Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **41** (1), 110-117 (2021).

Global bifurcation of positive and negative solutions from infinity in some quasi-linear elliptic problems with sign-chining weight

Shanay M. Hasanova*

Received: 09.09.2020 / Revised: 15.02.2021/ Accepted: 08.03.2021

Abstract. *In this paper we consider the nonlinear eigenvalue problem for elliptic partial differential equations with a sign-chaining weight function and study the global bifurcation from infinity of solutions to this problem. We prove the existence of two pairs of global continua emanating from the asymptotic bifurcation points corresponding to the principal eigenvalues of the linearized problem and contained in the classes of positive and negative functions.*

Keywords. elliptic partial differential equations, indefinite weight, principal eigenvalue, bifurcation from infinity, global continua

Mathematics Subject Classification (2010): 35J15, 35J25, 35J66, 35P05, 35P30, 47J10, 47J15.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N > 1$, with a smooth boundary $\partial\Omega$. We consider the following nonlinear eigenvalue problem

$$\begin{array}{l} Lu \equiv -\sum\limits_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + c(x)u = \lambda a(x)u + g(x,u,\nabla u,\,\lambda), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{array} \tag{1.1}$$

where λ is a real parameter, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}\right)$. We suppose that $a_{ij}(x) \in C^1\left(\overline{\Omega};\mathbb{R}\right)$, $i,j = 1,\,2,\,\ldots,N$, $c\,(x) \in C\left(\overline{\Omega};\mathbb{R}\right)$ and $c\,(x) \geq 0$, $x \in \overline{\Omega}$, $a(x) \in C\left(\overline{\Omega};\mathbb{R}\right)$ and takes both positive and negative values in Ω, and L is uniformly elliptic on $\overline{\Omega}$. The function $g \in C\left(\overline{\Omega} \times \mathbb{R}^{N+2};\mathbb{R}\right)$ satisfies the following conditions:

$$u\,g(x,u,v,\lambda) \leq 0 \ \text{ for any } \ (x,u,v,\lambda) \in \Omega \times \mathbb{R}^{N+2} \times \mathbb{R}; \tag{1.2}$$

$$g(x,u,v,\lambda) = o(|u| + |v|) \ \text{ as } \ |u| + |v| \to +\infty, \tag{1.3}$$

* Corresponding author

Sh.M. Hasanova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
E-mail: hshenay@mail.ru

uniformly in $x \in \Omega$ and in $\lambda \in \Lambda$, for every bounded interval $\Lambda \subset \mathbb{R}$.

Fundamental results on local and global bifurcation of solutions from zero and at infinity to nonlinear eigenvalue problems were obtained by M.A. Krasnoselskii [16], P.H. Rabinowitz [20-22], E.N. Dancer [11], A.P. Makhmudov and Z.S. Aliev [19], J. Lopez-Gomez [18], R.E Gaines and J.L. Mawhin [13] and others. By applying these results, the structure of continua of solutions of nonlinear eigenvalue problems for ordinary and partial differential equations of the second and fourth orders branching from bifurcation points and intervals of the lines $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{\infty\}$ has been investigated by many authors (see [3-9, 12, 15, 19-24] and references therein).

It is known that in the case $a(x) > 0,\ x \in \overline{\Omega}$, the linear eigenvalue problem which obtained from (1.1) by setting $g \equiv 0$ has unique simple principal eigenvalue λ_1 (the principal eigenvalue is the value $\lambda \in \mathbb{R}$ for which the linear problem (1.1) with $g \equiv 0$ admits a solution u such that $u(x) \neq 0$ for $x \in \Omega$) (see [17]). In [22] Rabinowitz proves the existence of global continua $\mathfrak{L}^+_{\lambda_1}$ and $\mathfrak{L}^-_{\lambda_1}$ in $\mathbb{R} \times C^{1,\alpha}$, $\alpha \in (0,1)$, (see Section 2) bifurcating from the point (λ_1, ∞) such that their intersection in some neighborhood of this point contains only positive and negative solutions components, respectively. Moreover, outside the neighborhood of the point (λ_1, ∞), each of these sets is either bounded, in this case it meets the line of trivial solutions, or meets another asymptotic bifurcation point, or has an unbounded projection onto the line of trivial solutions.

