

## The behavior of solutions of nonlinear degenerate parabolic equations in nonregular domains and removability of singularity on boundary

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**Abstract.** *In this paper the behavior of solutions of the initial-boundary problem of nonlinear degenerated parabolic equations of higher order in irregular domains and removability of singularity on boundary are studied. Analogies of those known as Saint-Venant's principle in theory of elasticity are obtained.*

**Keywords.** Nonlinear parabolic equations, degenerate, Saint-Venant's principle, removability of singularity.

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### 1 Introduction

The Saint-Venant's principle is well-known in theory of elasticity. In many papers analogies of Saint-Venant's principle is used for investigation of the behavior of solutions partial differential equations. The analogies of Saint-Venant's principle are some of energy estimations for partial differential equations.

This method firstly is used in [1, 2] and energy estimations for generalized solutions of second order linear equations were studied. We mention also papers [3–27].

Later in [3, 4, 14] quality properties of solutions nonlinear equations are investigated – a method was developed for obtaining "increasing lemma" depending on the geometry of a domain. Otherwise these "increasing lemma" are energy apriori estimates of type of Saint-Venant's principle.

In the study of quality properties of solutions of problems the role of the Saint-Venant's estimates is close to role of "increasing lemmas" of E.M. Landis [14] in analysis of properties continuously solutions of second order equations.

These estimates of type of Saint-Venant's principle allow us to study the behavior of solution in bounded domains with nonsmooth boundary, the behavior of solution in unbounded domains. The last gives of type of the Phragmén-Lindelöf theorems for behavior of solution in unbounded domains with noncompact boundary are obtained.

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The Saint-Venant's estimates also allow us to get uniqueness theorems of generalized solutions.

With help of the Saint-Venant's estimates we can also get some theorems about removability singularity on boundary.

Therefore, it is very important to obtain Saint-Venant's estimates based on the geometry of domains.

We investigate equations in bounded domains with nonsmooth boundary, in unbounded domains with noncompact boundaries, also removable singularity at boundary.

## 2 The behavior of solutions in domains with nonsmooth boundary

Let us consider bounded domain  $Q_T = \Omega \times (0, T)$ ,  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ ,  $0 < T < \infty$ ,  $\partial Q_T = \Gamma_0 \cup \Gamma_T \cup \Gamma$ ,  $\Gamma_0 = \Omega \times \{0\}$ ,  $\Gamma_T = \Omega \times \{T\}$ ,  $\Gamma = \partial\Omega \times (0, T)$ ,  $\Omega^-$  has a nonsmooth boundary. Nonsmoothness conditions will be given using a nonlinear frequency. Using a nonlinear frequency non-smooth domains are divided into two classes: narrow and wide domains.

The space  $L_p(0, T, W_{q,\omega}^m(\Omega_t))$  is determined as  $\left\{ u(x, t) : \int_0^T (\|u\|_{W_{q,\omega}^m(\Omega_t)})^p dt < \infty \right\}$ ,

where  $\Omega_t = Q_T \cap \{(x, \tau) : \tau = t\}$  and  $W_{q,\omega}^m(\Omega_t)$ - weighted Sobolev space such that weight  $\omega$  belongs to the Muckenhoupt class (see [8]). The generalized solution is determined from the space  $L_p(0, T, W_{q,\omega}^m(\Omega_t)) \cap W_2^1(0, T; L_2(\Omega))$ .

We consider initial-boundary problem

$$\frac{\partial u}{\partial t} - \sum_{|\alpha| \leq 2m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, t, u, Du, \dots, D_x^m u) = 0, \quad (2.1)$$

$$u|_{t=0} = 0 \quad (2.2)$$

$$D_x^\alpha u|_\Gamma = 0, \quad |\alpha| \leq m - 1, \quad (2.3)$$

where  $D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ ,  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ ,  $n \geq 1$ .

