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Nil clean divisor graph

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Abstract. *In this article, we introduce a new graph theoretic structure associated with a finite commutative ring, called nil clean divisor graph. For a ring R, nil clean divisor graph is denoted by $G_N(R)$, where the vertex set is $\{x \in R : x \neq 0, \exists\, y(\neq 0, \neq x) \in R$ such that xy is nil clean$\}$, two vertices x and y are adjacent if xy is a nil clean element. We prove some interesting results of nil clean divisor graph of a ring.*

Keywords. nil clean ring, weakly nil clean ring, nil clean divisor graph, idempotent divisor graph.

Mathematics Subject Classification (2010): 16N40, 16U99

1 Introduction

In this article, rings are finite commutative rings with non zero identity. Diesl [4], introduced the concept of nil clean ring as a subclass of clean ring in 2013. He defined that an element x of a ring R to be a nil clean element if it can be written as a sum of an idempotent element and a nilpotent element of R. R is called a nil clean ring if every element of R is nil clean. Also in 2015, Kosan and Zhou [8], developed the concept of weakly nil clean ring as a generalization of nil clean ring. An element x of a ring R is said to be a weakly nil clean if $x = n + e$ or $x = n - e$, where n is a nilpotent element and e is an idempotent element of R. The set of nilpotent elements, set of unit elements, nil clean elements and weakly nil clean elements of a ring R are denoted by $Nil(R)$, $U(R)$, $NC(R)$ and $WNC(R)$ respectively. By graph, we consider simple undirected graph. For a graph G, the set of edges and the set of vertices are denoted by $E(G)$ and $V(G)$ respectively. The concept of zero-divisor graph of a commutative ring was introduced by Beck [3] to discuss the coloring of rings. In 1999, Anderson and Livingston [1], introduced zero divisor graph $\Gamma(R)$ of a commutative ring R. They defined, the vertex set of $\Gamma(R)$ to be the set of all non-zero zero

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divisors of R and two vertices x and y are adjacent if $xy = 0$. Li et al.[9], developed a kind of graph structure of a ring R, called nilpotent divisor graph of R, whose vertex set is $\{x \in R : x \neq 0, \exists y(\neq 0) \in R$ such that $xy \in Nil(R)\}$ and two vertices x and y are adjacent if $xy \in Nil(R)$. In 2018, Kimball and LaGrange [7], generalized the concept of zero divisor graph to idempotent divisor graph. For any idempotent $e \in R$, they defined the idempotent divisor graph $\Gamma_e(R)$ associated with e, where $V(\Gamma_e(R)) = \{a \in R :$ there exists $b \in R$ with $ab = e\}$ and two vertices a and b are adjacent if $ab = e$.

In this article, we introduce nil clean divisor graph $G_N(R)$ associated with a finite commutative ring R. We define the nil clean divisor graph $G_N(R)$ of a ring R by taking $V(G_N(R)) = \{x \in R : x \neq 0, \exists y(\neq 0, \neq x) \in R$ such that $xy \in NC(R)\}$ as the vertex set and two vertices x and y are adjacent if and only if xy is a nil clean element of R. Clearly nil clean divisor graph is a generalization of both idempotent divisor graph and nilpotent divisor graph. The properties like girth, clique number, diameter and dominating number etc. of $G_N(R)$ have been studied.

To start with, we recall some preliminaries about graph theory. For a graph G, the degree of a vertex $v \in G$ is the number of edges incident to v, denoted by $deg(v)$. The neighbourhood of a vertex $v \in G$ is the set of all vertices incident to v, denoted by A_v. A graph G is said to be connected, if for any two distinct vertices of G, there is a path in G connecting them. Number of edges on the shortest path between vertices x and y is called the distance between x and y and is denoted by $d(x, y)$. If there is no path between x and y, then we say $d(x, y) = \infty$. The diameter of a graph G, denoted by $diam(G)$, is the maximum of distances of each pair of distinct vertices in G. If G is not connected, then we say $diam(G) = \infty$. Also girth of G is the length of the shortest cycle in G, denoted by $gr(G)$ and if there is no cycle in G, then we say $gr(G) = \infty$. A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by an edge.

A clique is a subset a of set of vertices of a graph such that its induced subgraph is complete. A clique having n number of vertices is called an n-clique. The maximal clique of a graph is a clique such that there is no clique with more vertices. The clique number of a graph G is denoted by $\omega(G)$ and defined as the number of vertices in a maximal clique of G.

