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Characterizations of Lipschitz functions via commutators generated by parametric Marcinkiewicz integral on generalized Orlicz-Morrey spaces

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Abstract. *In this paper, we give some new characterizations of the Lipschitz spaces via the Spanne-Guliyev and Adams-Guliyev type boundedness of the commutators associated with the parametric Marcinkiewicz integral on generalized Orlicz-Morrey spaces.*

Keywords. Parametric Marcinkiewicz integrals, generalized Orlicz-Morrey spaces, commutator, Lipschitz spaces

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1 Introduction

As a generalization of $L^p(\mathbb{R}^n)$, the Orlicz spaces were introduced by Birnbaum-Orlicz in [2] and Orlicz in [40], since then, the theory of the Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis.

The classical Morrey spaces were introduced by Morrey [35] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [21,34,36] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [22,23,43]).

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [3,4] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let T be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [8] states that the commutator operator $[b,T]f = T(bf) - bTf$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [5–7, 10, 12, 28, 29]).

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Suppose that S^{n-1} be the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^1(S^{n-1})$ and the following property

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$.

The parametric Marcinkiewicz integral is defined by Hörmander [32] as follows.

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2},$$

where $0 < \rho < n$. When $\rho = 1$, we simply denote it by $\mu_\Omega(f)$. It is well-known that the operator $\mu_\Omega(f)$ is defined by Stein in [45].

Let b be a locally integrable function on $\mathbb{R}^n$; the commutator generated by the parametric Marcinkiewicz integral μ_Ω^ρ and b is defined by

$$\mu_{\Omega,b}^\rho(f)(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} (b(x) - b(y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2}.$$

In [13], Deringoz et al. introduced generalized Orlicz-Morrey spaces as an extension of generalized Morrey spaces. Other definitions of generalized Orlicz-Morrey spaces can be found in [37] and [42]. In words of [27], our generalized Orlicz-Morrey space is the third kind and the ones in [37] and [42] are the first kind and second kind, respectively. According to the examples in [20], one can say that the generalized Orlicz-Morrey space of first kind and second kind are different and that second kind and third kind are different. However, it is not known that relation between first and second kind.

Boundedness of commutators of classical operators of harmonic analysis on generalized Orlicz-Morrey spaces were recently studied in various papers, see for example [14, 24,25]. In [15,17], the authors consider the boundedness of the parametric Marcinkiewicz integral operator and its commutator on generalized Orlicz-Morrey space of the third kind, see also [1,9,31,38,39,44]. In this paper, we give some new characterizations of the Lipschitz spaces via the Spanne-Guliyev and Adams-Guliyev type boundedness of the commutators associated with the parametric Marcinkiewicz integral on generalized Orlicz-Morrey spaces.

Everywhere in the sequel $B(x, r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x, r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$. By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

We recall the definition of Young functions.

Definition 2.1 *A function* $\Phi : [0, \infty) \to [0, \infty]$ *is called a Young function if* Φ *is convex, left-continuous,* $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$ *and* $\lim_{r\to\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \leq r \leq \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \leq r < \infty.$$

It is well known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \qquad \text{for } r \geq 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq k\Phi(r) \text{ for } r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \qquad r \geq 0,$$

for some $k > 1$.

We will also use the numerical characteristics

$$a_\Phi := \inf_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \qquad b_\Phi := \sup_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

of Young functions.

Remark 2.1 It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < a_\Phi \leq b_\Phi < \infty$, see [30].

Definition 2.2 *(Orlicz Space). For a Young function Φ, the set*

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\rm loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \leq r \leq 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^\Phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. The space $L^\Phi_{\rm loc}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_B \in L^\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

$L^\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \leq 1\right\}.$$

By elementary calculations we have the following.

Lemma 2.1 *Let Φ be a Young function and B a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{L^\Phi} = \frac{1}{\Phi^{-1}(|B|^{-1})}.$$

In the next sections where we prove our main estimates, we use the following lemma.

Lemma 2.2 *[13] For a Young function Φ, the following inequality is valid*

$$\int_{B(x,r)} |f(y)|dy \le 2|B(x,r)|\Phi^{-1}\left(|B(x,r)|^{-1}\right)\|f\|_{L^\Phi(B(x,r))},$$

where $\|f\|_{L^\Phi(B(x,r))} = \|f\chi_B\|_{L^\Phi}$.