Assume that the coefficients  $A_\alpha(x, t, \xi)$  are measurable with respect to  $(x, t) \in Q_T$ , continuously with respect to  $\xi \in \mathbb{R}^M$ , where  $M$  is the number of different multi-indices of length no more than  $m$  and satisfying the conditions

$$\sum_{|\alpha| \leq m} A_\alpha(x, t, \xi) \xi_\alpha^m \geq C \omega(x) |\xi^m|^p - C_1 \omega(x) \sum_{i=1}^{m-1} |\xi_i|^p - f_1(x, t) \quad (2.4)$$

$$|A_\alpha(x, t, \xi)| \leq C_2 \omega(x) \sum_{i=0}^m |\xi_i|^p + f_2(x, t), \quad (2.5)$$

where  $\xi = (\xi^0, \dots, \xi^m)$ ,  $\xi^i = (\xi_\alpha^i)$ ,  $|\alpha| = i$ ,  $p > 1$ ,

$$f_1 \in L_p(0, T, L_{p,loc}(\Omega_t)),$$

$$f_2 \in L_{1,loc}(\Omega_T).$$

The space  $W_{q,\omega}^m(\Omega_t)$  is the closure of  $\Omega_t$  the functions from  $C^m(\overline{\Omega})$  with respect to the norm

$$\|u\|_{W_{q,\omega}^m(\Omega_t)} = \left( \int_{\Omega_t} \omega(x) \sum_{|\alpha| \leq m} |D_x^\alpha u(x, t)|^q dx dt \right)^{1/q}.$$

Now we describe the geometry of  $Q_T$  with nonlinear frequency  $\lambda_p^p(r, \tau)$  of section  $\sigma(r, \tau) = S(r) \cap \Omega_\tau$ , where  $S(r) = Q_T \cap \partial Q_T(r)$ ,  $Q_T(r) = Q_T \cap B_r \times (0, T)$ ,  $B_r = B(0, r)$  is the ball with radius  $r$  centered at 0. Thus

$$\lambda_p^p(r, \tau) = \inf \left( \int_{\sigma(r, \tau)} \omega(x) |\nabla_s v|^p d\tau \right) \left( \int_{\sigma(r, \tau)} \omega(x) |v|^p d\tau \right)^{-1},$$

where the lower bound is taken by all continuously differentiable functions in the vicinity of  $\sigma(r, \tau)$  that vanish on  $\partial Q_T$ ;  $\nabla_s v(x)$  is a projection of the vector  $\nabla v(x)$  on a tangential plane to  $\sigma(r, \tau)$  at the point  $x$ . The function  $u \in L_p(0, T; \mathring{W}_{p, \omega, loc}^m(\Omega_t)) \cap W_2^1(0, T; L_{2, loc}(\Omega_t))$  is said to be a generalized solution of the problem (2.1)-(2.2) if the integral identity

$$\int_{Q_T} \frac{\partial u}{\partial t} \varphi dx dt + \int_{Q_T} \sum_{|\alpha| \leq m} A_\alpha(x, t, u, Du, \dots, D^m u) D^\alpha \varphi dx dt = 0 \quad (2.6)$$

is fulfilled for the arbitrary function  $\varphi \in L_p(0, T; \mathring{W}_{p, \omega, loc}^m(\Omega_t)) \cap L_2(Q_T)$ .

Generally we will consider classes of domains, for which the following estimate

$$\int_{S_r} \omega(x) |u(x, t)|^p dx dt \leq \lambda_p^{-p}(r) \int_{S_r} \omega(x) |\nabla u(x, t)|^p dx dt \quad (2.7)$$

holds. The necessary and sufficiently conditions on domains for holds estimate (2.7) is given in [9]. Using a nonlinear frequency we divide non-smooth domains into two classes:

1. The first class is the narrow domain class, i.d. complement of neighborhood of a point is sufficiently massive. For example, this class contains some cone with a vertex at this point. In terms of nonlinear frequency this class of domains satisfies the condition

$$r \lambda_p(r) > d_1 > 0, \forall r \in (0, r_0), r_0 > 0. \quad (A)$$

2. The second class is the broad domain class, i.d. those points, whose complement of neighborhood of a point is rather narrow. In terms of the nonlinear frequency, this class of regions satisfies the condition

$$r \lambda_p(r) < d_2 < \infty, \forall r \in (0, r_0). \quad (B)$$

Also the function  $\varphi(r)$  is determined at  $(0, r_0)$  by inequality

$$\inf_{r\psi(r) < |x| < r} \lambda_p(|x|) (r - r\psi(r)) \omega(x) \geq \mu > 0, \quad (2.8)$$

where  $0 < 1 - c_0 < \mu < 1$ .

