

Approximate solution of a class of nonlinear singular integral equations by the Newton-Kantorovich method

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Received: 23.03.2020 / Revised: 03.09.2020 / Accepted: 18.11.2020

Abstract. In the paper a class of nonlinear singular integral equation with a Cauchy kernel encountered in theory of boundary value problems for analytic functions is solved by the Newton-Kantorovich method.

Keywords. Newton-Kantorovich method, nonlinear singular integral equation, Fréchet derivative, Hölder condition

Mathematics Subject Classification (2010): 30E20, 30E25

1 Introduction

We consider a problem of finding the solution of a nonlinear singular integral equation (NSIE) of the form

$$f(t, u(t)) - \frac{1}{\pi} \int_{-1}^1 \frac{u(\tau)}{\tau - t} d\tau - D = 0, \quad (1)$$

satisfying the condition

$$u(-1) = u(1) = 0, \quad (2)$$

where $f(t, u)$ is the given function. It is required to find the function $u(t)$ and constant D for which (1)-(2) holds. Problem (1)-(2) is equivalent to the following Riemann-Hilbert problem [1]:

Find a holomorphic function $W(z) = u(z) + iv(z)$, $z = x + iy$ in the upper half-plane $y > 0$ of the complex plane z , that satisfies the Hölder condition in the half-plane $y \geq 0$, is bounded at infinity, and on the straight line $y = 0$ satisfies the boundary condition

$$\begin{cases} u(x) = 0, & \text{if } |x| \geq 1, \\ v(x) + f(x, u(x)) = 0, & \text{if } |x| < 1. \end{cases} \quad (3)$$

The equivalence of these problems follows directly from the representation

$$W(z) = \frac{1}{\pi i} \int_{-1}^1 \frac{u(\xi)}{\xi - z} d\xi - Di,$$

where D is a real number, $-Di$ is the value of the function $W(z)$ at infinity.

In the paper [1] it is shown that equivalence of problems (1)-(2) and may be used when reducing problem (1)-(2) for the function $f(x, u)$ of the form

$$f(x, u) = b_0(x) + b_1(x)u + \psi(u), \quad (4)$$

where $b_j(x)$ are the functions satisfying the Hölder condition, $j = 1, 2$, the $\psi(u)$ is a function with a bounded measurable derivative $\psi'(u)$ by means of the substitution

$$U = u, \quad V = v + \psi(u)$$

to the Riemann-Hilbert problem for solution the Beltrami-type quasilinear elliptic equation $W = U + iV$ with a linear boundary conditions

$$Re[G(x)W(x)] = g(x), \quad (5)$$

where

$$G(x) = \begin{cases} 1, & \text{if } |x| \geq 1, \\ b_1(x) - i, & \text{if } |x| < 1, \end{cases}$$

$$g(x) = \begin{cases} 0, & \text{if } |x| \geq 1, \\ -b_0(x), & \text{if } |x| < 1. \end{cases}$$

We note that equations of the form (1) are encountered also in contact theory of elasticity (see [4]).

It follows from singular integral equations that for solving problem (1)-(2) have the following relations [2]:

$$u(t) = -\frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{f(\tau, u(\tau))}{\sqrt{1-\tau^2}(\tau-t)} d\tau, \quad (6)$$

$$D = \frac{1}{\pi} \int_{-1}^1 \frac{f(\tau, u(\tau))}{\sqrt{1-\tau^2}} d\tau. \quad (7)$$

If the function $u^*(t)$ ($u^*(-1) = u^*(1) = 0$) is the solution of NSIE (6), then the pair (u^*, D^*) is the solution of the problem (1)-(2), where

$$D^* = \frac{1}{\pi} \int_{-1}^1 \frac{f(\tau, u^*(\tau))}{\sqrt{1-\tau^2}} d\tau.$$

