

Generating functions of the products of Gaussian numbers with Chebyshev polynomials of first and second kinds

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Abstract. *In this paper, we introduce a operator in order to derive some new symmetric properties of Gaussian Pell and Gaussian Pell Lucas numbers, and we give some new generating functions of the products of Gaussian Fibonacci numbers, Gaussian Lucas numbers, Gaussian Jacobsthal numbers, Gaussian Jacobsthal Lucas numbers, Gaussian Pell numbers and Gaussian Pell Lucas numbers with Chebyshev polynomials of first and second kinds.*

Keywords. symmetric functions, generating functions, Gaussian Fibonacci numbers, Gaussian Pell numbers.

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1 Introduction

The authors in [5] defined and studied the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers. They gave generating functions, Binet's formulas, explicit formulas and Q matrix of these numbers. They also presented explicit combinatorial and determinantal expressions, study negatively subscripted numbers and gave various identities. Similar to the Jacobsthal and Jacobsthal Lucas numbers they gave some interesting results for the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers.

In the papers [13, 16], second-order linear recurrence sequence $(U_n(a, b; p, q))_{n \geq 0}$ or briefly $(U_n)_{n \geq 0}$ is considered by the recurrence relation:

$$U_{n+2} = pU_{n+1} + qU_n, \quad \text{for } n \geq 0,$$

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with the initial conditions $U_0 = a$ and $U_1 = b$, special cases are listed in the table below:

a	b	p	q	$(U_n)_{n \geq 0}$	
i	1	1	1	$(GF_n)_{n \geq 0}$	Gaussian Fibonacci numbers
$2 - i$	$1 + 2i$	1	1	$(GL_n)_{n \geq 0}$	Gaussian Lucas numbers
$\frac{i}{2}$	1	1	2	$(GJ_n)_{n \geq 0}$	Gaussian Jacobsthal numbers
$2 - \frac{i}{2}$	$1 + 2i$	1	2	$(Gj_n)_{n \geq 0}$	Gaussian Jacobsthal Lucas numbers
i	1	2	1	$(GP_n)_{n \geq 0}$	Gaussian Pell numbers
$2 - 2i$	$2 + 2i$	2	1	$(GQ_n)_{n \geq 0}$	Gaussian Pell Lucas numbers

Table 1. Gaussian numbers.

Definition 1.1 [7] *The Chebyshev polynomials of second kind say $(U_n(x))_{n \in \mathbb{N}}$ is defined by the recurrence relation*

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad \text{for } n \geq 2$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$.

Definition 1.2 [7] *The Chebyshev polynomials of first kind say $(T_n(x))_{n \in \mathbb{N}}$ is defined by the recurrence relation*

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad \text{for } n \geq 2$$

with initial conditions $T_0(x) = 1$ and $T_1(x) = x$.

Binet's formulas are well known formulas in the theory of Fibonacci numbers. These formulas can also be carried out to the Gaussian Pell numbers. Let α and β be the roots of the characteristic equation $t^2 - 2t - 1 = 0$ of the recurrence relationship of the Gaussian Pell numbers. In the following theorem we give the Binet's formulas for Gaussian Pell and Gaussian Pell-Lucas numbers.

Theorem 1.1 [27] *Binet's formulas for Gaussian Pell and Gaussian Pell-Lucas numbers are given by*

$$GP_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i \frac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta}, \quad n \geq 0$$

and

$$GQ_n = (\alpha^n + \beta^n) - i(\alpha\beta^n + \beta\alpha^n), \quad n \geq 0$$

respectively.

2 Definitions and some Properties

In this section, we introduce a symmetric function and give some properties of this symmetric function. We also give some more useful definitions from the literature which are used in the subsequent sections.

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet A is a function of the letters which is invariant under permutation of the letters of A . Taking an extra indeterminate z , one has two fundamental series

$$\lambda_z(A) = \prod_{a \in A} (1 + az), \quad \sigma_z(A) = \frac{1}{\prod_{a \in A} (1 - az)}$$

the expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete functions $S_n(A)$

$$\lambda_z(A) = \sum_{n=0}^{+\infty} \Lambda_n(A) z^n, \quad \sigma_z(A) = \sum_{n=0}^{+\infty} S_n(A) z^n.$$

Let us now start at the following definition.

