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# Justification of collocation method for one class of curvilinear integral equations

**Mehpara N. Bakhshaliyeva**

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**Abstract.** In this work, we consider the curvilinear integral equation of the external Dirichlet boundary value problem for the Laplace equation. We divide the curve into elementary parts and establish a quadrature formula at some chosen control points for one class of curvilinear integrals. Then we replace the integral equation with the system of algebraic equations. We also establish the existence and uniqueness of the solution of this system, prove the convergence of its solution to the exact solution of the external Dirichlet boundary value problem for the Laplace equation at the control points, and determine the convergence rate of the method. Besides, we construct a sequence which converges to the solution of the external Dirichlet boundary value problem for the Laplace equation and provide an error estimate.

**Keywords.** Collocation method · curvilinear integral equations · Laplace equation · external Dirichlet boundary value problem.

**Mathematics Subject Classification (2010):** 45E05 · 31B10

## 1 Introduction and problem statement

Let $D \subset \mathbb{R}^2$ be a bounded domain with twice continuously differentiable boundary $L$, and $f$ be a given continuous function on $L$. Consider an external Dirichlet boundary value problem for the Laplace equation: find a function $u \in C^{(2)}\left(\mathbb{R}^2 \backslash \bar{D}\right) \bigcap C\left(\mathbb{R}^2 \backslash D\right)$ which satisfies the Laplace equation $\Delta u = 0$ in $\mathbb{R}^2 \backslash \bar{D}$, the Sommerfeld radiation condition

$$\left(\frac{x}{|x|}, gradu(x)\right) = o\left(\frac{1}{|x|^{1/2}}\right), \quad |x| \to \infty,$$

uniformly along all directions $x/|x|$, and the boundary condition

$$u(x) = f(x) \text{ on } L.$$

It is known that one of the methods for solving the external Dirichlet boundary value problem for the Laplace equation is to reduce it to the curvilinear integral equation (see [3]). Note that the main advantage of applying the integral equation method in the study of external boundary value problems is that such a method allows to reduce the problem for unbounded domain to the one for bounded domain of lesser dimension. Brakhage and

M.N. Bakhshaliyeva
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: mehpara.bakhshalieva@mail.ru

Werner in [2], Leis in [9], and Panich in [10] showed independently of each other that the combination of simple and double layer logarithmic potentials

$$u(x) = \int_L \left( \frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)} - i\eta \Phi(x,y) \right) \varphi(y)\, dL_y,\ x \in \mathbb{R}^2 \backslash \bar{D}$$

with continuous density $\varphi$ is a solution of the external Dirichlet boundary value problem for the Laplace equation if $\varphi$ is a solution of the integral equation

$$\varphi + A\varphi = 2f, \tag{1.1}$$

where $\mathbf{n}(x)$ is an outer unit normal at the point $x \in L$, $\Phi(x,\, y)$ is a fundamental solution of the Laplace equation, i.e.

$$\Phi(x,\, y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|},\ x, y \in \mathbb{R}^2,\ x \neq y,$$

$\eta \neq 0$ is an arbitrary real number,

$$A\varphi = K\,\varphi - i\eta\, S\,\varphi,$$

$$(S\varphi)(x) = 2 \int_L \Phi(x,y)\, \varphi(y)\, dL_y,\ x \in L,$$

$$(K\varphi)(x) = 2 \int_L \frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)}\, \varphi(y)\, dL_y,\ x \in L.$$

As the integral equations in closed form are very rarely solvable, it's vital to develop approximate methods for solving integral equations (with the corresponding theoretical justification, of course). Note that in [1, 4, 5, 6, 7] the justification of collocation method for integral equations of various boundary value problems for the Helmholtz equation in three-dimensional space has been given. This work deals with the justification of collocation method for the integral equation (1.1).