We will consider NSIE in the space \dot{H}_α ($0 < \alpha < \frac{1}{2}$):

$$\dot{H}_\alpha = \{\varphi \in H_\alpha[-1, 1] \mid \varphi(-1) = \varphi(1) = 0\},$$

where $H_\alpha[-1, 1]$ is a space of functions satisfying on $[-1, 1]$ the Hölder condition with the exponent α . Note that the norm in \dot{H}_α is determined in the following way:

$$\|\varphi\|_\alpha = \sup_{t_1, t_2 \in [-1, 1], t_1 \neq t_2} \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\alpha}.$$

As is known [3], the operator

$$(Av)(t) = \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{v(\tau)}{\sqrt{1-\tau^2}(\tau-t)} d\tau$$

acts from \dot{H}_α to \dot{H}_α ($0 < \alpha < \frac{1}{2}$) and is bounded.

We introduce the following operator

$$(Pu)(t) = u(t) + \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{f(\tau, u(\tau))}{\sqrt{1-\tau^2}(\tau-t)} d\tau.$$

Then NSIE (6) may be written in the form

$$Pu = 0.$$

Let the function $f(t, u)$ be determined in the domain $\Omega = \{(t, u) | -1 \leq t \leq 1, -\infty < u < +\infty\}$ and have the following properties:

- 1) $f(t, 0) \in \dot{H}_\alpha$,
- 2) $\exists l > 0$ such that for $\forall (t_1, u_1), (t_2, u_2) \in \Omega$ $|f(t_1, u_1) - f(t_2, u_2)| \leq l(|t_1 - t_2|^\alpha + |u_1 - u_2|)$.

Under these conditions, it is known [3] that the operator P acts from \dot{H}_α to \dot{H}_α and is bounded. We prove the following theorem.

Theorem 1 *Let in the domain Ω there exist $f'_u(t, u)$, $f''_{u^2}(t, u)$ and the following conditions, the fulfilled:*

- a) $f(t, 0), f'_u(t, 0), f''_{u^2}(t, 0) \in \dot{H}_\alpha$,
- b) $\exists l_0, l_1 > 0$ are such that for $\forall (t_1, u_1), (t_2, u_2) \in \Omega$

$$|f(t_1, u_1) - f(t_2, u_2)| \leq l_0(|t_1 - t_2|^\alpha + |u_1 - u_2|),$$

$$|f'_u(t_1, u_1) - f'_u(t_2, u_2)| \leq l_1(|t_1 - t_2|^\alpha + |u_1 - u_2|),$$

- c) $\forall u \in B(0, r), \forall t_1, t_2 \in [-1, 1]$

$$|f''_{u^2}(t_1, u(t_1)) - f''_{u^2}(t_2, u(t_2))| \leq A_r |t_1 - t_2|^\alpha,$$

where $B(0, r)$ is a ball of radius r in \dot{H}_α .

Then the operator P is Fréchet differentiable and the Fréchet derivative has a bounded inverse at each point of the space \dot{H}_α .

Proof. Let u_0 be an arbitrary fixed element of the space \dot{H}_α and h be any element of the some space.

Using the identities

$$f(\tau, u_0 + h) - f(\tau, u_0) = hf'_u(\tau, u_0) + h^2 \int_0^1 f''_{u^2}(\tau, u_0 + \theta h)(1 - \theta) d\theta,$$

we have

$$P(u_0 + h) - P(u_0) = h(t) + \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{f'_u(\tau, u_0(\tau))h(\tau)}{\sqrt{1-\tau^2}(\tau-t)} d\tau + w(t, h),$$

where

$$w(t, h) = \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{h^2(\tau) \int_0^1 f''_{u^2}(\tau, u_0(\tau) + \theta h(\tau))(1-\theta) d\theta}{\sqrt{1-\tau^2}(\tau-t)} d\tau.$$

By the theorem conditions

$$\psi(\tau) = h^2(\tau) \int_0^1 f''_{u^2}(\tau, u_0(\tau) + \theta h(\tau))(1-\theta) d\theta \in \dot{H}_\alpha.$$