If a is a sign-chaining weight function, then the linear problem (1.1) with $g \equiv 0$ has two simple positive and negative principal eigenvalues $\lambda_1^+ > 0$ and $\lambda_1^- < 0$, respectively (see [5, 10]). Note that in the case when the function g satisfies the $o(|u| + |\nabla u|)$ condition problem (1.1) was considered in [6] where it shown that there exist two pairs of unbounded continua bifurcating from points $(\lambda_1^+, 0)$ and $(\lambda_1^-, 0)$, and contained in the classes of positive and negative solutions, respectively. The purpose of this paper is to study the structure of global continua of solutions to problem (1.1) under conditions (1.2) and (1.3) branching from asymptotic bifurcation points corresponding to the principal eigenvalues of the linear problem (1.1) with $g \equiv 0$. We will prove the existence of two pairs of global continua $\mathcal{C}^+_{\lambda_1^+}, \mathcal{C}^-_{\lambda_1^+}$ and $\mathcal{C}^+_{\lambda_1^-}, \mathcal{C}^-_{\lambda_1^-} \subset \mathbb{R} \times \mathcal{C}^{1,\alpha}$ with properties that are deeper than the above properties obtained by Rabinowitz [22].

2 Preliminary and and the reduction of problem (1.1) to the operator equation

Let $E = \{u \in C^{1,\alpha}\left(\overline{\Omega}\right) : u = 0 \text{ on } \partial\Omega\}$ be the Banach space with the usual norm $||\cdot||_{1,\alpha} = ||\cdot||_{C^{1,\alpha}}$. A pair (λ, u) is called a solution to problem (1.1) if $u \in W^{2,p}(\Omega)$ and (λ, u) satisfies (1.1), where $\alpha \in (0,1)$ is given and p is a real number such that $p > N$ and $\alpha < 1 - N/p$. For this choice $W^{2,p}(\Omega)$ is compactly embedded in $C^{1,\alpha}(\overline{\Omega})$ (see [1, 14]), and consequently, any solution of (1.1) belongs to $\mathbb{R} \times E$. Therefore, we will study the structure of the solutions set of (1.1) in $\mathbb{R} \times E$ (the norm in $\mathbb{R} \times E$ is defined as $||(\lambda, u)|| = \left\{|\lambda|^2 + ||u||^2_{1,\alpha}\right\}^{\frac{1}{2}}$).

By σ (ν) we denote an element of $\{+\,,\,-\}$ that is, either $\sigma = +$ ($\nu = +$ respectively) or $\sigma = -$ ($\nu = -$ respectively).

For each σ and each ν let

$$\mathcal{P}^\nu_\sigma = \{u \in E : \sigma \int\limits_\Omega a u^2 dx > 0,\ \nu\, u > 0 \ \text{ in } \ \Omega, \ \text{and} \ \nu \frac{\partial u}{\partial n} < 0 \ \text{ on } \partial\Omega\},$$

where $\frac{\partial u}{\partial n}$ is the outward normal derivative of u on $\partial\Omega$. From the definition of the sets $\mathcal{P}_\sigma^+$, $\mathcal{P}_\sigma^-$ and $\mathcal{P}_\sigma = \mathcal{P}_\sigma^+ \cup \mathcal{P}_\sigma^-$ it is seen that for each σ they are open disjoint subsets in E. Moreover,

$$\partial\mathcal{P}_\sigma^\nu = \{u \in E : \text{ either } \sigma \int_\Omega au^2 dx = 0, \text{ or } \exists\, x_0 \in \Omega,\, u(x_0) = 0 \text{ or } \exists\, x_1 \in \partial\Omega,\, \tfrac{\partial u(x_1)}{\partial n} = 0\}.$$