## 2 Justification of collocation method

Let the curve $L$ be given by the parametric equation $x(t) = (x_1(t)\,, x_2(t))\,,\ t \in [a,b]$. Let's divide the interval $[a,b]$ into $n > 2M_1(b-a)/d$ equal parts: $t_k = a + \frac{(b-a)\,k}{n},\ k = \overline{0,n}$, where $M_1 = \max\limits_{t \in [a,b]} \sqrt{(x_1'(t))^2 + (x_2'(t))^2} < +\infty$ and $d$ is a radius of a standard circle (see [12]). As control points we consider $x(\tau_k)$, $k = \overline{1,n}$, where $\tau_k = a + \frac{(b-a)\,(2k-1)}{2n}$. Then the curve $L$ can be divided into elementary parts: $L = \bigcup_{l=1}^{n} L_l$, where $L_k = \{\, x(t) :\ t_{k-1} \leq t \leq t_k \}$.

It is known that (see [8])

(1) $\forall k \in \{1, 2, ..., n\}:\ r_k(n) \sim R_k(n)$, where $r_k(n) = \min\{\, |x(\tau_k) - x(t_{k-1})|\,,\, |x(t_k) - x(\tau_k)|\,\}$ and $R_k(n) = \max\{\, |x(\tau_k) - x(t_{k-1})|\,,\, |x(t_k) - x(\tau_k)|\,\}$;

(2) $\forall k \in \{1, 2, ..., n\}: R_k(n) \leq d/2$;

(3) $\forall k, j \in \{1, 2, ..., n\}: r_j(n) \sim r_k(n)$;

(4) $r(n) \sim R(n) \sim \frac{1}{n}$, where $R(n) = \max\limits_{k=\overline{1,n}} R_k(n)\,, r(n) = \min\limits_{k=\overline{1,n}} r_k(n)\,.$

By $C(L)$ we denote the space of all continuous functions on $L$ with the norm $\|\varphi\|_\infty = \max\limits_{x\in L} |\varphi(x)|$, and for the function $\varphi(x) \in C(L)$ we introduce a modulus of continuity of the form

$$\omega(\varphi,\,\delta) = \delta \sup_{\tau\ge\delta} \frac{\bar{\omega}(\varphi,\,\tau)}{\tau}, \quad \delta > 0,$$

where $\bar{\omega}(\varphi,\,\tau) = \max\limits_{\substack{|x-y|\le\tau \\ x,\,y\in L}} |\varphi(x) - \varphi(y)|$.

**Theorem 2.1.** *The expression*

$$A_n(x(\tau_l)) = \sum_{j=1}^{n} a_{lj}\varphi(x(\tau_j)) \tag{2.1}$$

*is a quadrature formula for the integral* $(A\varphi)(x)$ *at the control points* $x(\tau_l)\,,\quad l=\overline{1,\,n}$, *and the estimate*

$$\max_{l=\overline{1,\,n}} |(A\varphi)(x(\tau_l)) - A_n(x(\tau_l))| \le M\left(\omega(\varphi,\,1/n) + \|\varphi\|_\infty \frac{\ln n}{n}\right),$$

*is true, where*

$$a_{lj} = 0 \ if \ l = j;$$

$$a_{lj} = \frac{2(b-a)}{n}\left(\frac{\partial\Phi(x(\tau_l)\,,\,x(\tau_j))}{\partial\mathbf{n}(x(\tau_j))} - i\eta\,\Phi(x(\tau_l)\,,\,x(\tau_j))\right)\sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2},$$

*if* $l \ne j$. *Hereinafter* $M$ *denotes a positive constant which can be different in different inequalities.*

**Proof.** It was proved in [8] that the expressions

$$S_n(x(\tau_k)) = \frac{2(b-a)}{n}\sum_{\substack{j=1\\ j\ne k}}^{n} \Phi(x(\tau_k)\,,\,x(\tau_j))\,\sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2}\,\rho(x(\tau_j))$$

and

$$\tilde{K}_n(x(\tau_k)) = \frac{2(b-a)}{n}\sum_{\substack{j=1\\ j\ne k}}^{n} \frac{\partial\Phi(x(\tau_k)\,,\,x(\tau_j))}{\partial\mathbf{n}(x(\tau_k))}\,\sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2}\,\rho(x(\tau_j))\,,$$