Then we have

$$\begin{aligned} & \left\| \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{h^2(\tau) \int_0^1 f''_{u^2}(\tau, u_0(\tau) + \theta h(\tau))(1-\theta) d\theta}{\sqrt{1-\tau^2}(\tau-t)} d\tau \right\|_{\dot{H}_\alpha} \\ & \leq C_0 \left\| h^2(\tau) \int_0^1 f''_{u^2}(\tau, u_0(\tau) + \theta h(\tau))(1-\theta) d\theta \right\|_{\dot{H}_\alpha} \\ & \leq C_0 \|h\|_{\dot{H}_\alpha}^2 \left\| \int_0^1 f''_{u^2}(\tau, u_0(\tau) + \theta h(\tau))(1-\theta) d\theta \right\|_{\dot{H}_\alpha} \\ & \leq \frac{1}{2} C_0 \|h\|_{\dot{H}_\alpha}^2 \cdot A_{\|u_0\|+\|h\|}, \end{aligned}$$

where $C_0 = \|A\|_{\dot{H}_\alpha \rightarrow \dot{H}_\alpha}$.

Taking into account the last inequality, we have

$$\lim_{\|h\| \rightarrow 0} \frac{\|w(t, h)\|_{\dot{H}_\alpha}}{\|h\|_{\dot{H}_\alpha}} \leq \lim_{\|h\| \rightarrow 0} \frac{\frac{1}{2} C_0 \|h\|^2 \cdot A_{\|u_0\|+\|h\|}}{\|h\|_{\dot{H}_\alpha}} = 0$$

Thus, at any point $u_0 \in \dot{H}_\alpha$ there exists a Fréched derivative of the operator P and it has the form

$$P'(u_0)h = h(t) + \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{f'_u(\tau, u_0(\tau))}{\sqrt{1-\tau^2}(\tau-t)} h(\tau) d\tau. \quad (8)$$

We prove that there exists an inverse operator $[P'(u_0)]^{-1}$.

Let $u_0(t) \in \dot{H}_\alpha$ be an initial approximation. For $g(t) \in \dot{H}_\alpha$ we consider the equation

$$P'(u_0)h = h(t) + \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{f'_u(\tau, u_0(\tau))h(\tau)}{\sqrt{1-\tau^2}(\tau-t)} d\tau = g(t). \quad (9)$$

Taking into account condition a) of the theorem, it is easy to calculate that the index of equation (9) equals zero.

Then for any right part of $g(t)$, equation (9) has a unique solution in H_α , that has the following form [2].

$$h(t) = -\frac{b^2(t)g(t)}{1+b^2(t)} - \frac{z(t)}{1+b^2(t)} \cdot \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{b(\tau)g(\tau)}{\sqrt{1-\tau^2}z(\tau)(\tau-t)} d\tau + g(t), \quad (10)$$

where

$$\begin{aligned} z(t) &= \sqrt{1+b^2(t)} \exp\left(-\int_{-1}^1 \frac{\theta(\tau)}{\tau-t} d\tau\right), \quad \theta(t) = \frac{1}{\pi} \arg[1+ib(t)], \\ b(t) &= f'_u(t, u_0(t)). \end{aligned}$$

It can be seen from formula (10) that $h(\pm 1) = 0$, i.e. $h(t) \in \dot{H}_\alpha$.

Thus, we proved that equation (9), for any right part of from \dot{H}_α has a unique solution on \dot{H}_α , that is, there exists an inverse operator $[P'(u_0)]^{-1}$ and it is of the form

$$\begin{aligned} & [P'(u_0)]^{-1}g(t) \\ &= -\frac{b^2(t)g(t)}{1+b^2(t)} - \frac{z(t)}{1+b^2(t)} \cdot \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{b(\tau)g(\tau)}{\sqrt{1-\tau^2}z(\tau)(\tau-t)} d\tau + g(t). \end{aligned} \quad (11)$$