Let

$$J(r) = \int_{\Omega_r} \omega(x) |D^m u(x, t)|^p dx dt,$$

$$G(r) = \int_{\Omega_r} \left( \sum_{|\alpha| \leq m} \omega(x) (|F_\alpha(x, t)| + |f_2(x, t)|)^{\frac{p}{q}} \lambda_p^{-\frac{m-|\alpha|}{p-1} p}(|x|) + |f_1(x, t)| \right) dx dt.$$

As mentioned above, Saint-Venant type estimates are needed first to get the required estimates. Further using various lemmas we get estimates in bounded domains with non-smooth boundary, in unbounded domains with noncompact boundary and removable singularity

on boundary in bounded domains. In unbounded domains we get theorems of Phragmén-Lindelöf type.

Therefore, we give the following lemma for the case of bounded domains.

Let  $0 < \varphi_1(r) < c_0 < 1$  be a measurable function on interval  $(0, r_0)$  and  $r_0$  sufficiently small.

**Lemma 2.1** *Assume that  $J(r)$  is a continuous nondecreasing function on  $(0, r_0)$  and satisfies the inequality*

$$J(r(1 - \varphi_1(r))) \leq \lambda J(r) + h(r), \forall r \in (0, r_0),$$

where  $0 < \lambda < 1$  and  $h(r)$  is bounded.

Then the estimation

$$J(r) \leq C \exp \left( -\delta \ln \frac{1}{\lambda} \int_r^{r_0} \frac{d\tau}{\tau \varphi_1(\tau)} \right) (J(r_0) + h(r_0)) \quad (2.9)$$

is valid for  $J(r)$ . Here  $0 < \delta < 1 - c_0$ .

The proof of this lemma can be found in [3, 4, 11].

We define a function  $\psi(r)$  on  $(0, r_0)$  by the inequality

$$\inf_{r\psi(r) < |x| < r} \lambda_p(|x|)(r - r\psi(r))\omega(x) \geq \mu > 0, \quad (2.10)$$

where  $\mu$  is such that  $0 < 1 - c_0 < \psi(r) < 1$ .

**Theorem 2.1** *Let  $u \in L_p(0, T; \overset{\circ}{W}_{p,\omega,loc}^m(\Omega_t)) \cap W_2^1(0, T; L_2(\Omega_T))$  be the generalized solution of problem (2.1)-(2.3). Assume that coefficients satisfy the conditions (2.4) and (2.5), the domain satisfies the condition (A), the weight satisfies the Muckenhoupt condition, the function  $\psi(r)$  satisfies condition (2.10) and  $G(r)$  is bounded.*

Then for  $J(r)$  the following estimate holds

$$J \left( r \exp \left( -\frac{1 - \psi(r)}{1 - c_0 - \theta} \right) \right) \leq C \exp \left( -\theta \ln \frac{1}{\nu} \int_r^{r_0} \frac{d\tau}{\tau(1 - \psi(r))} \right) (J(r_0) + G(r_0)) \quad (2.11)$$

for every  $\nu > 0$  and  $\theta < 1 - c_0$ .

**Proof.** This theorem is Saint-Venant's type estimate. For proof we substitute special test functions to integral identity (2.6) and do some calculations (see also [4]).

**Corollary 2.1** *This estimate is new in the case of linear equations of following type*

$$\frac{\partial u}{\partial t} - \sum_{|\alpha| \leq m} a_\alpha(x, t) D^\alpha u = \sum_{|\alpha| \leq m} D^\alpha F_\alpha(x, t),$$

$$c_1 \omega(x) |\xi|^{2m} \leq \sum_{|\alpha|=m} a_\alpha \xi^\alpha \leq c_2 \omega(x) |\xi|^{2m}, \quad x \in \Omega, \xi \in \mathbb{R}^n,$$

$c_1, c_2 > 0$ ,  $F_\alpha \in L_2(\Omega_t)$ ,  $a_\alpha \in C^{|\alpha|-m}(\overline{Q_T})$  at  $|\alpha| > m$  and  $a_\alpha(x, t)$  is a measurable function such that  $|\alpha| \leq m$ .