2 Nil clean divisor graph

Motivated by the concepts of nilpotent divisor graph and idempotent divisor graph, we introduce nil clean divisor graph as follows:

Definition 2.1 *For a ring R, nil clean divisor graph, denoted by $G_N(R)$ is defined as a graph with vertex set $\{x \in R : x \neq 0, \exists y(\neq 0, \neq x) \in R$ such that $xy \in NC(R)\}$ and two vertices x and y are adjacent if $xy \in NC(R)$.*

From the above definition, we observe that nil clean divisor graph is a generalization of nilpotent divisor graph, which is again a generalization of zero divisor graph. For any idempotent $e \in R$, nil clean divisor graph of R is also a generalization of $\Gamma_e(R)$. As an example, the nil clean divisor graph $G_N(\mathbb{Z}_6)$ is shown below:

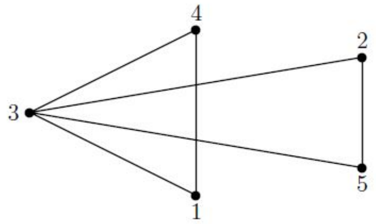

FIGURE 1. Nil clean divisor graph of $\mathbb{Z}_6$.

Theorem 2.1 *The nil clean divisor graph $G_N(R)$ is complete if and only if R is a nil clean ring.*

Proof. Let $G_N(R)$ is a complete and $x \in R$. If $x = 0$, then x is nil clean, if $x \neq 0$ then $x.1 = x$ is nil clean as $1 \in V(G_N(R))$. Converse is clear from the definition of nil clean divisor graph.

If $\mathbb{F}$ is a finite field of order n, then clearly $NC(\mathbb{F}) = \{0, 1\}$. Hence for any $x(\neq 0) \in \mathbb{F}$, x is adjacent to only x^{-1}, provided $x \neq x^{-1}$. Hence the nil-clean divisor graph of $\mathbb{F}$ is as follows:

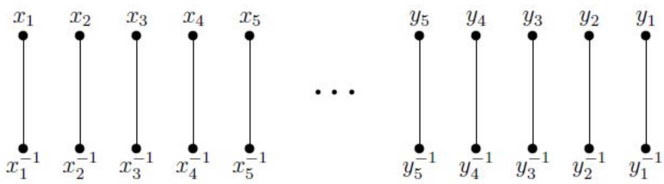

FIGURE 2. Nil clean divisor graph of $\mathbb{F}$.

Note that $x_i \neq x_i^{-1}$ and $y_i \neq y_i^{-1}$, otherwise we may get some isolated point as well in the graph.

Corollary 2.1 *For a field $\mathbb{F}$ of order n, where $n > 2$. If $A = \{a \in \mathbb{F} \ : \ a = a^{-1}\}$ then the following hold.*

1 *Diameter of $G_N(\mathbb{F})$ is infinite.*
2 *$Gr(G_N(\mathbb{F})) = \infty$ and $\omega(G_N(\mathbb{F})) = 2$.*
3 *$|V(G_N(\mathbb{F}))| = n - |A| - 1$.*

Theorem 2.2 *If R has a non trivial idempotent or a non trivial nilpotent element, then the girth of $G_N(R)$ is 3.*

Proof. If R has a non trivial idempotent e, then $\{0, 1, e, 1-e\} \subset NC(R)$ and we get a cycle $1-e-(1-e)-1$ in $G_N(R)$. Also if R has a non trivial nilpotent n, then $\{0, 1, n, n+1\} \subset NC(R)$. In this case $1 - n - (n+1) - 1$ is a cycle in $G_N(R)$.

Theorem 2.3 *If R has only trivial idempotents and trivial nilpotent, then girth of $G_N(R)$ is infinite.*

Proof. Since R has only trivial idempotents and trivial nilpotent so by Lemma 2.6 [2], R is a field. Hence the result.

Theorem 2.4 *Let R be a ring. Then the following hold.*

1 *Either R is a field or $G_N(R)$ is connected.*
2 *$diam(G_N(R)) = \infty$ or $diam(G_N(R)) \leq 3$.*
3 *$gr(G_N(R)) = \infty$ or $gr(G_N(R)) = 3$.*

Proof. Suppose R is a reduced ring.
Case (I): If R has no non trivial idempotent, then R is a field.
Case (II): If R has a non trivial idempotent, say $e \in Idem(R)$, then for any $x, y \in V(G_N(R))$, there exist $x_1, y_1 \in V(G_N(R))$, such that $xx_1, yy_1 \in NC(R) = Idem(R)$. So, we have a path $x - x_1e - y_1(1-e) - y$ from x to y.