are the quadrature formulas for the integrals $(S\varphi)(x)$ and $(K\varphi)(x)$, respectively, at the control points $x(\tau_k)\,,\quad k=\overline{1,\,n}$, and the relations

$$\max_{k=\overline{1,\,n}} |(S\varphi)(x(\tau_k)) - S_n(x(\tau_k))| \le M\left(\omega(\varphi,\,1/n) + \|\varphi\|_\infty \frac{\ln n}{n}\right),$$

$$\max_{k=\overline{1,\,n}} |(K\varphi)(x(\tau_k)) - K_n(x(\tau_k))| \le M\left(\omega(\varphi,\,1/n) + \|\varphi\|_\infty \frac{\ln n}{n}\right)$$

holds. Hence it follows that the expression (2.1) is a quadrature formula for the integral $(A\varphi)(x)$ at the control points $x(\tau_l)\,,\quad l=\overline{1,\,n}$, and the estimate

$$\max_{l=\overline{1,\,n}} |(A\varphi)(x(\tau_l)) - A_n(x(\tau_l))| \le M\left(\omega(\varphi,\,1/n) + \|\varphi\|_\infty \frac{\ln n}{n}\right)$$

holds. The theorem is proved.

Let $\mathbb{C}^n$ be a space of $n$–dimensional vectors $z^n = (z_1^n, z_2^n, \ldots, z_n^n)^{\mathrm{T}}$, $z_l^n \in \mathbb{C}$, $l = \overline{1,n}$, with the norm $\|z^n\| = \max\limits_{l=\overline{1,n}} |z_l^n|$, where "$a^{\mathrm{T}}$" denotes the transposition of the vector $a$. Using the quadrature formula (2.1), we replace the integral equation (1.1) by the system of algebraic equations with respect to $z_l^n$'s, approximate values of $\varphi\left(x\left(\tau_l\right)\right)$, $l = \overline{1,n}$, written as

$$(I^n + A^n)\; z^n = 2f^n, \tag{2.2}$$

where $I^n$ is a unit operator in the space $\mathbb{C}^n$, $A^n = (a_{lj})_{l,j=1}^n$, $f^n = p^n f$, and $p^n : C(L) \to \mathbb{C}^n$ is a linear bounded operator defined by the formula $p^n f = \left(f\left(x\left(\tau_1\right)\right), \; f\left(x\left(\tau_2\right)\right), \ldots, f\left(x\left(\tau_n\right)\right)\right)^{\mathrm{T}}$ and called a simple drift operator.

To justify the collocation method, we will use Vainikko's convergence theorem for linear operator equations (see [11]). To formulate that theorem, we need some definitions and a theorem from [11].

**Definition 2.1 ([11]).** *A system $Q = \{q^n\}$ of operators $q^n : C(S) \to \mathbb{C}^n$ is called a connecting system for $C(S)$ and $\mathbb{C}^n$ if*

$$\|q^n \varphi\| \to \|\varphi\|_\infty \;\; as \; n \to \infty, \; \forall \varphi \in C(S);$$

$$\left\|q^n\left(a\varphi + a'\varphi'\right) - \left(a\, q^n\varphi + a' q^n \varphi'\right)\right\| \to 0 \; as \; n \to \infty, \; \forall \varphi, \, \varphi' \in C(S), \; a, a' \in \mathbb{C}.$$

**Definition 2.2 ([11]).** *A sequence $\{\varphi_n\}$ of elements $\varphi_n \in \mathbb{C}^n$ is called $Q-$convergent to $\varphi \in C(S)$ if $\|\varphi_n - q^n\varphi\| \to 0$ as $n \to \infty$. We denote this fact by $\varphi_n \overset{Q}{\to} \varphi$.*

**Definition 2.3 ([11]).** *A sequence $\{\varphi_n\}$ of elements $\varphi_n \in \mathbb{C}^n$ is called $Q-$compact if every subsequence of it $\{\varphi_{n_m}\}$ contains a $Q-$convergent subsequence $\left\{\varphi_{n_{m_k}}\right\}$.*

**Proposition 2.1 ([11]).** *Let $q^n : C(S) \to \mathbb{C}^n$ be linear and bounded. Then the following conditions are equivalent:*