Definition 2.1 (see [1]) Let A and B be any two alphabets, then we give $S_n(A - B)$ by the following form:

$$\frac{\prod_{b \in B} (1 - bz)}{\prod_{a \in A} (1 - az)} = \sum_{n=0}^{+\infty} S_n(A - B) z^n = \sigma_z(A - B) \quad (2.1)$$

with the condition $S_n(A - B) = 0$ for $n < 0$.

Corollary 2.1 (see [1]). Taking $A = 0$ in (2.1) gives

$$\prod_{b \in B} (1 - bz) = \sum_{n=0}^{+\infty} S_n(-B) z^n = \lambda_z(-B). \quad (2.2)$$

Further, in the case $A = 0$ or $B = 0$, we have

$$\sum_{n=0}^{+\infty} S_n(A - B) z^n = \sigma_z(A) \times \lambda_z(-B). \quad (2.3)$$

Thus,

$$S_n(A - B) = \sum_{k=0}^n S_{n-k}(A) S_k(-B).$$

Definition 2.2 (see [34]) Let g be any function on \mathbb{R}^n , then we consider the divided difference operator as the following form

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1, \dots, x_i, x_{i+1}, \dots, x_n) - g^\sigma(x_1, \dots, x_i, x_{i+1}, \dots, x_n)}{x_i - x_{i+1}},$$

where g^σ is given by

$$g^\sigma(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = g(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n).$$

Definition 2.3 [8] Given an alphabet $E = \{e_1, e_2\}$, the symmetrizing operator $\delta_{e_1 e_2}^k$ is defined by

$$\delta_{e_1 e_2}^k f(e_1) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2}, \text{ for } k \in \mathbb{N}.$$

Definition 2.4 [39] The generalized hypergeometric functions ${}_p F_q(\cdot)$ are defined by

$${}_p F_q[\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; x] = \sum_{n=0}^{+\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n x^n}{(\beta_1)_n \dots (\beta_q)_n n!} \quad (2.4)$$

where $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q, x \in \mathbb{C}$, β_1, \dots, β_q are neither zero nor negative integers, and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1, & (n = 0) \\ \lambda(\lambda - 1) \dots (\lambda - n + 1), & (n \geq 1) \end{cases}.$$

For the special case that corresponds to $p = 2$ and $q = 1$ in (2.4), we can obtain ${}_2F_1(a, b; c; x)$ Gauss hypergeometric function

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}.$$

Theorem 2.1 [38] *The hypergeometric form of the Chebyshev polynomials of the first kind, can be written as follows:*

$$T_n(x) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right).$$

Theorem 2.2 [38] *The hypergeometric form of the Chebyshev polynomials of the second kind, can be written as follows:*

$$U_n(x) = (n+1) {}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right).$$

3 Generating Functions of the Products of Gaussian Numbers with Chebyshev Polynomials of First and Second Kinds

The following propositions are one of the key tools of the proof of our main result.

Proposition 3.1 [8] *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, then*

$$\sum_{n=0}^{+\infty} S_n(A) S_n(E) z^n = \frac{1 - a_1 a_2 e_1 e_2 z^2}{\left(\sum_{n=0}^{+\infty} S_n(-A) e_1^n z^n\right) \left(\sum_{n=0}^{+\infty} S_n(-A) e_2^n z^n\right)}. \quad (3.1)$$

Proposition 3.2 [15] *Given two alphabets $A = \{a_1, a_2\}$ and $E = \{e_1, e_2\}$, then*

$$\sum_{n=0}^{+\infty} S_{n-1}(A) S_n(E) z^n = \frac{(e_1 + e_2)z - e_1 e_2 (a_1 + a_2) z^2}{\left(\sum_{n=0}^{+\infty} S_n(-A) e_1^n z^n\right) \left(\sum_{n=0}^{+\infty} S_n(-A) e_2^n z^n\right)}. \quad (3.2)$$

In this part, we now derive the new generating functions of the products of Gaussian numbers and polynomials with Chebyshev polynomials of first and second kinds.