Remark 2.1 The eigenfunctions $u_1^+(x)$ and $u_1^-(x)$, $x \in \overline{\Omega}$, corresponding to the positive and negative principal eigenvalues $\lambda_1^+(x)$ and $\lambda_1^-(x)$, respectively, of the linear eigenvalue problem

$$\begin{aligned} &Lu(x) = \lambda\, a(x)\, u(x), \text{ in } \Omega, \\ &u(x) = 0, \text{ on } \partial\Omega, \end{aligned} \tag{2.1}$$

can be chose so that $u_1^+(x) > 0$ and $u_1^-(x) > 0$ for all $x \in \Omega$, $\frac{\partial u_1^+(x)}{\partial n} < 0$ and $\frac{\partial u_1^-(x)}{\partial n} < 0$ for all $x \in \partial\Omega$ (see [5, Lemmas 2.1-2.4, Theorem 2.1 and Remark 2.1]). Then it follows that for each σ we have $u_1^\sigma \in \mathcal{P}_\sigma^+$. It should be noted that u_1^σ is made unique by taking $||u_1^\sigma||_{1,\alpha} = 1$.

Let $(\lambda, u) \in \mathbb{R} \times E$. We consider the following inhomogeneous linear problem

$$\begin{aligned} &Lv = \lambda a(x)u + g(x, u, \nabla u, \lambda), \quad x \in \Omega, \\ &v = 0, \quad x \in \partial\Omega, \end{aligned} \tag{2.2}$$

Note that $\lambda = 0$ is not an eigenvalue of problem (2.1) and right hand side of problem (2.2) lies in L^p. Then it follows from L^p theory for uniformly elliptic partial differential equations [1, 2] that there exists a unique $v = \mathcal{G}(\lambda, u)$ which satisfies (2.2). Since E is compactly embedded in $W^{2,p}(\Omega)$ by (2.2) we have

$$||v||_{1,\alpha} \leq \text{ const } ||v||_{2,p} \leq \text{ const } ||\lambda a u + g(\cdot, u, \nabla u, \lambda)||_p, \tag{2.3}$$

where $||\cdot||_{2,p}$ and $||\cdot||_p$ are norms in $W^{2,p}$ and L^p, respectively. Then by the Arzela-Ascoli theorem it follows from (2.3) that $\mathcal{G}$ is compact on $\mathbb{R} \times E$.

By $w = \mathcal{L}u \in W^{2,p}(\Omega)$ we denote the solution of the following linear problem

$$\begin{aligned} &Lw = a(x)u, \quad x \in \Omega, \\ &w = 0, \quad x \in \partial\Omega. \end{aligned} \tag{2.4}$$

Following the above reasoning, we conclude that $\mathcal{L}$ is a compact linear operator on E. Note that according to Remark 2.1 the operator $\mathcal{L}$ has simple positive and negative principal characteristic values λ_1^+ and λ_1^-, respectively. Moreover, these characteristic values correspond to the unique eigenfunctions u_1^+ and u_1^- such that $u_1^\sigma \in \mathcal{P}_\sigma^+$ and $||u_1^\sigma||_{1,\alpha} = 1$ for each σ.

Let now $\mathcal{K}(\lambda, u) = \mathcal{G}(\lambda, u) - \mathcal{L}u$. Then the problem (1.1) is equivalent to the following nonlinear eigenvalue problem

$$u = \lambda\mathcal{L}u + \mathcal{K}(\lambda, u). \tag{2.5}$$

Lemma 2.1 *The following relation holds:*

$$K(\lambda, u) = o(||u||_{1,\alpha}) \text{ as } ||u||_{1,\alpha} \to +\infty, \tag{2.6}$$

uniformly in $\lambda \in \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$.