If R is not a reduced ring, then there exists $n \in Nil(R)$, such that $x - n - y$ is a path from x to y, for any $x, y \in V(G_N(R))$. Hence (1) and (2) follow from the above observations and Figure 2.
(3) If R is reduced, then either R is a field or there exists a non trivial idempotent $e \in R$, such that $1 - e - (1-e) - 1$ is a cycle. So, $gr(G_N(R)) = \infty$ or $gr(G_N(R)) = 3$. If R is a non reduced ring, then since nilpotent graph is a subgraph of nil clean divisor graph, so from Theorem 2.1 [9], $gr(G_N(R)) = 3$.

Corollary 2.2 *If R is not a reduced ring, then $diam(G_N(R)) \leq 2$.*

Corollary 2.3 *A ring R is a field if and only if nil clean divisor graph of R is bipartite.*

Proof. $\Rightarrow$ Trivial.
$\Leftarrow$ If nil clean divisor graph of R is bipartite then $gr(G_N(R)) \neq 3$. So from Theorem 2.4, $gr(G_N(R)) = \infty$ and hence R is a field.

Theorem 2.5 *For a ring R, the following are equivalent.*

1 *$G_N(R)$ is a star graph.*
2 *$R \cong \mathbb{Z}_5$.*

Proof. The result follows from the fact that $gr(G_N(R)) = \infty$ if and only if R is a field.

Theorem 2.6 *For any ring R, $\omega(G_N(R)) \geq max\{|Nil(R)|, |Idem(R)| - 1\}$.*

Proof. From the definition of nil clean divisor graph, we observe that $Nil(R)$ and $Idem(R)$ respectively induce a complete subgraph of $G_N(R)$.

Next we strudy about nil clean divisor graph of weakly nil clean ring.

Theorem 2.7 *Let R be a weakly nil clean ring which is not nil clean. Then*

1 *$\omega(G_N(R)) \geq [\frac{|R|}{2}]$, where $[x]$ is the greatest integer function.*
2 *If $|R| (> 3)$ is even then $diam(R) = 2$.*

Proof. As $x \in WNC(R)$ implies $-x \in NC(R)$, so if $|R|$ is even, then $|NC(R)| \geq \frac{|R|}{2}$ and if $|R|$ is odd, then $|NC(R)| \geq \frac{|R|+1}{2}$. Since R is commutative, so product of any two nil clean element is also a nil clean element. Hence $\omega(G_N(R)) \geq [\frac{|R|}{2}]$.

Since $|R| > 3$, so R is not a field and hence $G_N(R)$ is connected. As $|R \setminus \{0\}|$ is odd, so there exists an element $a \in R$ such that $x \in NC(R) \cap WNC(R)$. Hence for any $x, y \in R$, $x - a - y$ is a path in $G_N(R)$ and $diam(G_N(R)) = 2$ as R is not a nil clean ring.

3 Nil clean divisor graph of $\mathbb{Z}_{2p}$ and $\mathbb{Z}_{3p}$, for any odd prime p

In this section we study the structures of $G_N(\mathbb{Z}_{2p})$ and $G_N(\mathbb{Z}_{3p})$, for any odd prime p.

Lemma 3.1 *If $a \in V(G_N(\mathbb{Z}_{2p}))$, where p is an odd prime, then the following hold.*

1. *If $a = p$, then $deg(a) = 2p - 2$.*
2. *If $a \in \{1, p-1, p+1, 2p-1\}$, then $deg(a) = 2$.*
3. *Otherwise $deg(a) = 3$*

Proof. Clearly $NC(\mathbb{Z}_{2p}) = \{0, 1, p, p+1\}$.