For the case $A = \{a_1, -a_2\}$, $E = \{2e_1, -2e_2\}$ with replacing a_2 by $(-a_2)$, e_1 by $(2e_1)$ and e_2 by $(-2e_2)$ in (3.1) and (3.2) we have

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{1 - 4a_1 a_2 e_1 e_2 z^2}{(1 - 2a_1 e_1)(1 + 2a_2 e_1)(1 + 2a_1 e_2)(1 - 2a_2 e_2)}. \quad (3.3)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{2(e_1 - e_2)z + 4e_1 e_2 (a_1 - a_2) z^2}{(1 - 2a_1 e_1)(1 + 2a_2 e_1)(1 + 2a_1 e_2)(1 - 2a_2 e_2)}. \quad (3.4)$$

This case consists of three related parts. **Firstly**, the substitutions

$$\begin{cases} a_1 - a_2 = 1 \\ a_1 a_2 = 1 \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = x \\ 4e_1 e_2 = -1, \end{cases}$$

in (3.3) and (3.4) we obtain

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{1 + z^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4}. \quad (3.5)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{2xz - z^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4} \quad (3.6)$$

and we have the following theorems.

Theorem 3.1 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Fibonacci numbers with Chebyshev polynomials of second kind is given by

$$\sum_{n=0}^{+\infty} GF_n (n+1)_2 F_1\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right) z^n = \frac{i + 2x(1-i)z + (2i-1)z^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4}.$$

Proof. We know that

$$GF_n = iS_n(a_1 + [-a_2]) + (1-i)S_{n-1}(a_1 + [-a_2])$$

see [19]. We see that

$$\begin{aligned} & \sum_{n=0}^{+\infty} GF_n (n+1)_2 F_1\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right) z^n \\ &= \sum_{n=0}^{+\infty} (iS_n(a_1 + [-a_2]) + (1-i)S_{n-1}(a_1 + [-a_2])) S_n(2e_1 + [-2e_2]) z^n \\ &= i \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\ & \quad + (1-i) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\ &= \frac{i + iz^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4} \\ & \quad + \frac{2x(1-i)z - (1-i)z^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4} \\ &= \frac{i + 2x(1-i)z + (2i-1)z^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4}. \end{aligned}$$

This completes the proof.

Theorem 3.2 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Fibonacci numbers with Chebyshev polynomials of first kind is given by

$$\sum_{n=0}^{+\infty} GF_n {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right) z^n = \frac{i + x(1-2i)z + (2i(1-x^2) - 1)z^2 - (1-i)xz^3}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4}.$$

Proof. We have

$$GF_n = iS_n(a_1 + [-a_2]) + (1 - i)S_{n-1}(a_1 + [-a_2]),$$

see [19]. We see that

$$\begin{aligned} & \sum_{n=0}^{+\infty} GF_n {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2})z^n \\ &= \sum_{n=0}^{+\infty} (iS_n(a_1 + [-a_2]) + (1 - i)S_{n-1}(a_1 + [-a_2])) \\ & \times (S_n(2e_1 + [-2e_2]) - xS_{n-1}(2e_1 + [-2e_2]))z^n \\ &= i \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \\ & - ix \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n \\ & + (1 - i) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \\ & - (1 - i)x \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n \\ &= \frac{i(1+z^2)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} - \frac{ix(z+2xz^2)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} \\ & + \frac{(1-i)(2xz-z^2)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} - \frac{x(1-i)(z+z^3)}{1-2xz+(3-4x^2)z^2+2xz^3+z^4} \\ &= \frac{i+x(1-2i)z+(2i(1-x^2)-1)z^2-(1-i)xz^3}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}. \end{aligned}$$

This completes the proof.