Various versions of generalized Orlicz-Morrey spaces were introduced in [37], [42] and [13]. We used the definition of [13] which runs as follows.

Definition 2.3 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. We denote by $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Orlicz-Morrey space, the space of all functions $f \in L^\Phi_{\rm loc}(\mathbb{R}^n)$ for which*

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\Phi^{-1}(|B(x,r)|^{-1})\|f\|_{L^\Phi(B(x,r))} < \infty.$$

3 Auxiliary Results

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto \frac{1}{\Phi^{-1}(t^{-n})}\varphi(t)$ is almost increasing.

Lemma 3.1 *[16] Let B be a ball in $\mathbb{R}^n$. If $\varphi \in \mathcal{G}_\Phi$, then there exist $C > 0$ such that*

$$\frac{1}{\varphi(r_B)} \le \|\chi_B\|_{\mathcal{M}^{\Phi,\varphi}} \le \frac{C}{\varphi(r_B)},$$

where r_B denotes the radius of the ball.

In the next sections where we prove our main estimates, we use the following results.

Theorem 3.1 *[16,23] (Adams-Guliyev type result)*
Let $0 < \alpha < n$, $\Phi \in \mathcal{Y}$, $\gamma \in (0,1)$ and $\eta(t) \equiv \varphi(t)^\gamma$ and $\Psi(t) \equiv \Phi(t^{1/\gamma})$.
1. *If $\Phi \in \nabla_2$ and $\varphi(t)$ satisfies*

$$\sup_{r<t<\infty} \Phi^{-1}\big(t^{-n}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi(s)}{\Phi^{-1}\big(s^{-n}\big)} \le C\,\varphi(r), \tag{3.1}$$

then the condition

$$t^\alpha \varphi(t) + \int_t^\infty r^\alpha\,\varphi(r)\frac{dr}{r} \le C\varphi(t)^\gamma$$

for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

2. *If $\varphi \in \mathcal{G}_\Phi$, then the condition*

$$t^{\alpha}\varphi(t) \le C\varphi(t)^{\gamma} \tag{3.2}$$

for all $t > 0$, where $C > 0$ does not depend on t, is necessary for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

3. *Let $\Phi \in \nabla_2$. If $\varphi \in \mathcal{G}_\Phi$ satisfies the regularity condition*

$$\int_t^{\infty} r^{\alpha}\, \varphi(r)\frac{dr}{r} \le Ct^{\alpha}\varphi(t)$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition (3.2) *is necessary and sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

Theorem 3.2 *[16,23,24] (Spanne-Guliyev type result)*
Let $\Phi, \Psi \in \mathcal{Y}$ and $0 < \alpha < n$.

1. *Let $\Phi \in \nabla_2$. If the functions (Φ, Ψ) satisfy the condition*

$$r^{\alpha}\Phi^{-1}\big(r^{-n}\big) + \int_r^{\infty} \Phi^{-1}\big(t^{-n}\big)t^{\alpha}\frac{dt}{t} \le C\Psi^{-1}\big(r^{-n}\big), \tag{3.3}$$

then the condition

$$\int_t^{1} \operatorname*{ess\,inf}_{r<s<1} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-n}\big)}\Psi^{-1}\big(r^{-n}\big)\frac{dr}{r} \le C\,\varphi_2(t) \tag{3.4}$$

for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. *If the function $\varphi_1 \in \mathcal{G}_\Phi$, then the condition*

$$t^{\alpha}\varphi_1(t) \le C\varphi_2(t) \tag{3.5}$$

for all $t > 0$, where $C > 0$ does not depend on t, is necessary for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

3. *Let $\Phi \in \nabla_2$. Let also the functions (Φ, Ψ) satisfy the condition* (3.3). *If $\varphi_1 \in \mathcal{G}_\Phi$ satisfies the regularity type condition*

$$\int_t^{\infty} \frac{\Psi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-n}\big)}\, \varphi_1(r)\frac{dr}{r} \le Ct^{\alpha}\varphi_1(t)$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition (3.5) *is necessary and sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.*

4 Main Results

In this section, as an application of theorems of the previous section we consider the boundedness of $\mu^{\rho}_{\Omega,b}$ on generalized Orlicz-Morrey spaces when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given. Such a characterization was given in [26] for the boundedness of $[b, I_\alpha]$ and $[b, T]$ on generalized Orlicz-Morrey spaces.