Estimate the norm

$$\|[P'(u_0)]^{-1}\|_{\dot{H}_\alpha \rightarrow \dot{H}_\alpha}.$$

We have

$$\begin{aligned} \|[P'(u_0)]^{-1}\| &\leq \left\| \frac{1}{1+b^2(t)} \right\|_{H_\alpha} \cdot \|b(t)\|_{\dot{H}_\alpha}^2 \\ &+ \left\| \frac{1}{1+b^2(t)} \right\|_{H_\alpha} \cdot \|z(t)\|_{H_\alpha} C_0 \left\| \frac{1}{z(t)} \right\|_{H_\alpha} \cdot \|b(t)\|_{\dot{H}_\alpha} + 1. \end{aligned}$$

Estimate each norm separably. At first we estimate $\left\| \frac{1}{1+b^2(t)} \right\|_{H_\alpha}$.

Obviously, $\left\| \frac{1}{1+b^2(t)} \right\|_C \leq 1$.

In what follows,

$$\begin{aligned} \left| \frac{1}{1+b^2(t_1)} + \frac{1}{1+b^2(t_2)} \right| &\leq |b^2(t_2) - b^2(t_1)| |b(t_2) + b(t_1)| \leq 2\|b\|_C |b(t_2) - b(t_1)| \\ &\leq 2\|b\|_C l_1 (|t_1 - t_2|^\alpha + |u_0(t_1) - u_0(t_2)|) \leq 2\|b\|_C l_1 (1 + \|u_0\|) |t_1 - t_2|^\alpha \\ &\leq 2l_1 (\|u_0\| + 1) l_1 (1 + \|u_0\|) |t_1 - t_2|^\alpha. \end{aligned}$$

Thus,

$$\left\| \frac{1}{1+b^2(t)} \right\|_{H_\alpha} \leq 2l_1^2 (1 + \|u_0\|)^2 + 1 \equiv M_1.$$

In a similar way we find

$$\|b(t)\|_{\dot{H}_\alpha} \leq l_1 (1 + \|u_0\|) \equiv M_2.$$

Hence,

$$\left\| \frac{1}{1+b^2(t)} \right\|_{H_\alpha} \|b(t)\|^2 \leq l_1 (1 + \|u_0\|) [2l_1^2 (1 + \|u_0\|)^2 + 1] \equiv M_3.$$

Estimate the norm $z(t)$:

$$z(t) = \sqrt{1+b^2(t)} \exp \left(- \int_{-1}^1 \frac{\theta(\tau)}{\tau-t} d\tau \right).$$

Applying the Lagrange theorem, we get

$$|\sqrt{1+b^2(t_1)} - \sqrt{1+b^2(t_2)}| \leq l_1^2 (\|u_0\|_C + 1) (|t_1 - t_2|^\alpha + |u_0(t_1) - u_0(t_2)|). \quad (12)$$

On the other hand, we have

$$\|\sqrt{1+b^2(t_1)}\|_C = \sqrt{1+\|b(t)\|_C^2} \leq \sqrt{1+l_1(\|u_0\|_C+1)}. \quad (13)$$

It follows from (12) and (13) that

$$\|\sqrt{1+b^2(t)}\|_{H_\alpha} \leq l_1^2(1+\|u_0\|)^2 + \sqrt{1+l_1(\|u_0\|_C+1)} \equiv M_4. \quad (14)$$

Introduce a singular operator

$$(Bv)(t) = - \int_{-1}^1 \frac{v(\tau)}{\tau-t} d\tau.$$

It is known [3] that the operator B acts from \dot{H}_α to H_α and is bounded. Taking this into account, we have

$$\left\| - \int_{-1}^1 \frac{\theta(\tau)}{\tau-t} d\tau \right\|_{H_\alpha} \leq C_1 \|\theta(t)\|_{\dot{H}_\alpha}, \quad \text{where } C_1 = \|B\|_{\dot{H}_\alpha \rightarrow H_\alpha}.$$

Estimate $\|\theta(t)\|_{\dot{H}_\alpha}$.