*(1) the sequence $\{\varphi_n\}$ is $Q-$compact and the set of its $Q-$limit points is compact in $C(S)$;*

*(2) there exists a relatively compact sequence $\left\{\varphi^{(n)}\right\} \subset C(S)$ such that $\left\|\varphi_n - q^n\varphi^{(n)}\right\| \to 0$ as $n \to \infty$.*

**Definition 2.4 ([11]).** *A sequence of operators $\mathrm{E}^n : \mathbb{C}^n \to \mathbb{C}^n$ is called $QQ-$convergent to the operator $\mathrm{E} : C(S) \to C(S)$ if for every $Q-$convergent sequence $\{\varphi_n\}$ the relation $\varphi_n \overset{Q}{\to} \varphi \Rightarrow \mathrm{E}^n\varphi_n \overset{Q}{\to} \mathrm{E}\varphi$ holds. We denote this fact by $\mathrm{E}^n \overset{QQ}{\to} \mathrm{E}$.*

**Definition 2.5 ([11]).** *We say that a sequence of linear bounded operators $\mathrm{E}^n : \mathbb{C}^n \to \mathbb{C}^n$ converges compactly to the linear bounded operator $\mathrm{E} : C(S) \to C(S)$ if $\mathrm{E}^n \overset{QQ}{\to} \mathrm{E}$ and the following compactness condition holds:*

$$\varphi_n \in \mathbb{C}^n, \quad \|\varphi_n\| \le M \;\Rightarrow \{\mathrm{E}^n\varphi_n\} \;\; is \; Q - compact.$$

**Theorem 2.2 ([11]).** *Let the following conditions hold:*

1 $Ker\;\left(I + \mathrm{E}\right) = \{0\}$, *where $I$ is a unit operator in $C(S)$;*
2 $I^n + \mathrm{E}^n$' *s $(n \ge n_0)$ are Fredholm operators of index zero;*
3 $\vartheta_n \overset{Q}{\to} \vartheta$, $\vartheta_n \in \mathbb{C}^n$, $\vartheta \in C(S)$;
4 $\mathrm{E}^n \to \mathrm{E}$ *compactly.*

*Then the equation $(I + \mathrm{E})\;\varphi = \vartheta$ has a unique solution $\tilde{\varphi} \in C(S)$, the equation $(I^n + \mathrm{E}^n)\;\varphi_n = \vartheta_n \; (n \ge n_0)$ has a unique solution $\tilde{\varphi}_n \in \mathbb{C}^n$, and $\tilde{\varphi}_n \overset{Q}{\to} \tilde{\varphi}$ with*

$$c_1 \,\|\left(I^n + \mathrm{E}^n\right) q^n\tilde{\varphi} - \vartheta_n\| \le \|\tilde{\varphi}_n - q^n\tilde{\varphi}\| \le c_2 \,\|\left(I^n + \mathrm{E}^n\right) q^n\tilde{\varphi} - \vartheta_n\| \;,$$

*where*

$$c_1 = 1/\sup_{n\geq n_0} \| I^n + \mathrm{E}^n\| > 0\,, c_2 = \sup_{n\geq n_0} \left\| (I^n + \mathrm{E}^n)^{-1}\right\| < +\infty.$$

**Theorem 2.3.** *The equations* (1.1) *and* (2.2) *have unique solutions* $\varphi_* \in C(L)$ *and* $z_*^n \in \mathbb{C}^n$*, respectively,* $\| z_*^n - p^n\varphi_*\| \to 0$ *as* $n \to \infty$*, and the estimate*

$$\| z_*^n - p^n\varphi_*\| \leq M\left(\omega\,(f,\,1/n) + \|f\|_\infty \frac{\ln n}{n}\right)$$

*holds.*

**Proof.** It is proved in [3] that $Ker\ (I + A) = \{0\}$. Besides, the operators $I^n + A^n$ are Fredholm operators of index zero, and the operators $p^n : C(L) \to \mathbb{C}^n$ are linear and bounded. Taking into account the way the curve $L$ has been divided into elementary parts, we obtain

$$\lim_{n\to\infty} \|p^n g\| = \lim_{n\to\infty} \max_{l=\overline{1,n}} |g\,(x\,(\tau_l))| = \max_{x\in L} |g(x)| = \|g\|_\infty\,, \forall g \in C(L)\,.$$