Proof. By definition of the operator $\mathcal{K}$, continuity of the function g, and by the L^p estimates from (2.2) we get

$$\begin{aligned}||\mathcal{K}(\lambda,u)||_{1,\alpha}\leq&\ \text{const}\ ||\mathcal{K}(\lambda,u)||_{2,\,p}\\ \leq \text{const}\ ||g(\cdot,u,\nabla u,\lambda)||_p\leq&\ \text{const}\ \max_{x\in\overline{\Omega}}|g(x,u(x),\nabla u(x),\lambda)|.\end{aligned}\tag{2.7}$$

In view of condition (1.3) for the any small $\varepsilon>0$ there exists a positive number δ_ε such that for any $(u,s)\in\mathbb{R}\times\mathbb{R}^N$ satisfying the condition $|u|+|s|>\delta_\varepsilon$ the following relation holds:

$$|g(x,u,s,\lambda)|<\varepsilon(|u|+|s|)/2\tag{2.8}$$

uniformly in $x\in\overline{\Omega}$ and in $\lambda\in\Lambda$, for every fixed bounded interval $\Lambda\subset\mathbb{R}$.

On the other hand by the condition $g\in C\left(\overline{\Omega}\times\mathbb{R}^{N+2}\right)$ it follows that there exists positive constant A_ε such that

$$|g(x,u,s,\lambda)|\leq A_\varepsilon,\ \ (x,u,s,\lambda)\in\overline{\Omega}\times B_{\delta_\varepsilon}\times\Lambda,\tag{2.9}$$

where $B_{\delta_\varepsilon}=\{(u,s)\in\mathbb{R}\times\mathbb{R}^N:|u|+|s|\leq\delta_\varepsilon\}$.

Let us choose the number $\Delta_\varepsilon>\delta_\varepsilon$ large enough to satisfy the inequality

$$\frac{A_\varepsilon}{\Delta_\varepsilon}<\frac{\varepsilon}{2}.\tag{2.10}$$

Now we define the set S_{Δ_ε} as follows:

$$S_{\Delta_\varepsilon}=\{u\in E:||u||_{1,\,\alpha}\geq\Delta_\varepsilon\}.\tag{2.11}$$

Then by (2.8)-(2.11) for any $(\lambda,u)\in\Lambda\times S_{\Delta_\varepsilon}$ we have

$$\begin{aligned}\max_{x\in\overline{\Omega}}|g(x,u(x),\nabla u(x),\lambda)|\leq&\max_{\{x\in\overline{\Omega}\,:\,|u(x)|+|s(x)|\,\leq\,\delta_\varepsilon\}}|g(x,u(x),\nabla u(x),\lambda)|\\ +&\max_{\{x\in\overline{\Omega}\,:\,|u(x)|+|s(x)|\,>\,\delta_\varepsilon\}}|g(x,u(x),\nabla u(x),\lambda)|\leq A_\varepsilon+\tfrac{\varepsilon}{2}||u||_{1,\alpha}\\ <&\ \tfrac{\varepsilon}{2}\,\Delta_\varepsilon+\tfrac{\varepsilon}{2}||u||_{1,\alpha}<\varepsilon\,||u||_{1,\alpha},\end{aligned}\tag{2.12}$$

which, with regard to (2.7), we get (2.6). The proof of this lemma is complete.

3 Global bifurcation from infinity of solutions of problem (1.1)

By $\mathcal{C}$ we denote the set of solutions of problem (1.1). Let $(\lambda,u)\in\mathcal{C}$, $u\neq0$, and $v=\frac{u}{||u||^2_{1,\,\alpha}}$. Then we have $||v||_{1,\,\alpha}=\frac{1}{||u||_{1,\,\alpha}}$, and consequently, $u=||u||^2_{1,\,\alpha}v=\frac{v}{||v||^2_{1,\,\alpha}}$. Hence, following [22], dividing both sides of equality (2.5) by $||u||^2_{1,\,\alpha}$ we get

$$v=\lambda\mathcal{L}v+\frac{\mathcal{K}(\lambda,u)}{||u||^2_{1,\,\alpha}}=\lambda\mathcal{L}v+||v||^2_{1,\,\alpha}K\left(\lambda,\frac{v}{||v||^2_{1,\,\alpha}}\right).\tag{3.1}$$