$$\begin{aligned} |\theta(t_1) - \theta(t_2)| &\leq \frac{1}{\pi} |f'_u(t_1, u_0(t_1)) - f'_u(t_2, u_0(t_2))| \\ &\leq \frac{1}{\pi} l_1(1+\|u_0\|) |t_1 - t_2|^\alpha \Rightarrow \|\theta(t)\|_{\dot{H}_\alpha} \leq \frac{1}{\pi} l_1(1+\|u_0\|) \equiv M_5. \end{aligned}$$

Estimate the last one, we get

$$\left\| \exp \left(- \int_{-1}^1 \frac{\theta(\tau)}{\tau-t} d\tau \right) \right\|_{H_\alpha} \leq e^{\frac{c_1 l_1}{\pi}(1+\|u_0\|)} \left(1 + \frac{c_1 l_1}{\pi}(1+\|u_0\|) \right). \quad (15)$$

From estimations (14) and (15), for $\|z(t)\|$ we find

$$\|z(t)\| \leq M_3 \left(1 + \frac{c_1 l_1}{\pi}(1+\|u_0\|) \right) \cdot \exp \left(\frac{c_1 l_1}{\pi}(1+\|u_0\|) \right) \equiv M_6. \quad (16)$$

By the similar calculations we can prove the validity of the following estimation

$$\begin{aligned} \left\| \frac{1}{z(t)} \right\|_{H_\alpha} &\leq \left[1 + l_1^2(1+\|u_0\|)^2 \right] \left[1 + \frac{C_1 l_1}{\pi}(1+\|u_0\|) \right] \\ &\times \exp \left(\frac{C_1 l_1}{\pi}(1+\|u_0\|) \right) \equiv M_7. \end{aligned} \quad (17)$$

We put together are the obtained estimations and have

$$\|[P'(u_0)]^{-1}\|_{\dot{H}_\alpha \rightarrow \dot{H}_\alpha} \leq 1 + M_3 + C_0 M_1 M_2 M_6 M_7 \equiv M_0. \quad (18)$$

Now we estimate $\|[P'(u_0)]^{-1}P(u_0)\|_{\dot{H}_\alpha}$.

$$\|[P'(u_0)]^{-1}P(u_0)\| \leq \|[P'(u_0)]^{-1}\| \cdot \|P(u_0)\| \leq M_0 \|P(u_0)\|,$$

$$\|P(u_0)\| \leq \|u_0\| + C_0 \|f(t, u_0(t))\| \leq \|u_0\| + C_0 l_0(1+\|u_0\|).$$

Thus,

$$\|[P'(u_0)]^{-1}P(u_0)\| \leq M_0(\|u_0\| + C_0 l_0(1+\|u_0\|)).$$

Prove that the operator $P'(u)$ satisfies the Lipschits condition in the ball $u_0(t)$ centered at R .

$$\begin{aligned} P'(u_1)h - p'(u_2)h &= \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{f'_u(\tau, u_1(\tau)) - f'_u(\tau, u_2(\tau))}{\sqrt{1-\tau^2}(\tau-t)} h(\tau) d\tau \\ &= \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{[u_1(\tau) - u_2(\tau)]a(\tau)}{\sqrt{1-\tau^2}(\tau-t)} h(\tau) d(\tau), \end{aligned}$$

where

$$a(\tau) = \int_0^1 f''_{u_1}(\tau, u_1(\tau) + \theta(u_2(\tau) - u_1(\tau))) d\theta, \quad a(\tau) \in \dot{H}_\alpha.$$

Hence it follows that

$$\begin{aligned} \|P'(u_1) - P'(u_2)\| &= \sup_{\|h\|=1} \left\| \frac{\sqrt{1-t^2}}{\pi} \int_{-1}^1 \frac{[u_1(\tau) - u_2(\tau)]a(\tau)}{\sqrt{1-\tau^2}(\tau-t)} h(\tau) d(\tau) \right\| \\ &\leq C_0 \|u_1 - u_2\|_{\dot{H}_\alpha} \|a(\tau)\|_{\dot{H}_\alpha} \leq C_0 A_{\|u_0\|+3R} \|u_1 - u_2\|_{\dot{H}_\alpha}. \end{aligned} \quad (19)$$

Thus, we proved that all the conditions of applicability of the Newton-Kantorovich method the equation (6) are fulfilled, i.e. thus the proof of Theorem 1 is complete. .

