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Multilinear commutators of parabolic Calderón-Zygmund operators on generalized weighted variable parabolic Morrey spaces

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Abstract. *We established the boundedness of multilinear commutators of parabolic Calderón-Zygmund operators $T_{\mathbf{b}}$ on generalized weighted variable exponent parabolic Morrey spaces $M_w^{p(\cdot),\varphi}$ with the weight function w belonging to Muckenhoupt's class $A_{p(\cdot)}(\mathbb{R}^n)$. When $\mathbf{b} = (b_1, \dots, b_m)$, $b_i \in BMO(\mathbb{R}^n)$, $i = 1, \dots, m$ and $w \in A_{p(\cdot)}(\mathbb{R}^n)$, the sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the operator $T_{\mathbf{b}}$ from $M_w^{p(\cdot),\varphi_1}$ to $M_w^{p(\cdot),\varphi_2}$ are found.*

Keywords. Parabolic Calderón-Zygmund operator; Generalized weighted variable exponent parabolic Morrey space; Multilinear commutator; BMO.

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1 Introduction

The commutators of Calderón-Zygmund operator play an important role in studying the regularity of solutions of elliptic and parabolic partial differential equations of second order [13, 17, 29]. Let $b \in BMO(\mathbb{R}^n)$ and T be the Calderón-Zygmund singular integral operator. The commutator operator $[b, T]$ generated by b and T is defined $[b, T]f = T(bf) - bT(f)$. A well known result of Coifman, Rochberg and Weiss [11] states that the commutator operator $[b, T]$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. As the development of singular integral operators and their commutators, multilinear singular integral operators have been well-studied. It is known that multilinear operator, as a non-trivial extension of the commutator, is of

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great interest in harmonic analysis and has been widely studied by many authors [7–10,26–28,42]. In [15], the weighted $L_p\,(p>1)$-boundedness of the multilinear operator related to some singular integral operators is obtained and in [28], the boundedness of the Riesz potential, Calderón-Zygmund operators and their commutators in the generalized weighted variable exponent Morrey spaces $M_w^{p(\cdot),\varphi}$ were studied. In [46], the boundedness of the multilinear commutators in weighted Morrey spaces $L_w^{p,\kappa}$ for $1<p<\infty$ and $0<\kappa<1$ is obtained, where the symbol $\mathbf{b}$ belongs to bounded mean oscillation BMO^m, see also [47]. In [23], the boundedness of the classical operators and their commutators in spaces $M_w^{p,\varphi}$ was also studied, see also [29,31,33].

The classical Morrey spaces were originally introduced by Morrey in [39] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [12,17,39]. Guliyev, Mizuhara and Nakai [19,40,41] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [20,21,32,44]). Recently, Komori and Shirai [36] considered the weighted Morrey spaces $L_w^{p,\kappa}$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [23] gave a concept of generalized weighted Morrey space $M_w^{p,\varphi}$ which could be viewed as extension of both generalized Morrey space $M^{p,\varphi}$ and weighted Morrey space $L_w^{p,\kappa}$.

Variable exponent function spaces received considerable attentions in recent decades [14]. They are important not only in theory as generalizations of classical function spaces, but also for their wide applications in the fields of fluid dynamics, elasticity dynamics, the differential equations with nonstandard growth. The rich development can be found in many research works of the theory of variable exponent function spaces. We refer to [2,3,14,34,35,37] for the details. For example, Lebesgue spaces with variable exponent were studied in [45], Morrey spaces with variable exponent were studied in [3,5], generalized Morrey spaces with variable exponent were studied in [1,22,24], and generalized weighted Morrey spaces with variable exponent can be found in [4,23,28,31,33].