Let $\mathcal{H} : \mathbb{R} \times E \to E$ be the continuous operator defined by

$$\mathcal{H}(\lambda, v) = \begin{cases} ||v||_{1,\alpha}^2 K\left(\lambda, \frac{v}{||v||_{1,\alpha}^2}\right) & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases} \tag{3.2}$$

Then by (3.2) Eq. (3.1) can be rewritten in the following form

$$v = \lambda \mathcal{L} v + \mathcal{H}(\lambda, v). \tag{3.3}$$

Remark 3.1 In view of (3.2) it follows from Lemma 2.1 (see (2.6)) that

$$\mathcal{H}(\lambda, u) = o(||v||_{1,\alpha}) \text{ as } ||v||_{1,\alpha} \to 0, \tag{3.4}$$

uniformly in $\lambda \in \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$.

To study the asymptotic bifurcation of solutions to problem (1.1), we need the following statement.

Lemma 3.1 *The operator $\mathcal{H} : \mathbb{R} \times E \to E$ is completely continuous.*

Proof. Since the operator $\mathcal{H}$ is continuous, it suffices to show that this operator is compact.

Let $\varepsilon > 0$ be fixed, and let $(\lambda, v) \in \Lambda \times \overline{\mathcal{B}}_{\frac{1}{\Delta_\varepsilon}}$, where $\mathcal{B}_{\frac{1}{\Delta_\varepsilon}}$ is a ball in E with center 0 and radius $\frac{1}{\Delta_\varepsilon}$, and $\overline{\mathcal{B}}_{\frac{1}{\Delta_\varepsilon}}$ is its closure. Since $u = \frac{v}{||v||_{1,\alpha}}$ it follows that $(\lambda, u) \in \Lambda \times \overline{\mathcal{S}}_{\Delta_\varepsilon}$. Hence by (2.7) and (2.12) we get

$$||\mathcal{K}(\lambda, u)||_{2,p} \leq \text{const } \varepsilon \, ||u||_{1,\alpha}.$$

Consequently, we have

$$||\mathcal{H}(\lambda, v)||_{2,p} = \frac{\|K(\lambda, u)\|_{2,p}}{||u||_{1,\alpha}^2} \leq \frac{\text{const } \varepsilon}{||u||_{1,\alpha}} \leq \frac{\text{const } \varepsilon}{\Delta_\varepsilon}. \tag{3.5}$$

Since E is compactly embedded in $W^{2,p}(\Omega)$ it follows from (3.5) that $H(\Lambda \times \overline{\mathcal{B}}_{\frac{1}{\Delta_\varepsilon}})$ is relatively compact. Then for any $0 < r \leq \rho < +\infty$ the set

$$\mathcal{H}(\Lambda \times \overline{\mathcal{B}}_\rho \backslash \mathcal{B}_r)$$

is relatively compact in E, which implies that $\mathcal{H}$ is compact. The proof of this lemma is complete.

For any $\lambda \in \mathbb{R}$, we say that a subset $\mathcal{V} \subset \mathcal{C}$ meets (λ, ∞) (respectively $(\lambda, 0)$) if there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset \mathcal{V}$ such that $\lambda_n \to \lambda$ and $||u_n||_{1,\alpha} \to +\infty$ (respectively $||u_n||_{1,\alpha} \to 0$) as $n \to +\infty$. Furthermore, we will say that $\mathcal{V} \subset \mathcal{C}$ meets (λ, ∞) (respectively $(\lambda, 0)$) with respect to the set $\mathbb{R} \times \mathcal{P}_\sigma^\nu$ (from now $\sigma, \nu \in \{+, -\}$), if the sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty$ can be chosen so that $u_n \in \mathcal{P}_\sigma^\nu$ for all $n \in \mathbb{N}$ (see [6, 8, 23]).

Let $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the intervals $(0, +\infty)$ and $(-\infty, 0)$ respectively.