Theorem 2 Let $B(u_0, R)$ be a ball centered at u_0 of radius R in \dot{H}_α ($0 < \alpha < \frac{1}{2}$), and

$$h = M_0^2(\|u_0\| + C_0 l_0(1 + \|u_0\|)) C_0 A_{\|u_0\|+3R}.$$

Then, if $h < \frac{1}{4}$ and the function $f(t, u)$ satisfies the conditions of Theorem 1, then in the ball

$$\|u - u_0\| \leq M_0(\|u_0\| + C_0 l_0(1 + \|u_0\|)) \frac{1 - \sqrt{1-4h}}{2h}$$

equations (6) has a unique solution $u^*(t)$ and the sequence $\{u_k(t)\}$ in determined by the formula

$$u_{n+1}(t) = u_n(t) - [P'(u_0)]^{-1}(P(u_n)),$$

converges to this solution. Moreover we have the following estimation of the convergence:

$$\begin{aligned} \|u_n - u^*\|_{\dot{H}_\alpha} &\leq \frac{q^n}{1-q} \|[P'(u_0)]^{-1}P(u_0)\|_{\dot{H}_\alpha}, \\ q &= \frac{1 - \sqrt{1-4h}}{2} < \frac{1}{2}. \end{aligned}$$

It remains to prove that the sequence of constants

$$D_n = \frac{1}{\pi} \int_{-1}^1 \frac{f(\tau, u_n(\tau))}{\sqrt{1-\tau^2}} d\tau$$

converges to D^* . Estimate $|D_n - D^*|$. After simple calculations we have

$$\begin{aligned}
 |D_n - D^*| &= \left| \frac{1}{\pi} \int_{-1}^1 \frac{f(\tau, u_n(\tau))}{\sqrt{1-\tau^2}} d\tau - \frac{1}{\pi} \int_{-1}^1 \frac{f(\tau, u^*(\tau))}{\sqrt{1-\tau^2}} d\tau \right| \\
 &\leq \frac{1}{\pi} \int_{-1}^1 \frac{f(\tau, u_n(\tau)) - f(\tau, u^*(\tau))}{\sqrt{1-\tau^2}} d\tau \leq \frac{l_0}{\pi} \int_{-1}^1 \frac{u_n(\tau) - u^*(\tau)}{\sqrt{1-\tau^2}} d\tau \\
 &\leq \frac{l_0 q^n}{\pi(1-q)} \| [P'(u_0)]^{-1} P(u_0) \|_{\dot{H}_\alpha} \int_{-1}^1 \frac{d\tau}{\sqrt{1-\tau^2}} \\
 &= \frac{l_0 q^n}{\pi(1-q)} \| [P'(u_0)]^{-1} P(u_0) \|_{\dot{H}_\alpha} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

References

1. Junghanns P., Silbermann B. Numerical analysis of the quadrature method for solving linear and nonlinear Singular integral equations, *Karl-Mark-Stadt*, 895p. (1988).
2. Muskhelishvili N.I. Singular integral equations, *M.*, "Nauka", (1968).
3. Huseynov A.I., Mukhtarov Kh.Sh. Introduction to theory of nonlinear singular integral equations, *M.*, "Nauka", (1980).
4. Obnosov Yu.V. Some nonlinear boundary value problems of theory of analytic functions solvable in the closed form, *Scientific works of the jubilee seminar on boundary value problems. Minsk*, "Universitetskoe" publishing house, (1985).
5. Kolmogorov A.N., Fomin S.V. Elements of the theory of functions and functional analysis, *M.* "Nauka", (1976).