\frac{|d|}{p}} \|w_1(r) r^{\frac{|d|}{p}}\|_{L_{\theta_1}(0,t)} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)}\big\|_{L_\infty(0,\infty)}$$
$$+ \big\|t^{-\frac{|d|}{p}} \|w_2(r) r^{\frac{|d|}{p}}\|_{L_{\theta_2}(0,t)} \|w_2\|^{-1}_{L_{\theta_2}(t,\infty)}\big\|_{L_\infty(0,\infty)}$$
$$+ \sup_{0<t<\infty} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \Big\| w_2(r) \Big(\frac{r}{t+r}\Big)^{\frac{|d|}{p_2}} \Big\|_{L_{\theta_2}(0,\infty)} \Big) \tag{4.11}$$

uniformly in $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$.

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