Theorem 3.1 *For each σ and each ν there exists an unbounded continua $\mathcal{C}_\sigma^\nu$ of $\mathcal{C}$ containing $(\lambda_1^\sigma, \infty)$ for which the following hold:*

(i) *there exists a neighborhood $\mathcal{Q}_\sigma$ of $(\lambda_1^\sigma, \infty)$ such that*

$$\mathcal{C}_\sigma^\nu \subset \mathbb{R}^\sigma \times E, \;\; (\mathcal{C}_\sigma^\nu \cap Q_\sigma) \subset ((\mathbb{R}^\sigma \times S_\sigma^\nu) \cup \{(\lambda_1^\sigma, \infty)\}),$$

(ii) *either the set $\mathcal{C}_\sigma^\nu \setminus (\mathbb{R}^\sigma \times S_\sigma^\nu)$ meets $\mathbb{R}^\sigma \times \{\infty\}$ for some $\lambda \in \mathbb{R}$, or the set $\mathcal{C}_\sigma^\nu$ meets $\mathcal{R}^\sigma = \mathbb{R}^\sigma \times \{0\}$ for some $\lambda \in \mathbb{R}$, or the natural projection $Pr_{\mathcal{R}^\sigma}(\mathcal{C}_\sigma^\nu)$ of $\mathcal{C}_\sigma^\nu$ on $\mathcal{R}^\sigma$ is unbounded.*

Proof. In previous section, we reduced problem (1.1) to operator equation (2.5) in the space E. Next using the inversion

$$(\lambda, u) \to \left(\lambda, \frac{u}{||u||^2_{1,\alpha}}\right) = (\lambda, v) \tag{3.6}$$

we turned the bifurcation at infinity problem (2.5) into the bifurcation from zero problem (3.3) with completely continuous operators $\mathcal{L}$ and $\mathcal{H}$ (see Lemmas 2.1, 3.1 and Remark 3.1). However, to problem (3.3) we cannot immediately apply the global bifurcation result of [6] to obtain the desired result. This is due to the fact that if (λ, v) is a solution to problem (3.3) and $v \in \partial\mathcal{P}^\nu_\sigma$, then it does not follow that $u \equiv 0$ (see [6, Lemma 3.1 and Corollary 3.2]). Therefore, we will apply the standard global bifurcation theory which developed in [11] and [20] combining the results of [6].

Let $\mathfrak{C}$ be the closure of the set of nontrivial solutions to problem (3.3). Then by [20, Lemma 1.24 and Theorem 1.25; 11, Theorem 2; 6 Theorem 3.6; 7, Theorem 3.1] for each σ there exist continua $\mathfrak{C}^+_\sigma$ and $\mathfrak{C}^-_\sigma$ of $\mathfrak{C}$ containing $(\lambda^\sigma_1, 0)$ for which the following hold:

(a) there exists a neighborhood $\mathfrak{Q}_\sigma$ of $(\lambda^\sigma_1, 0)$ in $\mathbb{R}^\sigma \times E$ such that $\mathfrak{C}^+_\sigma \cap \mathfrak{Q}_\sigma \subset \mathbb{R}^\sigma \times S^+_\sigma$ and $\mathfrak{C}^-_\sigma \cap \mathfrak{Q}_\sigma \subset \mathbb{R}^\sigma \times S^-_\sigma$,

(b) either $\mathfrak{C}^+_\sigma$ and $\mathfrak{C}^-_\sigma$ are both unbounded, or $\mathfrak{C}^+_\sigma \cap \mathfrak{C}^-_\sigma \neq \{\mathcal{R}, \emptyset\}$, or $\mathfrak{C}^+_\sigma$ and $\mathfrak{C}^-_\sigma$ are bounded and $\mathfrak{C}^+_\sigma \cap \mathfrak{C}^-_\sigma = \{(\lambda, 0)\}$ for some $\lambda \in \mathcal{R}^\sigma$, $\lambda \neq \lambda^\sigma_1$.