Theorem 3.3 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Lucas numbers with Chebyshev polynomials of second kind is given by

$$\sum_{n=0}^{+\infty} GL_n (n+1) {}_2F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2})z^n = \frac{2-i+2x(3i-1)z+(3-4i)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}.$$

Proof. We know that

$$GL_n = (2 - i)S_n(a_1 + [-a_2]) + (3i - 1)S_{n-1}(a_1 + [-a_2]),$$

see [19]. We see that

$$\begin{aligned} & \sum_{n=0}^{+\infty} GL_n (n+1) {}_2F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2})z^n \\ &= \sum_{n=0}^{+\infty} ((2 - i)S_n(a_1 + [-a_2]) + (3i - 1)S_{n-1}(a_1 + [-a_2]))S_n(2e_1 + [-2e_2])z^n \end{aligned}$$

$$\begin{aligned}
&= (2-i) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad + (3i-1) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&= \frac{(2-i) + (2-i)z^2}{1-2xz + (3-4x^2)z^2 + 2xz^3 + z^4} \\
&\quad + \frac{2x(3i-1)z - (3i-1)z^2}{1-2xz + (3-4x^2)z^2 + 2xz^3 + z^4} \\
&= \frac{2-i + 2x(3i-1)z + (3-4i)z^2}{1-2xz + (3-4x^2)z^2 + 2xz^3 + z^4}.
\end{aligned}$$

This completes the proof.

Theorem 3.4 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Lucas numbers with Chebyshev polynomials of first kind is given by

$$\begin{aligned}
&\sum_{n=0}^{+\infty} GL_n {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n \\
&= \frac{(2-i) + (4i-3)xz + (3-4x^2 + 2i(x^2-2))z^2 + (1-3i)xz^3}{1-2xz + (3-4x^2)z^2 + 2xz^3 + z^4}.
\end{aligned}$$

Proof. We know that

$$GL_n = (2-i)S_n(a_1 + [-a_2]) + (3i-1)S_{n-1}(a_1 + [-a_2]),$$

see [19]. We see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} GL_n {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n &= \sum_{n=0}^{+\infty} ((2-i)S_n(a_1 + [-a_2]) + (3i-1)S_{n-1}(a_1 + [-a_2])) \\
&\quad \times (S_n(2e_1 + [-2e_2]) - xS_{n-1}(2e_1 + [-2e_2])) z^n \\
&= (2-i) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad - (2-i)x \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
&\quad + (3i-1) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad + (1-3i)x \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
&= \frac{(2-i)(1+z^2)}{1-2xz + (3-4x^2)z^2 + 2xz^3 + z^4} - \frac{x(2-i)(z+2xz^2)}{1-2xz + (3-4x^2)z^2 + 2xz^3 + z^4} \\
&\quad + \frac{(3i-1)(2xz-z^2)}{1-2xz + (3-4x^2)z^2 + 2xz^3 + z^4} - \frac{x(3i-1)(z+z^3)}{1-2xz + (3-4x^2)z^2 + 2xz^3 + z^4}
\end{aligned}$$

$$= \frac{2 - i + (4i - 3)xz + (3 - 4x^2 + 2i(x^2 - 2))z^2 + (1 - 3i)xz^3}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4}.$$

This completes the proof.

Secondly, the substitutions

$$\begin{cases} a_1 - a_2 = 1 \\ a_1 a_2 = 2 \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = x \\ 4e_1 e_2 = -1, \end{cases}$$

in (3.3) and (3.4) we give

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n = \frac{1 + 2z^2}{1 - 2xz + (5 - 8x^2)z^2 + 2xz^3 + 4z^4}. \quad (3.7)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n = \frac{2xz - z^2}{1 - 2xz + (5 - 8x^2)z^2 + 2xz^3 + 4z^4}, \quad (3.8)$$

and we have the following theorems.