1. If $a = p$, then for any $y \in V(G_N(\mathbb{Z}_{2p}))$, either $yp = p$ or $yp = 0$. Hence every element of $V(G_N(\mathbb{Z}_{2p}))$ is adjacent to p.
2. It is easy to observe that, $A_1 = \{p, p+1\}$, $A_{p-1} = \{p, 2p-1\}$, $A_{p+1} = \{1, p\}$ and $A_{2p-1} = \{p-1, p\}$.
3. Let $a \in \mathbb{Z}_{2p} \setminus \{0, 1, p-1, p, p+1, 2p-1\}$.
Case (I): Let a be an even number. If $ax = 0$ in $\mathbb{Z}_{2p}$, then it has two solutions 0 and p. If $ax = 1$ in $\mathbb{Z}_{2p}$, then it has no solution, since $gcd(2p, a) = 2 \nmid 1$. If $ax = p$ in $\mathbb{Z}_{2p}$, then also it has no solution, since $gcd(2p, a) = 2 \nmid p$. If $ax = p+1$ in $\mathbb{Z}_{2p}$, then it has two distinct solutions x_1 and x_2 in $\mathbb{Z}_{2p}$, since $gcd(2p, a) = 2 \mid p+1$. Hence we conclude that $A_a = \{p, x_1, x_2\}$.
Case (II): Let a be an odd number. If $ax = 0$ in $\mathbb{Z}_{2p}$, then it has a unique solution $x = 0$. If $ax = 1$ in $\mathbb{Z}_{2p}$, then it has unique odd solution $x = y_1$ in $\mathbb{Z}_{2p}$, since $gcd(2p, a) = 1 \mid 1$. If $ax = p$ in $\mathbb{Z}_{2p}$, then it has unique solution $x = p$, since $gcd(2p, a) = 1 \mid p$. If $ax = p+1$ in $\mathbb{Z}_{2p}$, then it has unique even solution $x = y_2$ in $\mathbb{Z}_{2p}$, since $gcd(2p, a) = 1 \mid p+1$. Hence $A_a = \{p, y_1, y_2\}$
From the above cases it follows $deg(a) = 3$.

Remark 3.1 In the proof of Lemma 3.1 (3), Case(I), since $ax_1 = ax_2$ in $\mathbb{Z}_{2p}$, so $x_1 - x_2 = 0$ or p, but $x_1 - x_2 \neq 0$ as x_1 and x_2 are distinct. Hence if x_1 is odd, then x_2 is even and if x_1 is even, then x_2 is odd.

From Lemma 3.1 and Remark 3.1, for any prime $p > 2$, the nil clean divisor graph of $\mathbb{Z}_{2p}$ is the following:

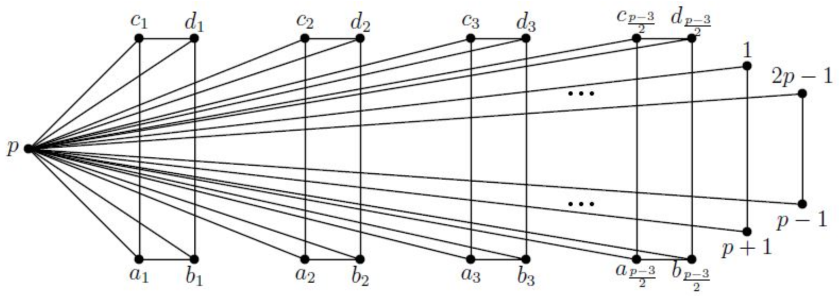

FIGURE 3. Nil clean divisor graph of $\mathbb{Z}_{2p}$.

In Figure 3, a_i and b_i are even numbers from $\mathbb{Z}_{2p} \setminus \{0, 1, p-1, p, p+1, 2p-1\}$ such that $a_i b_i = p+1$, for $1 \leq i \leq \frac{p-3}{2}$. Also $c_i = a_i + p$ and $d_i = b_i + p$, for $1 \leq i \leq \frac{p-3}{2}$. From the above observations we conclude the following:

Theorem 3.1 *The following hold for nil clean divisor graph $G_N(\mathbb{Z}_{2p})$, for any odd prime p.*

1. *Clique number of $G_N(\mathbb{Z}_{2p})$ is 3.*
2. *Diameter of $G_N(\mathbb{Z}_{2p})$ is 2.*

3 *Girth of* $G_N(\mathbb{Z}_{2p})$ *is* 3.
4 $\{p\}$ *is the unique smallest dominating set for* $G_N(\mathbb{Z}_{2p})$*, that is, dominating number of the graph is* 1.

Next we discuss about nil clean divisor graph of $\mathbb{Z}_{3p}$.