We recall the definition of Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$.

Definition 4.1 *Let $0 < \beta \le 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,*

$$|b(x) - b(y)| \le C|x - y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta(\mathbb{R}^n)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}$.

To prove the theorems, we need auxiliary results. The first one is the following characterizations of Lipschitz space, which is due to DeVore and Sharply [18].

Lemma 4.1 *Let $0 < \beta \le 1$, we have*

$$\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \approx \sup_B \frac{1}{|B|^{1+\beta/n}} \int_B |f(x) - f_B| dx.$$

Lemma 4.2 *Let $0 < \beta \le 1$ and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then the following pointwise estimate holds:*

$$\mu^\rho_{\Omega,b}(f)(x) \lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} I_\beta(|f|)(x).$$

Proof.

$$\begin{aligned}
\mu^\rho_{\Omega,b}(f)(x) &\le \Big(\int_0^\infty \Big(\int_{|x-y|\le t} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |b(x) - b(y)| \, |f(y)| dy \Big)^2 \frac{dt}{t^{2\rho+1}} \Big)^{1/2} \\
&\le \int_{\mathbb{R}^n} \frac{|\Omega((x-y)')|}{|x-y|^{n-\rho}} |b(x) - b(y)| \, |f(y)| \Big(\int_{|x-y|}^\infty \frac{dt}{t^{2\rho+1}} \Big)^{1/2} dy \\
&\le C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\rho}} \frac{|x-y|^\beta}{|x-y|^\rho} dy \\
&\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} I_\beta(|f|)(x).
\end{aligned}$$

The following result concerning the boundedness of commutator of parametric Marcinkiewicz integral operator $\mu^\rho_{\Omega,b}$ on L^p spaces.

Theorem 4.1 *Let Ω is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^\infty(S^{n-1})$, $0 < \rho < n$, $0 < \beta \le 1$, $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, $1 < p < q < \infty$ and $1/p - 1/q = \beta/n$. Then, there is a constant C independent of f such that*

$$\|\mu^\rho_{\Omega,b}(f)\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

The following interpolation result is from [11, Theorem 2.2].

Lemma 4.3 *Let $\alpha \in [0, 1)$, $p_i, q_i \in (0, \infty)$ satisfy $1/q_i = 1/p_i - \alpha$ for $i \in \{1, 2\}$, $p_1 < p_2$ and T be a sublinear operator of weak type (p_i, q_i) for $i \in \{1, 2\}$. Then T is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$, where Φ and Ψ are Young function satisfying that $1 < p_1 < a_\Phi \le b_\Phi < p_2 < \infty$, $1 < q_1 < a_\Psi \le b_\Psi < q_2 < \infty$ and, for all $t \in (0, \infty)$, $\Psi^{-1}(t) = \Phi^{-1}(t)\, t^{-\alpha}$.*

As a consequence of Lemma 4.3 and Theorem 4.1, we get the following result.

Corollary 4.1 *Let Ω is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^\infty(S^{n-1})$, $0 < \beta \le 1$, $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, Φ, Ψ be a Young function and $0 < \rho < n$. If $\Phi, \Psi \in \Delta_2 \cap \nabla_2$ and $\Psi^{-1}(t^{-n}) = \Phi^{-1}(t^{-n})\, r^\beta$, then $\mu^\rho_{\Omega,b}$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

The following Adams-Guliyev type boundedness of the commutators associated with the parametric Marcinkiewicz integral on generalized Orlicz-Morrey spaces is valid.