We recall the definition of parabolic metric. Let $a_1,\dots,a_n$ be fixed real numbers, $a_i\ge 1$, $1\le i\le n$. For fixed $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, the function $F(x,\rho)=\sum_{i=1}^n(x_i^2/\rho^{2a_i})$ is a decreasing function in ρ $(\rho>0)$. We denote by $\rho(x)$ the unique solution of the equation $F(x,\rho)=1$. In 1966, Besov, Il'in, Lizorkin [6] and Fabes, Riviere [16] proved that $\rho(x)$ is a metric on $\mathbb{R}^n$. For $\mu>0$ and $x\in\mathbb{R}^n$, define the dilation on $\mathbb{R}^n$ by

$$A_\mu:(x_1,x_2,\dots,x_n)\to(\mu^{a_1}x_1,\mu^{a_2}x_2,\dots,\mu^{a_n}x_n)$$

Then it is immediate to that $\rho(A_\mu x)=\mu\rho(x)$ and $\rho(tx)\le\rho(x)$ when $t\le 1$. One has the polar decomposition $x=A_\rho x'$ with $x'=(x_1/\rho(x)^{a_1},\dots,x_n/\rho(x)^{a_n})\in\mathbb{S}^{n-1}$, $\rho=\rho(x)$ and $dx=\rho^{a-1}J(x')d\rho d\sigma(x')$, where $|a|=\sum_{i=1}^n a_i$ and $J(x')=\sum_{i=1}^n(x_i')^2$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([6, **?**]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids $\mathcal{E}(x,r)=\{y\in\mathbb{R}^n:\rho(x-y)<r\}$, with the Lebesgue measure $|\mathcal{E}(x,r)|=v_n r^{|a|}$, where v_n is the volume of the unit ellipsoid in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}(x,r)=\mathbb{R}^n\setminus\mathcal{E}(x,r)$ be the complement of $\mathcal{E}(x,r)$. If $a=(1,\dots,1)$, then clearly $\rho(x)=|x|$ and $\mathcal{E}(x,r)=B(x,r)$. Note that in the standard parabolic case $P_0=\operatorname{diag}(1,\dots,1,2)$ we have

$$\rho(x)=\sqrt{\frac{|x'|^2+\sqrt{|x'|^4+x_n^2}}{2}},\qquad x=(x',x_n).$$

Let a measurable function $K(x,\xi):\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ be a variable kernel, $\mathbf{b}=(b_1,...,b_m)$ and b_j, $1\le j\le m$ be locally integrable functions when we consider parabolic

Calderón-Zygmund integral and its multilinear commutators as defined by

$$Tf(x) = \int_{\mathbb{R}^n} K(x, x-y) f(y) dy, \tag{1.1}$$

$$T_{\mathbf{b}} f(x) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} (b_j(x) - b_j(y)) K(x, x-y) f(y) dy, \tag{1.2}$$

where $K(x, y)$ is variable parabolic Calderón-Zygmund kernel. That is, $K(x, \xi)$ is called variable parabolic Calderón-Zygmund kernel if:

i) $K(x, \cdot)$ is a parabolic Calderón-Zygmund kernel for a.a. $x \in \mathbb{R}^n$:
a) $K(x, \cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$,
b) $K(x, (\mu^{a_1}\xi_1, \mu^{a_2}\xi_2, \dots, \mu^{a_n}\xi_n)) = \mu^{-|a|} K(x, \xi), \quad \forall \mu > 0, \xi \in \mathbb{R}^n$,
c) $\int_{\mathbb{S}^{n-1}} K(x, \xi) d\sigma_\xi = 0\,, \quad \int_{\mathbb{S}^{n-1}} |K(x, \xi)| d\sigma_\xi < +\infty$,
ii) $\left\| D_\xi^\beta K \right\|_{L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})} \le M(\beta) < \infty$ for every multi-index β.

A weight function is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(x) dx$, the Lebesgue measure of E by $|E|$, and the characteristic function of E by χ_E.

Given an open set $E \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : E \to [1, \infty)$, $p'(\cdot)$ is the conjugate exponent defined by $p'(\cdot) = p(\cdot)/(p(\cdot)-1)$. For a measurable subset $E \subset \mathbb{R}^n$, we denote $p^-(E) = \text{ess inf}\{p(x) : x \in E\}$, $p^+(E) = \text{ess sup}\{p(x) : x \in E\}$. Especially, we denote $p^- = p^-(\mathbb{R}^n)$ and $p^+ = p^+(\mathbb{R}^n)$. The set $\mathcal{P}(E)$ consists of all $p(\cdot) : E \to [1, \infty)$ satisfying $p^-(E) > 1$, $p^+(E) < \infty$. Denote by $\mathcal{P}_0(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that $0 < p^- \le p(x) \le p^+ < \infty$, $x \in \mathbb{R}^n$. Let $\mathcal{P}_1(\mathbb{R}^n)$ be the set of all measurable function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that $1 \le p^- \le p(x) \le p^+ < \infty$, $x \in \mathbb{R}^n$. Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (1, \infty)$ such that $1 < p^- \le p(x) \le p^+ < \infty$, $x \in \mathbb{R}^n$.