Theorem 3.5 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal numbers with Chebyshev polynomials of second kind is given by

$$\sum_{n=0}^{+\infty} GJ_n (n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2})z^n = \frac{i + 2x(2-i)z + (3i-2)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}.$$

Proof. We know that

$$GJ_n = \frac{i}{2}S_n(a_1 + [-a_2]) + (1 - \frac{i}{2})S_{n-1}(a_1 + [-a_2]),$$

see [19]. We see that

$$\begin{aligned} & \sum_{n=0}^{+\infty} GJ_n (n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2})z^n \\ &= \sum_{n=0}^{+\infty} (\frac{i}{2}S_n(a_1 + [-a_2]) + (1 - \frac{i}{2})S_{n-1}(a_1 + [-a_2])) \times S_n(2e_1 + [-2e_2])z^n \\ &= \frac{i}{2} \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \\ & \quad + (1 - \frac{i}{2}) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \\ &= \frac{i + 2iz^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4} \\ & \quad + \frac{2x(2-i)z - (2-i)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4} \\ &= \frac{i + 2x(2-i)z + (3i-2)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}. \end{aligned}$$

This completes the proof.

Theorem 3.6 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal numbers with Chebyshev polynomials of first kind is given by

$$\begin{aligned} & \sum_{n=0}^{+\infty} GJ_n {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n \\ &= \frac{i + (2x - 3ix)z - 4(4x + i(1 + 2x + 2x^2))z^2 - 4(2-i)xz^3}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}. \end{aligned}$$

Proof. We know that

$$GJ_n = \frac{i}{2} S_n(a_1 + [-a_2]) + (1 - \frac{i}{2}) S_{n-1}(a_1 + [-a_2]),$$

see [19]. We see that

$$\begin{aligned} & \sum_{n=0}^{+\infty} GJ_n {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n = \sum_{n=0}^{+\infty} \left(\frac{i}{2} S_n(a_1 + [-a_2]) + \left(1 - \frac{i}{2}\right) S_{n-1}(a_1 + [-a_2]) \right) \\ & \quad \times (S_n(2e_1 + [-2e_2]) - x S_{n-1}(2e_1 + [-2e_2])) z^n \\ &= \frac{i}{2} \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n - \frac{ix}{2} \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\ & \quad + \left(1 - \frac{i}{2}\right) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\ & \quad - x \left(1 - \frac{i}{2}\right) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\ &= \frac{i(1 + 2z^2)}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4} - \frac{ix(z + 4x)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4} \\ & \quad + \frac{(2-i)(2xz - z^2)}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4} - \frac{x(2-i)(z + 2z^3)}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4} \\ &= \frac{i + 2x(1-i)z + (i(3 - 4x^2) - 2)z^2 + 2x(i-2)z^3}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}. \end{aligned}$$

This completes the proof.

Theorem 3.7 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal-Lucas numbers with Chebyshev polynomials of second kind is given by

$$\sum_{n=0}^{+\infty} GJ_n (n+1) {}_2F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) z^n = \frac{4 - i + 2x(5i - 2)z + (10 - 7i)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}.$$

Proof. We know that

$$Gj_n = \left(2 - \frac{i}{2}\right) S_n(a_1 + [-a_2]) + \left(\frac{5}{2}i - 1\right) S_{n-1}(a_1 + [-a_2])$$

see [19]. We see that

$$\begin{aligned} \sum_{n=0}^{+\infty} Gj_n (n+1) {}_2F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) z^n &= \sum_{n=0}^{+\infty} \left(\left(2 - \frac{i}{2}\right) S_n(a_1 + [-a_2]) \right) \\ &+ \left(\frac{5}{2}i - 1 \right) S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\ &= \left(2 - \frac{i}{2}\right) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\ &+ \left(\frac{5}{2}i - 1\right) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\ &= \frac{4 - i + (8 - 2i)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4} \\ &+ \frac{2x(5i - 2)z - (5i - 2)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4} \\ &= \frac{4 - i + 2x(5i - 2)z + (10 - 7i)z^2}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}. \end{aligned}$$

This completes the proof.