Theorem 4.2 *(Adams-Guliyev type result) Let Ω is a homogeneous function of degree zero on $\mathbb{R}^n$, $0<\beta\le 1$, $0<\gamma<1$, $b\in L^1_{\rm loc}(\mathbb{R}^n)$, $\Phi\in\mathcal{Y}$, $\eta(t)\equiv\varphi(t)^\gamma$ and $\Psi(t)\equiv\Phi(t^{1/\gamma})$.*

1. *If $\Omega\in L^\infty(S^{n-1})$, $\Phi\in\nabla_2$ and $\varphi(t)$ satisfies* (3.1) *and*

$$\int_t^\infty r^\beta\,\varphi(r)\frac{dr}{r}\le Ct^\beta\varphi(t), \tag{4.1}$$

$$t^\beta\varphi(t)\le C\varphi(t)^\gamma$$

hold for all $t>0$, where $C>0$ does not depend on t, then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $\mu^\rho_{\Omega,b}$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

2. *If $\varphi\in\mathcal{G}_\Phi$, Ω satisfy*

$$|\Omega(x')-\Omega(y')|\lesssim\big(\log(2/|x'-y'|)\big)^{-\mu},\mu>1,x',y'\in S^{n-1} \tag{4.2}$$

and the condition

$$\varphi(t)^\gamma\le t^\beta\varphi(t)$$

holds for all $t>0$, where $C>0$ does not depend on t, then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $\mu^\rho_{\Omega,b}$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

3. *If Ω satisfy condition* (4.2)*, $\Phi\in\nabla_2$, $\varphi\in\mathcal{G}_\Phi$, condition* (4.1) *holds and $\varphi(t)^\gamma\approx t^\beta\varphi(t)$, then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $\mu^\rho_{\Omega,b}$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

Proof. (1) The first statement of the theorem follows from Theorem 3.1 and Lemma 4.2.

(2) We shall now prove the second part. We use the idea given in [33] (see also [19,26, 41]). Choose $z_0\in\mathbb{R}^n$ and $\delta>0$ such that in the neighborhood $\{z:|z-z_0|<\sqrt{n}\delta\}$, function $|z|^{n-\beta}$ can be represented as a Fourier series which absolutely converges. That is

$$|z|^{n-\beta}=\sum_{k=0}^\infty a_k e^{i\nu_k\cdot z}.$$

Let $z_1=\frac{z_0}{\delta}$. For any ball $B=B(x_0,r)$, let $y_0=x_0-2rz_1$ and $B'=B(y_0,r)$. Then for $x\in B$ and $y\in B'$, we have that

$$\left|\frac{x-y}{2r}-z_1\right|\le\left|\frac{x-x_0}{2r}\right|+\left|\frac{y-y_0}{2r}\right|\le 1.$$

Since Ω is homogeneous function of degree zero, and satisfies (4.2), then there exists a positive constant A with $0<A<1$, for $x,y\in\mathbb{R}^n$, $x\ne y$,

$$\Omega(x-y)=\Omega((x-y)')\ge C\big(\log(2/A)\big)^{-\mu}. \tag{4.3}$$

Now set $s(x)=[sgn(b(x)-b_{B'})]\chi_B(x)$, then

$$\int_B|b(x)-b_{B'}|dx=\int_B(b(x)-b_{B'})s(x)dx=\frac{1}{|B'|}\int_B\int_{B'}(b(x)-b(y))s(x)dydx$$
$$\approx\delta^{\beta-n}\,r^{-\beta}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\beta}}\left|\frac{\delta(x-y)}{2r}\right|^{n-\beta}s(x)\chi_B(x)\chi_{B'}(y)dydx$$
$$\approx r^{-\beta}\sum_{k=0}^\infty a_k\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\beta}}e^{i\nu_k\cdot\frac{\delta}{2r}(x-y)}s(x)\chi_B(x)\chi_{B'}(y)dydx.$$

Taking

$$g_k(y) = e^{-i(\delta/2r)\nu_k\cdot y}\chi_{B'}(y) \ \text{ and } \ h_k(x) = e^{i(\delta/2r)\nu_k\cdot x}s(x)\chi_B(x),$$

applying the Minkowski inequality, Hölder inequality and (4.3), we obtain

$$\int_B |b(x) - b_{B'}|dx \approx r^{-\beta}\sum_{k=0}^{\infty} a_k \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^{n-\beta}}\, g_k(y)h_k(x)dydx$$

$$\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_{\mathbb{R}^n}\big|[b,I_\beta]g_k(x)\big|\,|h_k(x)|dx$$

$$= Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B\Big|\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\beta}}\, g_k(y)dy\Big|dx$$