Lemma 3.2 *In* $G_N(\mathbb{Z}_{3p})$*; where* $p \equiv 2(mod\, 3)$*, the following hold.*

1 $deg(3k) = 5$ *if* $3k \notin \{p+1, 2p-1\}$*, for* $1 \le k \le p-1$.
2 $deg(p+1) = deg(2p-1) = 4$.

Proof. Here $NC(\mathbb{Z}_{3p}) = \{0, 1, p+1, 2p\}$. Observe that $3k.x \equiv 1(mod\, 3p)$ and $3k.x \equiv 2p(mod\, 3p)$ has no solution, as $gcd(3k, 3p) = 3$ does not divide 1 and $2p$. The congruence $3k.x \equiv 0(mod\, 3p)$ has three incongruent solutions $\{0, p, 2p\}$ in $\mathbb{Z}_{3p}$. Also $3k.x \equiv p+1(mod\, 3p)$ has three distinct incongruent solutions in $\mathbb{Z}_{3p}$, as $gcd(3k, 3p) = 3$ divides $p+1$.

1 As $x^2 \equiv p+1(mod\, 3p)$, has two solutions $p+1$ and $2p-1$, hence if $3k \notin \{p+1, 2p-1\}$, then $deg(3k) = 6-1 = 5$, as $0 \notin V(G_N(\mathbb{Z}_{3p}))$.
2 If $3k \in \{p+1, 2p-1\}$, then $deg(3k) = 6-2$, as $0 \notin V(G_N(\mathbb{Z}_{3p}))$ and we do not consider any loop.

Lemma 3.3 *In* $G_N(\mathbb{Z}_{3p})$*, where* $p \equiv 2(mod\, 3)$ *the following hold.*

1 $deg(p) = deg(2p) = 2p-2$.
2 *For* $x \in \{1, p-1, 3p-1, 2p+1\}$, $deg(x) = 2$.
3 *For* $x \in \mathbb{Z}_{3p} \setminus L$, $deg(x) = 3$*, where* $L = \{3k \ : \ 1 \le k \le p-1\} \cup \{1, p-1, 2p+1, 3p-1, p, 2p\}$.

Proof. Here $NC(\mathbb{Z}_{3p}) = \{0, 1, p+1, 2p\}$.

1 Clearly $p.x \equiv 1(mod\, 3p)$ and $p.x \equiv p+1(mod\, 3p)$ have no solution as $gcd(3p, p)$ does not divide 1 and $p+1$. Also $p.x \equiv 0(mod\, 3p)$ has p incongruent solutions $\{3k \ : \ 0 \le k \le p-1\}$ and $p.x \equiv 2p(mod\, 3p)$ has p incongruent solutions $\{3k+2 \ : \ 0 \le k \le p-1\}$. Since $0 \notin V(G_N(\mathbb{Z}_{3p}))$ and p is of the form $3i+2$, for some $0 \le i \le p-1$, hence $deg(p) = 2p-2$. Now $2p.x \equiv 0(mod\, 3p)$ has p incongruent solutions $\{3k \ : \ 0 \le k \le p-1\}$ and $2p.x \equiv 2p(mod\, 3p)$ has p incongruent solutions $\{3k+1 \ : \ 0 \le k \le p-1\}$. But $2p.x \equiv 1(mod\, 3p)$ and $2p.x \equiv p+1(mod\, 3p)$ have no solutions. Hence $deg(2p) = 2p-2$, since $2p$ is of the form $3i+1$, for some $1 \le i \le p-1$.
2 Since $x \equiv a(mod\, 3p)$, has only one solution a, hence $deg(1) = 2$. Also $(3p-1).x \equiv c(mod\, 3p)$ has only one solution $(3p-1)a$, hence $deg(3p-1) = 2$, as $0 \notin V(G_N(\mathbb{Z}_{3p}))$ and $3p-1 \in U(\mathbb{Z}_{3p})$. Equation $(p-1).x \equiv 1(mod\, 3p)$ and $(2p+1).x \equiv c(mod\, 3p)$ have a unique solutions, where $c \in \{0, 1, 2p, p+1\}$. Since $p-1, 2p+1 \in U(\mathbb{Z}_{3p})$, so $deg(p-1) = deg(2p+1) = 2$.
3 Let $a \in \mathbb{Z}_{3p} \setminus L$. As $gcd(a, 3p) = 1$, so $a.x \equiv 0(mod\, 3p)$ has a unique solution $x = 0$. Also $a.x \equiv c(mod\, 3p)$, where $c \in \{1, 2p, p+1\}$ has a unique solution. Hence $deg(a) = 3$.