We define the variable exponent Lebesgue space $L^{p(\cdot)}(E)$ as the set of real-valued measurable functions f on E such that, for some $\varepsilon > 0$, $\int_E \left(\varepsilon |f(x)|\right)^{p(x)} dx < \infty$. This is a Banach function space with respect to the Luxemburg-Nakano norm,

$$\|f\|_{L^{p(\cdot)}(E)} = \inf \Big\{ \lambda > 0 : \int_E \Big(\frac{f(x)|}{\lambda} \Big)^{p(x)} dx \le 1 \Big\}.$$

The space $L^{p(\cdot)}_{\text{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(\Omega) := \Big\{ f \text{ is measurable} : \ f \in L^{p(\cdot)}(E) \text{ for all compact subsets } E \subset \Omega \Big\}.$$

The weighted Lebesgue space $L^{p(\cdot)}_w(E)$ is defined by as the set of all measurable functions for which $\|f\|_{L^{q(\cdot)}_w(\Omega)} = \|wf\|_{L^{q(\cdot)}(\Omega)} < \infty$.

Next we define some classes of variable exponent functions. Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the parabolic maximal operator M is defined by

$$Mf(x) = \sup_{\mathcal{E} \ni x} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |f(y)| dy,$$

where $|\mathcal{E}|$ is the Lebesgue measure of the ellipsoid $\mathcal{E}$.

The set $\mathcal{B}(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying that the parabolic maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

An important subset of $\mathcal{B}(\mathbb{R}^n)$ is the class of globally log-Hölder continuous functions $p(\cdot) \in LH(\mathbb{R}^n)$, with $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Recall that $p(\cdot) \in LH(\mathbb{R}^n)$, if $p(\cdot)$ satisfies

$$|p(x) - p(y)| \leq \frac{C}{-\log(\rho(x-y))}, \quad \rho(x-y) \leq 1/2,$$
$$|p(x) - p(y)| \leq \frac{C}{\log(e+\rho(x))}, \quad \rho(y) \geq \rho(x).$$

Let us define the class $A_{p(\cdot)}(\mathbb{R}^n)$ (see [37]) to consist of those weights w for which

$$\sup_{\mathcal{E}} |\mathcal{E}|^{-1} \|w\|_{L^{p(\cdot)}(\mathcal{E}(x,r))} \|w^{-1}\|_{L^{p'(\cdot)}(\mathcal{E}(x,r))} < \infty.$$

We define the generalized weighed variable exponent Morrey spaces as follows.

Definition 1.1 *Let $p(\cdot) \in \mathcal{P}_1(\mathbb{R}^n)$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_w^{p(\cdot),\varphi}$ the generalized weighted variable exponent parabolic Morrey space, the space of all functions $f \in L_w^{p(\cdot),\mathrm{loc}}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_w^{p(\cdot),\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x,r))} \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x,r))},$$

where $L_w^{p(\cdot)}(\mathcal{E}(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x,r))} \equiv \|f\chi_{\mathcal{E}(x,r)}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}.$$

Remark 1.1 (1) If $w \equiv 1$, then $M^{p(\cdot),\varphi}(1) = M^{p(\cdot),\varphi}$ is the generalized variable exponent parabolic Morrey space, see for example, [22].

(2) If $\varphi(x,r) \equiv w(\mathcal{E}(x,r))^{\frac{\kappa-1}{p(x)}}$, then $M_w^{p(\cdot),\varphi} = L_w^{p(\cdot),\kappa}$ is the weighted variable exponent parabolic Morrey space.