**Corollary 2.2** Let  $u \in L_p(0, T; \overset{\circ}{W}_{p,\omega,loc}^m(\Omega_t)) \cap W_2^1(0, T; L_2(\Omega_t))$  be generalized solution of problem (2.1)-(2.3) and  $0 \in \partial\Omega$ . The domain  $\Omega$  in the neighborhood of the point has a boundary such that  $\lambda_p(r) > \lambda^{(0)}r^{-1}$  for any  $r \in (0, r_0)$ ,  $\lambda^{(0)} > 0$ . Then there exists  $\gamma_0 > 0$  such that if for  $G(r)$  the following estimate holds

$$G(r) \leq Cr^{\gamma_0+\varepsilon}G(r_0), \quad \forall r \in (0, r_0)$$

for sufficiently small  $\varepsilon$ , the following estimate holds

$$\omega(x) |D^j u(x, t)| \leq C |x|^{m-\frac{n}{p}-j+\gamma_0} (J(r) + G(r))^{\frac{1}{p}}, \quad (2.12)$$

$$j = 0, 1, \dots, \left[ m - \frac{n}{p} \right].$$

The proof of this corollary follows from estimate (2.11) and embedding theorems.

### 3 The removable singularity of solutions

First we state an auxiliary lemma.

**Lemma 3.1** Let  $I(r)$  be a non-negative and non-growing function on interval  $(0, r_0)$ ,  $r_0 > 0$ , function satisfying condition

$$I(r) \leq \theta I(r\varepsilon(r)) + G(r\varepsilon(r)), \quad 0 < \theta < 1, \quad (3.1)$$

where  $\varepsilon(r)$  is a measurable function,  $0 < c_0 < \varepsilon(r) < 1$  such that

$$k(r) = (\varphi(r))^{-1} \inf_{r\varepsilon(r) < \tau < r} \varphi(\tau) \geq \nu > 0, \quad \varphi(r) \equiv 1 - \varepsilon(r) \quad (3.2)$$

and function  $G(r)$  is a measurable and locally bounded.

Then the following holds:

- 1)  $I(r_i) < cG(r_i)$  for some sequence  $r_i \rightarrow 0$ ;
- 2) or  $I(r)$  is growing fast enough as  $r \rightarrow 0$ , namely

$$I(r) \geq C \exp \left( \ln(\theta + \delta)^{-1} \int_r^{r_0} \frac{d\tau}{\tau(1 - \varepsilon(r))} \right) I(r_0), \quad (3.3)$$

where  $0 < \delta < 1 - \theta$ .

See [10] for the proof of this lemma.

We will define

$$I(r) = \int_{\Omega \setminus \Omega_r} \omega(x) |D^m u|^p dx dt.$$

For small  $r$ , for the behavior  $I(r)$  following theorem holds. This theorem gives apriori estimates of energy integrals.

**Theorem 3.1** Let  $u \in L_p(0, T; \overset{\circ}{W}_{p,\omega,loc}^m(\Omega, \Gamma)) \cap W_2^1(0, T; L_2(\Omega_T))$  be generalized solution of problem (2.1)-(2.3). Assume that coefficients satisfy the conditions (2.4) and (2.5), the domain satisfies the condition (A), the weight satisfies the Muckenhoupt condition, the function  $\psi(r)$  satisfies condition (2.10) and  $k(r)$  satisfies condition (3.2). Then for  $I(r)$  the following holds:

- 1)  $I(r_i) \leq c(1 + G(r_i))$  for some sequence  $r_i \rightarrow 0$ ;  
 2) or  $I(r)$  is growing fast enough as  $r \rightarrow 0$ , namely

$$I(r) \leq C \exp \left( \ln \frac{1}{k_0 + \delta} \int_r^{r_0} \frac{d\tau}{\tau \psi(r)} \right), \quad (3.4)$$

where  $k_0 = \text{const}$ .

Also the following theorem holds.

**Theorem 3.2** Let  $u \in L_p(0, T; \overset{\circ}{W}_{p,\omega,loc}^m(\Omega, \Gamma)) \cap W_2^1(0, T; L_2(\Omega_T))$  be generalized solution of problem (2.1)-(2.3). In the conditions of Theorem 3.1 the following inequality holds

$$I(r) \leq C \exp \left( -c \int_r^{r_0} \frac{\psi(\tau) \tau^{-1}}{1 + \psi(\tau)} d\tau \right), \quad \forall r < r_0. \quad (3.5)$$

Then singularity set  $\Gamma$  of solution  $u(x, t)$  removable, i.d.  $u \in L_p(0, T; \overset{\circ}{W}_{p,\omega}^m(\Omega)) \cap W_2^1(0, T; L_2(\Omega))$ .

The proof of Theorems 3.1 and 3.2 similarly to correspondingly theorems from [4, 10, 11].