It is obvious that inversion (3.6) transforms $\mathcal{C}$ into $\mathfrak{C}$ (heuristically, it swaps points at $u = \infty$ (respectively, $u = 0$) with points at $u = 0$ (respectively, $u = \infty$)). Let $\mathcal{C}^\nu_\sigma$ and $\mathcal{Q}_\sigma$ is the inverse image of $\mathfrak{C}^\nu_\sigma$ and $\mathfrak{Q}_\sigma$, respectively, under inversion (3.6). Then it follows from properties (a) and (b) of the set $\mathfrak{C}^\nu_\sigma$ that for $\mathcal{C}^\nu_\sigma$ the statements (i) and (ii) of the theorem hold. The proof of this theorem is complete.

Corollary 3.1 *If we assume additionally that the function $g(x,u,s,\lambda) \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R})$, $s = (s_1, s_2, \ldots, s_n)$, has the form*

$$g(x,u,s,\lambda) = g_1(x,u,s,\lambda)u + g_{2,1}(x,u,s,\lambda)s_1 + g_{2,2}(x,u,s,\lambda)s_2 + \ldots + g_{2,N}(x,u,s,\lambda)s_N,$$

and

$$g_1(x,u,s,\lambda) \le 0,\ g_{2,i}(x,u,s,\lambda) \le 0,\ (x,u,s,\lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}.$$

Then for each σ and each ν there exists a subcontinuum $\mathcal{T}^\nu_\sigma$ of the set $\mathcal{C}^\nu_\sigma \backslash Q_\sigma$ that lies in $\mathbb{R}^\sigma \times S^\nu_\sigma$ and either unbounded in $\mathbb{R} \times E$ or meets $\mathcal{R}^\sigma$ for some $\lambda \in \mathbb{R}$.

Proof. It follows from [22, Remark 2.36] that each element of $\mathcal{C}$ is a classical solution of problem (1.1). By statement (i) of Theorem 3.1, we have

$$(\mathcal{C}^\nu_\sigma \cap Q_\sigma) \subset ((\mathbb{R}^\sigma \times S^\nu_\sigma) \cup \{(\lambda^\sigma_1, \infty)\}).$$

By $\mathcal{T}^\nu_\sigma$ we denote a maximal subcontinuum of $\mathcal{C}^\nu_\sigma$ which lies in $\mathbb{R}^\sigma \times S^\nu_\sigma$. If $\mathcal{C}^\nu_\sigma \backslash Q_\sigma$ is bounded, then there exists $(\tilde{\lambda}, \tilde{u}) \in \partial\mathcal{T}^\nu_\sigma$ such that $u \in \mathcal{P}^\nu_\sigma$. By (1.1) $(\tilde{\lambda}, \tilde{u})$ solves the following nonlinear problem

$$\begin{aligned} &Lu + \sum_{i=1}^N b_i(x)u_{x_i} = \lambda a(x)u + g_1(x,u,\nabla u,\lambda)u, \quad x \in \Omega, \\ &v = 0, \quad x \in \partial\Omega, \end{aligned}$$

where

$$b_i(x) = -\,g_{2,i}(x, \tilde{u}(x), \nabla\tilde{u}(x), \tilde{\lambda}) \ge 0,\ x \in \Omega.$$

Then it follows from [6, Corollary 3.2] that $u \equiv 0$. Now, the assertion of the corollary follows from Theorem 3.1. The proof of the corollary is complete.

Corollary 3.2 *If g_1, $g_{2,i}$, $i = 1, 2, \dots, N$, are as in Corollary 3.1 and $g_1(x, 0, 0, \lambda) = 0$, $g_{2,i}(x, 0, 0, \lambda) = 0$, $x \in \Omega$, $\lambda \in \mathbb{R}$, and the set $\mathcal{C}_\sigma^\nu$ meets $\mathcal{R}^\sigma = \mathbb{R}^\sigma \times \{0\}$ for some $\lambda \in \mathbb{R}$, then $\lambda = \lambda_1^\sigma$.*