From Lemma 3.2 and Lemma 3.3, for any prime $p > 3$ with $p \equiv 2(mod\, 3)$, the nil clean divisor graph of $\mathbb{Z}_{3p}$ is the following:

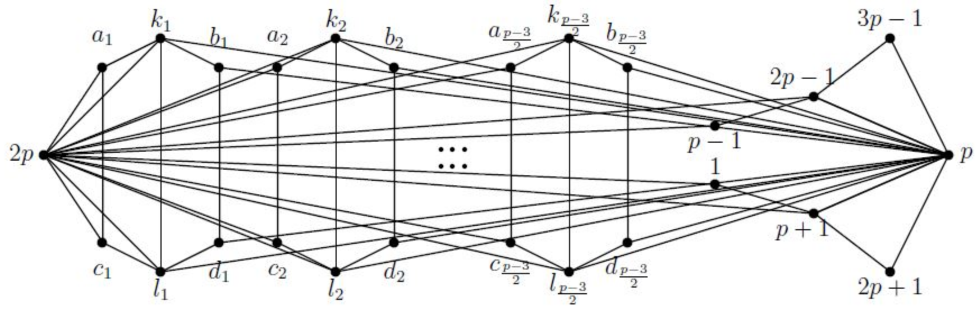

FIGURE 4. Nil clean divisor graph of $\mathbb{Z}_{3p}$, where $p \equiv 2(mod\, 3)$.

In Figure 3, $\{l_i, k_i\} \subseteq \{3k \ : \ 1 \le k \le p-1\}$, $a_i c_i \equiv 1(mod\, 3p)$, $b_i d_i \equiv 1(mod\, 3p)$ and $a_i k_i \equiv c_i l_i \equiv b_i k_i \equiv d_i l_i \equiv p+1(mod\, 3p)$, for $1 \le i \le \frac{p-3}{2}$. Also $a_i \equiv c_i \equiv 1(mod\, 3)$ and $b_i \equiv d_i \equiv 2(mod\, 3)$, for $1 \le i \le \frac{p-3}{2}$.

Theorem 3.2 *For any pri me p, where $p \equiv 2(mod\, 3)$, the following hold:*

1. *Girth of $G_N(\mathbb{Z}_{3p})$ is 3.*
2. *Clique number of $G_N(\mathbb{Z}_{3p})$ is 3.*
3. *Diameter of $G_N(\mathbb{Z}_{3p})$ is 3.*
4. *$\{p, 2p\}$ is the unique smallest dominating set for $G_N(\mathbb{Z}_{3p})$, that is, dominating number of the graph is 2.*

Proof. Clearly $NC(\mathbb{Z}_{3p}) = \{0, 1, p+1, 2p\}$.

1. Since $p - (p+1) - (2p+1) - p$ is a cycle of $G_N(\mathbb{Z}_{3p})$, so girth of $G_N(\mathbb{Z}_{3p})$ is 3.
2. If possible, let $\omega((G_N(\mathbb{Z}_{3p})) = 4$. Then there exists $A = \{z_i \ : \ 1 \le i \le 4\} \subset V(G_N(\mathbb{Z}_{3p}))$ such that A forms a complete subgraph of $G_N(\mathbb{Z}_{3p})$. If $x \in \mathbb{Z}_{3p} \backslash \{p, 2p, 3k \ : \ 1 \le k \le p-1\}$, then $deg(x) \le 3$, otherwise x is adjacent to either p or $2p$, x^{-1} and $3i$, for some $1 \le i \le p-1$. But x^{-1} is also adjacent to $3j$, for some $1 \le j \le p-1$ such that $i \ne j$. So $A \subseteq \{p, 2p, 3k \ : \ 1 \le k \le p-1\}$. Suppose $z_1 = 3k$, for some $1 \le k \le p-1$. From Figure 3, $A_{z_1} \subseteq \{p, 2p, 3i+1, 3j+2, 3s\}$, where $1 \le i, j, s \le p-1$, also $3s \notin A_{3i+1}$, $3s \notin A_{3j+2}$, $3i+1 \notin A_{3j+2}$, $p \notin A_{2p}$, $2p \notin A_{3j+2}$ and $p \notin N_{3i+1}$. Therefore $z_i \notin \{3k \ : \ 1 \le k \le p-1\}$, a contradiction. Hence $\omega((G_N(\mathbb{Z}_{3p})) = 3$, as $\{p, 2p-1, 3p-1\}$ forms a complete subgraph of $G_N(\mathbb{Z}_{3p})$.
3. From Figure 3; 1 and 2 are connected by a path $1 - (p+1) - p - 2$, so by Theorem 2.4, $diam(G_N(\mathbb{Z}_{3p})) = 3$.
4. Since every element of $G_N(\mathbb{Z}_{3p}) \setminus \{p, 2p\}$ is adjacent to either p or $2p$. Hence proof follows from Figure 3.