(3) If $\varphi(x,r) \equiv v(\mathcal{E}(x,r))^{\frac{\kappa}{p(x)}} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x,r))}$, then $M_w^{p(\cdot),\varphi} = L_{v,w}^{p(\cdot),\kappa}$ is the two weighted variable exponent parabolic Morrey space.

(4) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-|a|}{p(x)}}$ with $0 < \lambda < |a|$, then $M_w^{p(\cdot),\varphi} = L^{p(\cdot),\lambda}(\mathbb{R}^n)$ is the variable exponent parabolic Morrey space, see for example, [3].

(5) If $\varphi(x,r) \equiv \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x,r))}$, then $M_w^{p(\cdot),\varphi} = L_w^{p(\cdot)}(\mathbb{R}^n)$ is the weighted variable exponent Lebesgue space.

The goal of this paper is to prove the boundedness of the multilinear commutator of parabolic Calderón-Zygmund operator $T_{\mathbf{b}}$ from one parabolic generalized weighted variable exponent parabolic Morrey space $M_w^{p(\cdot),\varphi_1}$ to another $M_w^{p(\cdot),\varphi_2}$ for $\mathbf{b} = (b_1,\ldots,b_m)$, $b_i \in BMO(\mathbb{R}^n)$, $i = 1,\ldots,m$. By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Main results

In this section we give main results. The following statement for the isotropic case $a = (1,\ldots,1)$ was proved in [28] and for the anisotropic case it is proved similarly.

Theorem 2.1 *Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}(\mathbb{R}^n)$ and (φ_1,φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\|w\|_{L^{p(\cdot)}(\mathcal{E}(x,s))}}{\|w\|_{L^{p(\cdot)}(\mathcal{E}(x,t))}} \frac{dt}{t} \le C\,\varphi_2(x,r), \tag{2.1}$$

where C does not depend on x and r. Then the operator T is bounded from $M_w^{p(\cdot),\varphi_1}$ to $M_w^{p(\cdot),\varphi_2}$.

In the following theorem we give a Guliyev's type local estimate (see, for example, [19,22,28]) which are the fundamental tools to prove the following main theorem about the boundedness of the parabolic multilinear commutator operators $T_{\mathbf{b}}$ on the parabolic generalized weighted variable exponent Morrey spaces $M_w^{p(\cdot),\varphi}$.

Theorem 2.2 *Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}(\mathbb{R}^n)$, $\mathbf{b} = (b_1,\ldots,b_m)$, $b_i \in BMO(\mathbb{R}^n)$, $i = 1,\ldots,m$, and $T_{\mathbf{b}}$ be a parabolic multilinear commutators defined as* (1.2). *Then*

$$\begin{aligned}\|T_{\mathbf{b}}f\|_{L_w^{p(\cdot)}(\mathcal{E}(x_0,r))} &\le C \prod_{j=1}^m \|b_j\|_{BMO}\, \|w\|_{L^{p(\cdot)}(\mathcal{E}(x_0,t))} \\ &\times \int_{2r}^\infty \ln^m\left(e+\frac{t}{r}\right) \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x_0,t))}\, \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t}\end{aligned}$$

holds for any ball $\mathcal{E} = \mathcal{E}(x_0,r)$ and for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$, where C does not depend on f, $x_0 \in \mathbb{R}^n$ and $r > 0$.

In the following, we give the main theorem about the boundedness of the multilinear commutator operator $T_{\mathbf{b}}$ on the parabolic generalized weighted variable exponent Morrey spaces $M_w^{p(\cdot),\varphi}$.

Theorem 2.3 *Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}(\mathbb{R}^n)$, and (φ_1,φ_2) satisfies the condition*

$$\int_r^\infty \ln^m\left(e+\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\|w\|_{L^{p(\cdot)}(\mathcal{E}(x,s))}}{\|w\|_{L^{p(\cdot)}(\mathcal{E}(x,t))}} \frac{dt}{t} \le C\,\varphi_2(x,r), \tag{2.2}$$

where C does not depend on x and r. Let $\mathbf{b} = (b_1,\ldots,b_m)$, $b_i \in BMO(\mathbb{R}^n)$, $i = 1,\ldots,m$. Then the operator $T_{\mathbf{b}}$ is bounded from $M_w^{p(\cdot),\varphi_1}$ to $M_w^{p(\cdot),\varphi_2}$. Moreover,

$$\|T_{\mathbf{b}}f\|_{M_w^{p(\cdot),\varphi_2}} \lesssim \prod_{j=1}^m \|b_j\|_{BMO}\, \|f\|_{M_w^{p(\cdot),\varphi_1}}.$$

Remark 2.1 Note that Theorems 2.2 and 2.3 were proved in [18] in the case $a = (1,\ldots,1)$ and $w \equiv 1$, and in [30] in the case $a = (1,\ldots,1)$ and $w \in A_{p(\cdot)}(\mathbb{R}^n)$.