Consequently, the system of simple drift operators $P = \{p^n\}$ is a connecting system for the spaces $C(L)$ and $\mathbb{C}^n$. Then, $f^n \overset{P}{\to} f$, and, by error estimate for the quadrature formula (2.1), we obtain $I^n + A^n \overset{PP}{\to} I + A$. By Definition 2.5, it remains to verify the compactness condition, which, due to Proposition 2.1, is equivalent to the following condition: $\forall\,\{z^n\}\,,\ z^n \in \mathbb{C}^n\,,\ \|z^n\| \leq M$, there exists relatively compact sequence $\{A_n z^n\} \subset C(L)$ such that

$$\|A^n z^n - p^n\,(\ A_n z^n)\,\| \to 0 \text{ as } n \to \infty.$$

As $\{A_n z^n\}$, we consider the sequence

$$(A_n z^n)\ (x) = (K_n z^n)\ (x) - i\eta\ (S_n z^n)\ (x),$$

where

$$(K_n z^n)\ (x) = 2\sum_{j=1}^{n} z_j^n \int_{L_j} \frac{\partial\Phi\,(x,y)}{\partial\mathbf{n}\ (y)} dL_y\,,\ \ x \in L,$$

$$(S_n z^n)\ (x) = 2\sum_{j=1}^{n} z_j^n \int_{L_j} \Phi\,(x,y)\ dL_y\,,\ \ x \in L.$$

Let $L_d(x) = \{y \in L : \ |y - x| < d\}$. Consider arbitrary points $x',\ \ x'' \in L$ such that $|x' - x''| = \delta < d/2$. Obviously,

$$\left|\,(K_n z^n)\ \left(x'\right) - (K_n z^n)\ \left(x''\right)\,\right| \leq 2\ \ \|z^n\| \int_L \left|\frac{\partial\Phi\,(x',y)}{\partial\mathbf{n}\ (y)} - \frac{\partial\Phi\,(x'',y)}{\partial\mathbf{n}\ (y)}\right| dL_y$$

$$\leq 2\ \|z^n\| \int_{L_{\delta/2}(x')} \left|\frac{\partial\Phi\,(x',y)}{\partial\mathbf{n}\ (y)}\right|\ dL_y + 2\ \|z^n\| \int_{L_{\delta/2}(x'')} \left|\frac{\partial\Phi\,(x'',y)}{\partial\mathbf{n}\ (y)}\right|\ dL_y$$

$$+2\ \|z^n\| \int_{L_{\delta/2}(x')} \left|\frac{\partial\Phi\,(x'',y)}{\partial\mathbf{n}\ (y)}\right| dL_y + 2\ \|z^n\| \int_{L_{\delta/2}(x'')} \left|\frac{\partial\Phi\,(x',y)}{\partial\mathbf{n}\ (y)}\right|\ dL_y$$

$$+2\ \ \|z^n\| \int_{L_d(x')\setminus\left(L_{\delta/2}(x')\bigcup L_{\delta/2}(x'')\right)} \left|\frac{\partial\Phi\,(x',y)}{\partial\mathbf{n}\ (y)} - \frac{\partial\Phi\,(x'',y)}{\partial\mathbf{n}\ (y)}\right| dL_y$$

$$+2\ \ \|z^n\| \int_{L\setminus L_d(x')} \left|\frac{\partial\Phi\,(x',y)}{\partial\mathbf{n}\ (y)} - \frac{\partial\Phi\,(x'',y)}{\partial\mathbf{n}\ (y)}\right| dL_y.$$