Lemma 3.4 *In $G_N(\mathbb{Z}_{3p})$, where $p \equiv 1(mod\, 3)$, the following hold.*

1. *$deg(3k) = 5$ if $3k \notin \{p-1, 2p+1\}$, for $1 \le k \le p-1$.*
2. *$deg(p-1) = deg(2p+1) = 4$.*

Proof. Proof is similar to the proof of Lemma 3.2.

Lemma 3.5 *In $\mathbb{Z}_{3p}$, where $p \equiv 1(mod\, 3)$, the following hold.*

1. *$deg(p) = deg(2p) = 2p - 2$.*
2. *For $x \in \{1, p+1, 3p-1, 2p-1\}$, $deg(x) = 2$.*
3. *For $x \in \mathbb{Z}_{3p} \setminus L$, $deg(x) = 3$, where $L = \{3k \ : \ 1 \le k \le p-1\} \cup \{1, p, 2p, p+1, 2p-1, 3p+1\}$.*

Proof. Proof is similar to the proof Lemma 3.3.

From Lemma 3.4 and Lemma 3.5, the nil clean divisor graph of $\mathbb{Z}_{3p}$, where $p \equiv 1(mod\,3)$ is the following:

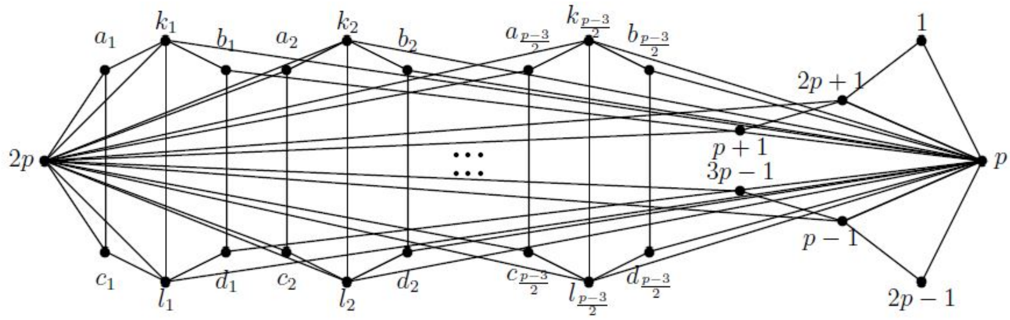

FIGURE 5. Nil clean divisor graph of $\mathbb{Z}_{3p}$, where $p \equiv 1(mod\ 3)$.

In Figure 3, $\{l_i, k_i\} \subseteq \{3k\ :\ 1 \leq k \leq p-1\}$, $a_ic_i \equiv 1(mod\ 3p)$, $b_id_i \equiv 1(mod\ 3p)$ and $a_ik_i \equiv c_il_i \equiv b_ik_i \equiv d_il_i \equiv 2p+1(mod\ 3p)$, for $1 \leq i \leq \frac{p-3}{2}$. Also $a_i \equiv c_i \equiv 2(mod\,3)$ and $b_i \equiv d_i \equiv 1(mod\,3)$, for $1 \leq i \leq \frac{p-3}{2}$. Hence we get the following theorem:

Theorem 3.3 *If $p \equiv 1(mod\ 3)$ then*

1 *Girth of $G_N(\mathbb{Z}_{3p})$ is* 3.
2 *Clique number of $G_N(\mathbb{Z}_{3p})$ is* 3.
3 *Diameter of $G_N(\mathbb{Z}_{3p})$ is* 3.
4 $\{p, 2p\}$ *is the unique smallest dominating set for $G_N(\mathbb{Z}_{3p})$, that is, dominating number of the graph is* 2.

Proof. Since Figure 3 and Figure 3 are similar, hence the proof is similar to the proof of Theorem 3.2.

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