3 Some preliminary results

In this section, we state some results about the variable L^p spaces. The following generalized Hölder inequality on variable Lebesgue spaces can be found in [14].

Lemma 3.1 *Let $q(\cdot), q_1(\cdot) \ldots, q_m(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ so that $1/q(\cdot) = 1/q_1(\cdot) + \cdots 1/q_m(\cdot)$. Then for any $f_j \in L^{q_j(\cdot)}(\mathbb{R}^n)$, $i = 1, \ldots, m$,*

$$\|f_1 \cdots f_m\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \cdots \|f_m\|_{L^{q_m(\cdot)}(\mathbb{R}^n)}.$$

Lemma 3.2 *[34] Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$. Then $M : L^{p(\cdot)}_w(\mathbb{R}^n) \to L^{p(\cdot)}_w(\mathbb{R}^n)$ if and only if $w \in A_{p(\cdot)}(\mathbb{R}^n)$.*

Definition 3.1 *$BMO(\mathbb{R}^n)$ is the Banach space modulo constants with the norm $\|\cdot\|_*$ defined by*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}| dy < \infty,$$

where $b \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and

$$b_{\mathcal{E}(x,r)} = \frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} b(y) dy.$$

The following results are proved by Ye and Zhu in [46] and in the isotropic case $a_i = 1$, $1 \le i \le n$ by Perez and Trujillo-Gonzalez [42].

Lemma 3.3 *Let $1 < p < \infty$ and $w \in A_p$ and suppose that $\mathbf{b} = (b_1, \ldots, b_m)$, $b_i \in BMO(\mathbb{R}^n)$, $i = 1, \ldots, m$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} \big|T_{\mathbf{b}} f(x)\big|^p w(x) dx \le C \prod_{j=1}^m \|b_j\|_{BMO} \int_{\mathbb{R}^n} |f(x)|^p w(x) dx.$$

Let $\mathbf{b} = (b_1, \ldots, b_m)$, $b_i \in BMO(\mathbb{R}^n)$, $i = 1, \ldots, m$. By the generalized Hölder's inequality in Orlicz spaces (see [43, page 58]) and John-Nirenberg's inequality, we have (see also [38, (2.14)])

$$\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} \big|b_1(x) - (b_1)_{\mathcal{E}}\big| \ldots \big|b_m(x) - (b_m)_{\mathcal{E}}\big| |g(x)| dx \lesssim \prod_{j=1}^m \|b_j\|_{BMO} \|g\|_{L(\log L)^m, \mathcal{E}}. \quad (3.1)$$

For $\Phi(t) = t\big(1 + \log^+ t\big)$, the Φ-average of a function f on a parabolic ball $\mathcal{E}$ is defined by

$$\|f\|_{L(\log L), \mathcal{E}} = \|f\|_{\Phi, \mathcal{E}} := \inf\Big\{\lambda > 0 : \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \le 1\Big\}.$$

The maximal function associated to $\Phi(t) = t\big(1 + \log^+ t\big)$ is defined by

$$M_{L(\log L)}(f)(x) = \sup_{\mathcal{E}\ni x} \|f\|_{L(\log L), \mathcal{E}},$$

where the supremum is taken over all the parabolic balls containing x.