Using the formula for calculation of a curvilinear integral, we obtain

$$\int_{L_{\delta/2}(x')} \left| \frac{\partial \Phi\left(x', y\right)}{\partial \mathbf{n}\ (y)} \right| \ dL_y$$

$$= \frac{1}{2\pi} \int_{L_{\delta/2}(x')} \frac{\left|\left(x'-y, \mathbf{n}\ (y)\right)\right|}{\left|x'-y\right|^2}\ dL_y \leq M \int_{L_{\delta/2}(x')}\ dL_y \leq M\delta,$$

$$\int_{L_{\delta/2}(x'')} \left| \frac{\partial \Phi\left(x'', y\right)}{\partial \mathbf{n}\ (y)} \right| \ dL_y \leq M\delta,$$

$$\int_{L_{\delta/2}(x')} \left| \frac{\partial \Phi\left(x'', y\right)}{\partial \mathbf{n}\ (y)} \right| \ dL_y$$

$$= \frac{1}{2\pi} \int_{L_{\delta/2}(x')} \frac{\left|\left(x''-y, \mathbf{n}\ (y)\right)\right|}{\left|x''-y\right|^2}\ dL_y \leq M \int_{L_{\delta/2}(x')}\ dL_y \leq M\delta$$

and

$$\int_{L_{\delta/2}(x'')} \left| \frac{\partial \Phi\left(x', y\right)}{\partial \mathbf{n}\ (y)} \right| \ dL_y \leq M\delta.$$

As for every $y \in L_d(x') \backslash (L_{\delta/2}(x') \bigcup L_{\delta/2}(x''))$

$$\left|x'-y\right| \leq \left|x'-x''\right| + \left|x''-y\right| \leq 3\ \left|x''-y\right|$$

and

$$\left|x''-y\right| \leq 3\ \left|x'-y\right|,$$

taking into account

$$\frac{\partial \Phi\left(x', y\right)}{\partial \mathbf{n}\ (y)} - \frac{\partial \Phi\left(x'', y\right)}{\partial \mathbf{n}\ (y)} = \frac{1}{2\pi}\left( \frac{\left(x'-y, \mathbf{n}\ (y)\right)}{\left|x'-y\right|^2} - \frac{\left(x''-y, \mathbf{n}\ (y)\right)}{\left|x''-y\right|^2} \right)$$

$$= \frac{\left(x'-y, \mathbf{n}\ (y)\right)\ \left(\left|x''-y\right| - \left|x'-y\right|\right)\ \left(\left|x''-y\right| + \left|x'-y\right|\right)}{2\pi\ \left|x'-y\right|^2 \left|x''-y\right|^2} + \frac{\left(x'-x'', \mathbf{n}\ (y)\right)}{2\pi\ \left|x''-y\right|^2},$$

we obtain

$$\left| \frac{\partial \Phi\left(x', y\right)}{\partial \mathbf{n}\ (y)} - \frac{\partial \Phi\left(x'', y\right)}{\partial \mathbf{n}\ (y)} \right| \leq \frac{M\delta}{\left|x'-y\right|}, \forall y \in L_d(x') \backslash (L_{\delta/2}(x') \bigcup L_{\delta/2}(x'')).$$

Therefore,

$$\int_{L_d(x') \backslash \left(L_{\delta/2}(x') \bigcup L_{\delta/2}(x'')\right)} \left| \frac{\partial \Phi\left(x', y\right)}{\partial \mathbf{n}\ (y)} - \frac{\partial \Phi\left(x'', y\right)}{\partial \mathbf{n}\ (y)} \right| dL_y$$

$$\leq M\delta \int_{L_d(x') \backslash \left(L_{\delta/2}(x') \bigcup L_{\delta/2}(x'')\right)} \frac{dL_y}{\left|x'-y\right|} \leq M\delta \int_{\delta}^{d} \frac{dt}{t} \leq M\delta\ \left|\ln \delta\right|.$$

It is clear that

$$\int_{L \backslash L_d(x')} \left| \frac{\partial \Phi\left(x', y\right)}{\partial \mathbf{n}\ (y)} - \frac{\partial \Phi\left(x'', y\right)}{\partial \mathbf{n}\ (y)} \right| dL_y \leq M\delta.$$

By summing up the above estimates, we obtain

$$\left|\ \left(K_n z^n\right)\ \left(x'\right) - \left(K_n z^n\right)\ \left(x''\right)\ \right| \leq M\delta\ \left|\ln \delta\right|.$$

Similarly we can show that

$$\left| \left(S_n z^n\right) \left(x'\right) - \left(S_n z^n\right) \left(x''\right) \right| \le M\delta \left|\ln \delta\right|.$$

As a result,

$$\left| \left(A_n z^n\right) \left(x'\right) - \left(A_n z^n\right) \left(x''\right) \right| \le M\delta \left|\ln \delta\right|, \tag{2.3}$$

consequently, $\{A_n z^n\} \subset C(L)$.