Theorem 3.8 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal-Lucas numbers with Chebyshev polynomials of first kind is given by

$$\begin{aligned} &\sum_{n=0}^{+\infty} Gj_n {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n \\ &= \frac{4 - i + 6x(i-1)z + ((10 - 16x^2) + i(4x^2 - 7))z^2 + 2x(2 - 5i)z^3}{2 - 4xz + (10 - 16x^2)z^2 + 4xz^3 + 8z^4}. \end{aligned}$$

Proof. We know that

$$Gj_n = \left(2 - \frac{i}{2}\right) S_n(a_1 + [-a_2]) + \left(\frac{5}{2}i - 1\right) S_{n-1}(a_1 + [-a_2]),$$

see [19]. We see that

$$\begin{aligned} \sum_{n=0}^{+\infty} Gj_n {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n &= \sum_{n=0}^{+\infty} \left(\left(2 - \frac{i}{2}\right) S_n(a_1 + [-a_2]) + \left(\frac{5}{2}i - 1\right) S_{n-1}(a_1 + [-a_2]) \right) \\ &\times (S_n(2e_1 + [-2e_2]) - xS_{n-1}(2e_1 + [-2e_2])) z^n \\ &= \left(2 - \frac{i}{2}\right) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \end{aligned}$$

$$\begin{aligned}
& -\left(2 - \frac{i}{2}\right) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
& + \left(\frac{5}{2}i - 1\right) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
& - x \left(\frac{5}{2}i - 1\right) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
= & \frac{(4-i)(1+2z^2)}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4} - \frac{x(4-i)(z+4xz^2)}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4} \\
& + \frac{(5i-2)(2xz-z^2)}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4} - \frac{x(5i-2)(z+2z^3)}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4} \\
= & \frac{4-i+6x(i-1)z + ((10-16x^2)+i(4x^2-7))z^2 + 2x(2-5i)z^3}{2-4xz+(10-16x^2)z^2+4xz^3+8z^4}.
\end{aligned}$$

This completes the proof.

Thirdly, the substitutions

$$\begin{cases} a_1 - a_2 = 2 \\ a_1 a_2 = 1 \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = x \\ 4e_1 e_2 = -1, \end{cases}$$

in (3.3) and (3.4) we give

$$\begin{aligned}
\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n &= \frac{1+z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}. \quad (3.9) \\
\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n &= \frac{2xz-2z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}, \quad (3.10)
\end{aligned}$$

and we have the following theorems.

Theorem 3.9 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell numbers with Chebyshev polynomials of second kind is given by

$$\sum_{n=0}^{+\infty} GP_n(n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) z^n = \frac{i+2x(1-2i)z+(5i-2)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

Proof. We know that

$$GP_n = iS_n(a_1 + [-a_2]) + (1-2i)S_{n-1}(a_1 + [-a_2]),$$

see [19]. We see that

$$\begin{aligned}
& \sum_{n=0}^{+\infty} GP_n(n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) z^n \\
= & \sum_{n=0}^{+\infty} (iS_n(a_1 + [-a_2]) + (1-2i)S_{n-1}(a_1 + [-a_2])) \times S_n(2e_1 + [-2e_2]) z^n
\end{aligned}$$

$$\begin{aligned}
&= i \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad + (1 - 2i) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&= \frac{i + iz^2}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4} + \frac{2x(1 - 2i)z - 2(1 - 2i)z^2}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4} \\
&= \frac{i + 2x(1 - 2i)z + (5i - 2)z^2}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4}.
\end{aligned}$$

This completes the proof.

Theorem 3.10 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell numbers with Chebyshev polynomials of first kind is given by

$$\sum_{n=0}^{+\infty} GP_{n2} F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n = \frac{i + (x - 4xi)z + (i(5 - 2x^2) - 2)z^2 + (2i - 1)xz^3}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4}.$$

Proof. We know that

$$GP_n = iS_n(a_1 + [-a_2]) + (1 - 2i)S_{n-1}(a_1 + [-a_2]) \quad [19].$$

We see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} GP_n {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n &= \sum_{n=0}^{+\infty} (iS_n(a_1 + [-a_2]) + (1 - 2i)S_{n-1}(a_1 + [-a_2])) \\
&\quad \times (S_n(2e_1 + [-2e_2]) - xS_{n-1}(2e_1 + [-2e_2])) z^n \\
&= i \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n - ix \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
&\quad + (1 - 2i) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad - (1 - 2i)x \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
&= \frac{i(1 + z^2)}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4} - \frac{ix(2z + 2xz^2)}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4} \\
&\quad + \frac{(1 - 2i)(2xz - 2z^2)}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4} - \frac{x(1 - 2i)(z + z^3)}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4} \\
&= \frac{i + (x - 4xi)z + (i(5 - 2x^2) - 2)z^2 + (2i - 1)xz^3}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4}.
\end{aligned}$$

This completes the proof.