It is not difficult to check the following pointwise equivalence (see (21) in [21])

$$M_{L(\log L)}(f)(x) \approx M^2 f(x), \ \text{ where } \ M^2 = M \circ M.$$

Let $M^\sharp$ be the sharp parabolic maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |\mathcal{E}(x,r)|^{-1} \int_{\mathcal{E}(x,r)} |f(y) - f_{\mathcal{E}(x,r)}| dy.$$

Definition 3.2 *We define the $BMO(\mathbb{R}^n)$ space as the set of all locally integrable functions f with finite norm*

$$\|b\|_{BMO} = \sup_{x\in\mathbb{R}^n} M^{\sharp}b(x) = \sup_{x\in\mathbb{R}^n,\, r>0} |\mathcal{E}(x,r)|^{-1} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}|dy.$$

Definition 3.3 *We define the $BMO_{p(\cdot),w}(\mathbb{R}^n)$ space as the set of all locally integrable functions f with finite norm*

$$\|b\|_{BMO_{p(\cdot),w}} = \sup_{x\in\mathbb{R}^n,\, r>0} \frac{\|(b(\cdot) - b_{\mathcal{E}(x,r)})\chi_{\mathcal{E}(x,r)}\|_{L^{p(\cdot)}_w}}{\|\chi_{\mathcal{E}(x,r)}\|_{L^{p(\cdot)}_w}}.$$

Theorem 3.1 *[35] Let $p(\cdot) \in LH(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n)$ and w be a Lebesgue measurable function. If $w \in A_{p(\cdot)}(\mathbb{R}^n)$, then the norms $\|\cdot\|_{BMO_{p(\cdot),w}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

The following theorem for the isotropic case $a = (1,\dots,1)$ was proved in [45] and for the anisotropic case it is proved similarly.

Lemma 3.4 *Let $\mathbf{b} = (b_1,\dots,b_m)$, $b_i \in BMO(\mathbb{R}^n)$, $i = 1,\dots,m$. Then there exists a constant $C > 0$ such that for $p(\cdot) \in LH(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}(\mathbb{R}^n)$, we have*

$$\|T_{\mathbf{b}}f\|_{L^{p(\cdot)}_w} \lesssim \prod_{j=1}^m \|b_j\|_{BMO}\, \|f\|_{L^{p(\cdot)}_w}.$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s)\, w(s)\, ds, \quad H^*_w g(t) := \int_t^\infty \Big(1+\ln\frac{s}{t}\Big)^m g(s)\, w(s)\, ds, \quad 0 < t < \infty,$$

where w is a weight.

Theorem 3.2 *[25] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w g(t) \le C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)\, ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 3.3 *[23] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H^*_w g(t) \le C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \Big(1+\ln\frac{s}{t}\Big)^m \frac{w(s)\, ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

4 Proof of main results

Proof of Theorem 2.2. Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}(\mathbb{R}^n)$, $\mathbf{b} = (b_1, \dots, b_m)$, $b_i \in BMO(\mathbb{R}^n)$, $i = 1, \dots, m$.

For arbitrary $x_0 \in \mathbb{R}^n$ and $r > 0$, set $\mathcal{E} = \mathcal{E}(x_0, r)$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2\mathcal{E})}$. Hence

$$\|T_{\mathbf{b}}f\|_{L_w^{p(\cdot)}(\mathcal{E})} \le \|T_{\mathbf{b}}f_1\|_{L_w^{p(\cdot)}(\mathcal{E})} + \|T_{\mathbf{b}}f_2\|_{L_w^{p(\cdot)}(\mathcal{E})}.$$

From the boundedness of $T_{\mathbf{b}}$ in $L_w^{p(\cdot)}$ (see Lemma 3.4) it follows that:

$$\begin{aligned}\|T_{\mathbf{b}}f_1\|_{L_w^{p(\cdot)}(\mathcal{E})} &\le \|T_{\mathbf{b}}f_1\|_{L_w^{p(\cdot)}} \\ &\lesssim \prod_{j=1}^m \|b_j\|_{BMO} \|f_1\|_{L_w^{p(\cdot)}} = \prod_{j=1}^m \|b_j\|_{BMO} \|f\|_{L_w^{p(\cdot)}(2\mathcal{E})}.\end{aligned}$$