The relative compactness of the sequence $\{A_n z^n\}$ follows from the Arzela theorem. In fact, the uniform boundedness follows immediately from the condition $\|z^n\| \le M$, and the equicontinuity follows from the estimate (2.3). Besides, proceeding as in [8], we obtain

$$\|A^n z^n - p^n (A_n z^n)\| \to 0 \text{ as } n \to \infty.$$

So it follows that all conditions of Theorem 2.2 are satisfied. Then the equations (1.1) and (2.2) have unique solutions $\varphi_* \in C(L)$ and $z_*^n \in \mathbb{C}^n$, respectively, and

$$c_1 \delta_n \le \| z_*^n - p^n \varphi_* \| \le c_2 \delta_n,$$

where

$$c_1 = 1/\sup_n \| I^n + A^n \| > 0,\ c_2 = \sup_n \left\| (I^n + A^n)^{-1} \right\| < +\infty,$$

$$\delta_n = \| (I^n + A^n)(p^n \varphi_*) - 2f^n \|.$$

Taking into account the error estimates for the quadrature formula (2.1), we obtain

$$\delta_n = \| (I^n + A^n)(p^n \varphi_*) - 2p^n f \| = \| p^n \varphi_* + A^n (p^n \varphi_*) - p^n (\varphi_* + A\varphi_*) \|$$

$$= \| A^n (p^n \varphi_*) - p^n (A\varphi_*) \| = \max_{l=\overline{1,n}} \left| \sum_{j=1}^n a_{lj} \varphi_* (x(\tau_j)) - (A\varphi_*)(x(\tau_l)) \right|$$

$$\le M \left( \omega(\varphi_*, 1/n) + \|\varphi_*\|_\infty \frac{\ln n}{n} \right).$$

Besides, in view of the ineqalities

$$\omega(S\varphi_*, h) \le M \|\varphi_*\| h |\ln h|$$

and

$$\omega(K\varphi_*, h) \le M \|\varphi_*\| h |\ln h|,$$

(see [3]), we have

$$\omega(\varphi_*, 1/n) = \omega(2f - A\varphi_*, 1/n) \le \omega(2f, 1/n) + \omega(A\varphi_*, 1/n)$$

$$\le M \left( \omega(f, 1/n) + \|\varphi_*\|_\infty \frac{\ln n}{n} \right).$$

Then, by

$$\|\varphi_*\|_\infty = 2 \left\| (I + A)^{-1} f \right\|_\infty \le 2 \left\| (I + A)^{-1} \right\| \|f\|_\infty,$$

we get the validity of the theorem.

**Corollary 2.1.** *Let $x_0 \in \mathbb{R}^2 \backslash \bar{D}$ and $z_*^n = (z_1^*, z_2^*, \ldots, z_n^*)^{\mathrm{T}}$ be a solution of the system of algebraic equations (2.2). Then the sequence*

$$u_n(x_0) = \frac{b-a}{n} \sum_{j=1}^n \left( \frac{\partial \Phi(x_0, x(\tau_j))}{\partial \mathbf{n}(x(\tau_j))} - i\eta \Phi(x_0, x(\tau_j)) \right) \sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2} z_j^*$$

*converges to the value* $u(x_0)$ *of the solution* $u(x)$ *of the external Dirichlet boundary value problem for the Laplace equation at the point* $x_0$*, and*

$$|u_n(x_0) - u(x_0)| \leq M \left( \omega\ (f,\, 1/n) + \|f\|_\infty \frac{\ln n}{n} \right).$$

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