$$\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B\frac{dx}{|x-y_0|^{\rho-\beta}}\Big|\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\rho}}\, g_k(y)dy\Big|$$

$$\le Cr^{-\beta}\frac{(\log(2/A))^{\mu}}{(\log(2/A))^{\mu}}\sum_{k=0}^{\infty}|a_k|\int_B\frac{dx}{|x-y_0|^{\rho-\beta}}\Big|\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\rho}}\, g_k(y)dy\Big|$$

$$\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B\frac{dx}{|x-y_0|^{\rho-\beta}}\Big|\frac{1}{(\log(2/A))^{\mu}}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\rho}}\, g_k(y)dy\Big|$$

$$\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B\frac{dx}{|x-y_0|^{\rho-\beta}}\Big|\int_{\mathbb{R}^n}\frac{\Omega((x-y)')}{|x-y|^{n-\rho}}\,(b(x)-b(y))\, g_k(y)dy\Big|$$

$$\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B|x-y_0|^{\beta}dx\Big|\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}\,(b(x)-b(y))\, g_k(y)dy\Big|$$

$$\times\Big(\int_{|x-y_0|\le t,|x-y|\le t}\frac{dt}{t^{2\rho+1}}\Big)\Big(\int_{|x-y_0|\le t,|x-y|\le t}\frac{dt}{t^{2\rho+1}}\Big)^{-\frac{1}{2}}$$

$$\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B|x-y_0|^{\beta}dx\Big(\int_{|x-y_0|}^{\infty}\Big|\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}\,(b(x)-b(y))\, g_k(y)\chi_{B(x,t)}(y)dy\Big|$$

$$\times\frac{dt}{t^{2\rho+1}}\Big)\Big(\int_{|x-y_0|\le t,|x-y|\le t}\frac{dt}{t^{2\rho+1}}\Big)^{-\frac{1}{2}}$$

$$\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B|x-y_0|^{\beta}dx\Big(\int_{|x-y_0|}^{\infty}\Big|\int_{B(x,t)}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}\,(b(x)-b(y))\, g_k(y)dy\Big|^2\frac{dt}{t^{2\rho+1}}\Big)^{\frac{1}{2}}$$

$$\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B|x-y_0|^{\beta}\mu^{\rho}_{\Omega,b}(g_k)(x)dx.$$

If $y_0\in B=B(x_0,r)$, then $|x-y_0|<r$ and applying Lemma 2.2, we have

$$\int_B|b(x)-b_{B'}|dx \lesssim r^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B|x-y_0|^{\beta}\mu^{\rho}_{\Omega,b}(g_k)(x)dx$$

$$\lesssim\sum_{k=0}^{\infty}|a_k|\int_B\mu^{\rho}_{\Omega,b}g_k(x)dx\lesssim\sum_{k=0}^{\infty}|a_k|\,|B|\Psi^{-1}\left(|B|^{-1}\right)\,\|\mu^{\rho}_{\Omega,b}g_k\|_{L^{\Psi}(B)}$$

$$\lesssim \sum_{k=0}^{\infty} |a_k|\,|B|\,\eta(r)\,\|\mu_{\Omega,b}^{\rho} g_k\|_{\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty} |a_k|\,|B|\,\eta(r)\,\|g_k\|_{\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)}$$
$$\lesssim \sum_{k=0}^{\infty} |a_k|\,|B|\,\varphi(r)^{\gamma-1} \leq \sum_{k=0}^{\infty} |a_k|\,r^{n+\beta},$$

since $\|g_k\|_{\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)} \lesssim \varphi(r)^{-1}$ by Lemma 3.1.

Thus we have obtained

$$\frac{1}{|B|^{1+\frac{\beta}{n}}}\int_B |b(x)-c_0|dx \leq \frac{2}{|B|^{1+\frac{\beta}{n}}}\int_B |b(x)-b_{B'}|dx \lesssim \sum_{k=0}^{\infty} |a_k| \lesssim 1.$$

If $y_0 \notin B = B(x_0,r)$ and $|x_0-y_0| \leq 2r$, then with an argument that used in the case $y_0 \in B = B(x_0,r)$, it is not difficult to obtain that

$$\frac{1}{|B|^{1+\frac{\beta}{n}}}\int_B |b(x)-c_0|dx \leq \frac{2}{|B|^{1+\frac{\beta}{n}}}\int_B |b(x)-b_{B'}|dx \lesssim \sum_{k=0}^{\infty} |a_k| \lesssim 1.$$