These theorems generalize the corresponding results in [12, 14, 16–19].

#### 4 The behavior solutions in unbounded domains. Theorem of types Phragmén-Lindelöf

Now we consider unbounded domains. First we define a measurable function  $\psi(r) : 1 < \psi(r) < \infty, \forall r > r_0 > 0$  and  $\varphi_0$  is continuous, non-growing function such that  $\varphi_0(r) \geq r^{-1} \sup(h(r)) - 1$  at  $r > r_0$ . Here the upper bound is taken over all non-decreasing function  $h(r) : h(r) \leq r\psi(r)$  at  $r_0 < r < \infty$ .

**Lemma 4.1** Assume that on  $(r_0, \infty)$  non-negative, continuous function  $I(r)$  satisfies inequality

$$I(r) \leq \theta I(r\psi(r)), \quad \psi(r) = 1 + \varphi_0(r) \quad (4.1)$$

for all  $r \in (r_0, \infty)$  and  $0 < \theta < 1$ . Then for any  $r \in (r_0, \infty)$  the following estimate holds

$$I \left( r \exp \left( -\frac{\varphi_0(r)}{1 - \nu} \right) \right) \geq \theta \exp \left( \nu \ln \theta^{-1} \int_{r_0}^r \frac{d\tau}{\tau \varphi_0(\tau)} \right) I(r_0) \quad (4.2)$$

for all  $\nu \in (0, 1)$ .

The proof of Lemma 4.1 is similar to the proof of the corresponding result in [11].

The unbounded domains which satisfy isoperimetric conditions divided into two classes, depending on behavior nonlinear frequency function  $\lambda_p(r)$  at  $r \rightarrow \infty$ . First class is narrow domains and in our terms this condition is

$$r\lambda_p(r) > c > 0, \quad \forall r > r_0 > 0. \quad (A_1)$$

Second class is wide domains and in our terms this condition is

$$r\lambda_p(r) \leq c < \infty, \quad \forall r > r_0 > 0. \quad (B_1)$$

Also we define the function  $\psi(r)$  for  $h_0 > 0$  by inequality

$$\inf_{r < \tau < r\psi(r)} r\lambda_p(\tau) (\psi(r) - 1) \geq h_0, \psi(r) > 1, \forall r > r_0. \quad (4.3)$$

Let

$$J(r) = \int_{\Omega_r} \omega(x) |D^m u(x, t)|^p dx dt,$$

$$G(r) = \int_{\Omega_r} \left( \sum_{|\alpha| \leq m} (|F_\alpha(x, t)| + |f_2(x, t)|)^{\frac{p}{p-1}} \lambda_p^{-\frac{m-|\alpha|}{p-1} p} (|x|) + |f_1(x, t)| \right) dx dt.$$

The following theorem is true.

**Theorem 4.1** Let  $u \in L_p(0, T; \overset{\circ}{W}_{p, \omega(x), loc}^m(\Omega_t)) \cap W_2^1(0, T; L_2(\Omega_T))$  be generalized solution of problem (2.1)-(2.3),  $\omega(x)$  be in Muckenhoupt classes. Assume that coefficients satisfy the conditions (2.4) and (2.5). Moreover domain  $\Omega$  is narrow enough in the sense that  $\lambda_p(r) > \delta^{-1}$  at  $(r_0, \infty)$ . Let  $\psi(r)$  be any function which satisfy condition (4.3) and  $\varphi_0(r)$  be the function under construction by  $\psi(r)$  as in Lemma 4.1. Then for  $J(r)$  the following hold:

1.

$$\lim_{r \rightarrow \infty} \frac{J(r)}{G(r)} < \infty; \quad (4.4)$$

2. Or

$$J\left(r \exp\left(\frac{\varphi_0(r)}{1-\nu}\right)\right) \geq \theta \exp\left(\nu \ln \frac{1}{\theta} \int_{r_0}^r \frac{d\tau}{\tau \varphi_0(\tau)}\right) J(r_0) \quad (4.5)$$

for all  $\nu \in (0, 1)$  and for all  $r > r_0$  at big enough  $r_0$ .

The proof of Theorem 4.1 is similar the proof of the corresponding result in [11].

**Corollary 4.1** This is Phragmén-Lindelöf type theorem. Also we can give this type theorems for wide classes of domains and for integrals of functions.

**Corollary 4.2** Many examples can be given showing the sharpness of the results obtained in different domains.

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