References

1. Adams, R.A.: Sobolev Spaces, Academic Press, *New York*, (1975).
2. Agmon, A., Douglis, A., Nirenberg, L.: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I*, Comm. Pure. Appl. Math. **12**, 623-727 (1959).
3. Aliyev, Z.S.: *Some global results for nonlinear fourth order eigenvalue problems*, Cent. Eur. J. Math. **12**(12), 1811-1828, (2014).
4. Aliyev, Z.S.: *Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order*, Sb. Math. **207**(12), 1625-1649 (2016).
5. Aliyev Z.S., Hasanova, Sh.M.: *Global bifurcation of a second order nonlinear elliptic problem with an indefinite weight function*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **42**(1), 116-125 (2016).
6. Aliyev Z.S., Hasanova, Sh.M.: *Global bifurcation of positive solutions of semi-linear elliptic partial differential equations with indefinite weight*, Z. Anal. Anwend. **38**(1), 1-15 (2019).
7. Aliyev Z.S., Hasanova, Sh.M.: *Global bifurcation of positive solutions from zero in nonlinearizable elliptic problems with indefinite weight*, J. Math. Anal. Appl. **491**(1), (2020); doi.org/10.1016/j.jmaa.2020.
8. Aliyev Z.S., Mustafayeva, N.A.: *Bifurcation of solutions from infinity for certain nonlinear eigenvalue problems of fourth-order ordinary differential equations*, Electron. J. Differ. Equ. **2018**(98), 1-19 (2018).
9. H. Berestycki, *On some nonlinear Sturm-Liouville problems*, J. Diff. Equat. **26**(1), 375-390 (1977).
10. Brown, K.J., Lin, C.C.: *On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight-function*, J. Math. Anal. Appl. **75**, 112-120 (1980).
11. Dancer, E.N.: *On the structure of solutions of non-linear eigenvalue problems*, Indiana Univ. Math. J. **23**, 1069-1076 (1974).
12. Healey, T.J., Kielhöfer, H: *Symmetry and nodal properties in the global bifurcation analysis of quasi-linear elliptic equations*, Arch. Ration. Mech. Anal. **113**, 299-311 (1991).
13. Gaines, R.E., Mawhin, J.L.: Coincidence degree and nonlinear differential equations, *Springer-Verlag, Berlin, New York* (1977).
14. Gilbarg D., Trudinger, N.S.: Elliptic Partial Differential Equations of the Second Order, *Springer, Berlin* (1977).
15. Kielhöfer, H.: Bifurcation Theory. An Introduction with Applications to PDEs, *in: Appl. Math. Scien., V. 156, Springer-Verlag, New York* (2004).
16. Krasnoselskii, M.A.: Topological methods in the theory of nonlinear integral equations, *Pergamon, London* (1974).
17. Krein, M.G., Rutman, M.A.: Linear operators leaving invariant a cone in a Banach space, *Amer. Math. Soc. Transl.* **10**(1), 199-325 (1962).
18. López-Gómez, J.: Spectral Theory and Nonlinear Functional Analysis, *Chapman and Hall/CRC, Boca Raton* (2001).
19. Makhmudov, A.P., Aliev, Z.S.: *Global bifurcation of solutions of certain nonlinearizable eigenvalue problems*, Diff. Equat. **25**(1), 71-76 (1989).

20. Rabinowitz, P.H.: *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7**(3), 487-513 (1971)
21. Rabinowitz, P.H.: Théorie du degré topologique et applications dans les problémes aux limites non linéaires, *(rédrigé par H. Berestycki), Lecture Notes, Analyse Numérique, Université Paris* 6 (1975).
22. Rabinowitz, P.H.: *On bifurcation from infinity*, J. Differential Equations, **14**(3), 462-475 (1973).
23. Rynne, B.P.: *Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable*, J. Math. Anal. Appl. **228**(1), 141-156 (1998).
24. Rynne, B.P., Youngson, M.A.: *Bifurcation of positive solutions from zero or infinity in elliptic problems which are not linearizable*, Nonlinear Anal. **44**(1), 21-31 (2001).