Theorem 3.11 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell-Lucas numbers with Chebyshev polynomials of second kind is given by

$$\sum_{n=0}^{+\infty} GQ_n (n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) z^n = \frac{2-2i+2x(6i-2)z+(6-14i)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}.$$

Proof. We know that

$$GQ_n = (2-2i)S_n(a_1+[-a_2]) + (6i-2)S_{n-1}(a_1+[-a_2]),$$

see [19]. We see that

$$\begin{aligned} \sum_{n=0}^{+\infty} GQ_n (n+1)_2 F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) z^n &= \sum_{n=0}^{+\infty} ((2-2i)S_n(a_1+[-a_2]) \\ &\quad + (6i-2)S_{n-1}(a_1+[-a_2])) S_n(2e_1+[-2e_2]) z^n \\ &= (2-2i) \sum_{n=0}^{+\infty} S_n(a_1+[-a_2]) S_n(2e_1+[-2e_2]) z^n \\ &\quad + (6i-2) \sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2]) S_n(2e_1+[-2e_2]) z^n \\ &= \frac{2-2i+(2-2i)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4} \\ &\quad + \frac{2x(6i-2)z-2(6i-2)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4} \\ &= \frac{2-2i+2x(6i-2)z+(6-14i)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}. \end{aligned}$$

This completes the proof.

Theorem 3.12 For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell-Lucas numbers with Chebyshev polynomials of first kind is given by

$$\begin{aligned} \sum_{n=0}^{+\infty} GQ_n {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n \\ = \frac{2(1-i)+x(10i-6)z+(6-4x^2+i(4x^2-14))z^2+x(2-6i)z^3}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}. \end{aligned}$$

Proof. We know that

$$GQ_n = (2-2i)S_n(a_1+[-a_2]) + (6i-2)S_{n-1}(a_1+[-a_2]),$$

see [19]. We see that

$$\begin{aligned} \sum_{n=0}^{+\infty} GQ_n {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) z^n &= \sum_{n=0}^{+\infty} ((2-2i)S_n(a_1+[-a_2]) + (6i-2)S_{n-1}(a_1+[-a_2])) \\ &\quad \times (S_n(2e_1+[-2e_2]) - xS_{n-1}(2e_1+[-2e_2])) z^n \end{aligned}$$

$$\begin{aligned}
&= (2 - 2i) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad - x(2 - 2i) \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_{n-1}(a_1 + [-a_2]) z^n \\
&\quad + (6i - 2) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad - x(6i - 2) \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
&= \frac{(2 - 2i)(1 + z^2)}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4} - \frac{x(2 - 2i)(2z + 2xz^2)}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4} \\
&\quad + \frac{(6i - 2)(2xz - 2z^2)}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4} - \frac{x(6i - 2)(z + z^3)}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4} \\
&= \frac{2(1 - i) + x(10i - 6)z + (6 - 4x^2 + i(4x^2 - 14))z^2 + x(2 - 6i)z^3}{1 - 4xz + (6 - 4x^2)z^2 + 4xz^3 + z^4}.
\end{aligned}$$

This completes the proof.

4 Conclusion

In this paper, by making use of Eqs. (3.1) and (3.2), we have derived some new generating functions of the products of Gaussian Fibonacci numbers, Gaussian Lucas numbers, Gaussian Jacobsthal numbers, Gaussian Jacobsthal Lucas numbers, Gaussian Pell numbers and Gaussian Pell Lucas numbers with Chebyshev polynomials of first and second kinds. The derived theorems and corollaries are based on symmetric functions and products of these numbers and polynomials.

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