If $y_0 \notin B = B(x_0,r)$ and $|x_0-y_0| > 2r$. Since $y_0 = x_0 - 2rz_1$ and $z_1 = \frac{z_0}{\delta}$, then $\frac{|x_0-y_0|}{2r} = |z_1| = \frac{|z_0|}{\delta} < \sqrt{n}$, that is, $|x_0-y_0| \leq 2\sqrt{n}r$. Thus

$$\int_B |b(x)-b_{B'}|dx \lesssim r^{-\beta}\sum_{k=0}^{\infty} |a_k| \int_B |x-y_0|^{\beta}\mu_{\Omega,b}^{\rho}(g_k)(x)dx$$
$$\lesssim r^{-\beta}\sum_{k=0}^{\infty} |a_k||x_0-y_0| \int_B \mu_{\Omega,b}^{\rho} g_k(x)dx \lesssim \sum_{k=0}^{\infty} |a_k|\,|B|\Psi^{-1}\left(|B|^{-1}\right)\,\|\mu_{\Omega,b}^{\rho} g_k\|_{L^{\Psi}(B)}$$
$$\lesssim \sum_{k=0}^{\infty} |a_k|\,|B|\,\eta(r)\,\|\mu_{\Omega,b}^{\rho} g_k\|_{\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty} |a_k|\,|B|\,\eta(r)\,\|g_k\|_{\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)}$$
$$\lesssim \sum_{k=0}^{\infty} |a_k|\,|B|\,\varphi(r)^{\gamma-1} \leq \sum_{k=0}^{\infty} |a_k|\,r^{n+\beta}.$$

Thus, we get

$$\frac{1}{|B|^{1+\frac{\beta}{n}}}\int_B |b(x)-c_0|dx \leq \frac{2}{|B|^{1+\frac{\beta}{n}}}\int_B |b(x)-b_{B'}|dx \lesssim \sum_{k=0}^{\infty} |a_k| \lesssim 1,$$

which completes the proof of second parts of the theorem.

(3) The third statement of the theorem follows from the first and second parts of the theorem.

Similar to the reasoning used in the proof of Theorem 4.2, one can also obtain the following Spanne-Guliyev type boundedness of the commutators associated with the parametric Marcinkiewicz integral on generalized Orlicz-Morrey spaces.

Theorem 4.3 *(Spanne-Guliyev type result) Let Ω is a homogeneous function of degree zero on $\mathbb{R}^n$, $0<\beta\leq 1$, $b \in L^1_{\rm loc}(\mathbb{R}^n)$, $\Phi, \Psi \in \mathcal{Y}$.*

1. *If $\Omega \in L^{\infty}(S^{n-1})$, $\Phi \in \nabla_2$, (Φ,Ψ) satisfy the condition* (3.4) *and*

$$r^{\beta}\Phi^{-1}\big(r^{-n}\big) + \int_r^{\infty} \Phi^{-1}\big(t^{-n}\big)t^{\beta}\frac{dt}{t} \leq C\Psi^{-1}\big(r^{-n}\big), \tag{4.4}$$

hold for all $t > 0$, where $C > 0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $\mu^\rho_{\Omega,b}$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. *If $\varphi \in \mathcal{G}_\Phi$, Ω satisfy* (4.2) *and the condition*

$$\varphi_2(t) \le Ct^\beta \varphi_1(t)$$

holds for all $t > 0$, where $C > 0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $\mu^\rho_{\Omega,b}$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

3. *Let Ω satisfy condition* (4.2), *$\Phi \in \nabla_2$, (Φ, Ψ) satisfy the condition* (4.4) *and $\varphi_2(t) \approx t^\beta \varphi_1(t)$. If $\varphi_1 \in \mathcal{G}_\Phi$ satisfies the regularity type condition*

$$\int_t^\infty \frac{\Psi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-n}\big)} \varphi_1(r) \frac{dr}{r} \le Ct^\beta \varphi_1(t)$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $\mu^\rho_{\Omega,b}$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

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