For the term $\|T_{\mathbf{b}}f_2\|_{L_w^{p(\cdot)}(\mathcal{E})}$, without loss of generality, we can assume $m = 2$. Thus, the operator $T_{\mathbf{b}}f_2$ can be divided into four parts

$$\begin{aligned}T_{\mathbf{b}}f_2(x) &= \big(b_1(x) - (b_1)_{\mathcal{E}}\big)\big(b_2(x) - (b_2)_{\mathcal{E}}\big)\int_{\mathbb{R}^n} K(x,y)f_2(y)dy \\ &+ \int_{\mathbb{R}^n} K(x,y)\big(b_1(y) - (b_1)_{\mathcal{E}}\big)\big(b_2(y) - (b_2)_{\mathcal{E}}\big)f_2(y)dy \\ &- \big(b_1(x) - (b_1)_{\mathcal{E}}\big)\int_{\mathbb{R}^n} K(x,y)\big(b_2(y) - (b_2)_{\mathcal{E}}\big)f_2(y)dy \\ &- \big(b_2(x) - (b_2)_{\mathcal{E}}\big)\int_{\mathbb{R}^n} K(x,y)\big(b_1(y) - (b_1)_{\mathcal{E}}\big)f_2(y)dy \\ &= I_1(x) + I_2(x) + I_3(x) + I_4(x).\end{aligned}$$

For $x \in B$ we have

$$\begin{aligned}|T_{\mathbf{b}}f_2(x)| &\le |I_1(x) + |I_2(x)| + |I_3(x)| + |I_4(x)| \\ &\lesssim \big|b_1(x) - (b_1)_{\mathcal{E}}\big|\big|b_2(x) - (b_2)_{\mathcal{E}}\big|\int_{\complement(2\mathcal{E})} \frac{|f(y)|}{\rho(x_0-y)^{|a|}}dy \\ &+ \int_{\complement(2\mathcal{E})} \big|b_1(y) - (b_1)_{\mathcal{E}}\big|\big|b_2(y) - (b_2)_{\mathcal{E}}\big|\frac{|f(y)|}{\rho(x_0-y)^{|a|}}dy \\ &+ \big|b_1(x) - (b_1)_{\mathcal{E}}\big|\int_{\complement(2\mathcal{E})} \big|b_2(y) - (b_2)_{\mathcal{E}}\big|\frac{|f(y)|}{\rho(x_0-y)^{|a|}}dy \\ &+ \big|b_2(x) - (b_2)_{\mathcal{E}}\big|\int_{\complement(2\mathcal{E})} \big|b_1(y) - (b_1)_{\mathcal{E}}\big|\frac{|f(y)|}{\rho(x_0-y)^{|a|}}dy.\end{aligned}$$

Then

$$\begin{aligned}\|T_{\mathbf{b}}f_2\|_{L^{p(\cdot)}_w(\mathcal{E})} &\lesssim \Big\| \int_{\complement_{(2\mathcal{E})}} \prod_{j=1}^{2} \big|b_i(y) - (b_i)_{\mathcal{E}}\big| \frac{|f(y)|dy}{\rho(x_0-y)^{|a|}} \Big\|_{L^{p(\cdot)}_w(\mathcal{E})} \\ &+ \Big\| \big|b_1(\cdot) - (b_1)_{\mathcal{E}}\big| \left(\int_{\complement_{(2\mathcal{E})}} \big|b_2(y) - (b_2)_{\mathcal{E}}\big| \frac{|f(y)|dy}{\rho(x_0-y)^{|a|}} \right) \Big\|_{L^{p(\cdot)}_w(\mathcal{E})} \\ &+ \Big\| \big|b_2(\cdot) - (b_2)_{\mathcal{E}}\big| \left(\int_{\complement_{(2\mathcal{E})}} \big|b_1(y) - (b_1)_{\mathcal{E}}\big| \frac{|f(y)|dy}{\rho(x_0-y)^{|a|}} \right) \Big\|_{L^{p(\cdot)}_w(\mathcal{E})} \\ &+ \Big\| \int_{\complement_{(2\mathcal{E})}} \prod_{j=1}^{2} \big|b_i(\cdot) - (b_i)_{\mathcal{E}}\big| \frac{|f(y)|dy}{\rho(x_0-y)^{|a|}} \Big\|_{L^{p(\cdot)}_w(\mathcal{E})} \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Let us estimate I_1.

$$\begin{aligned} I_1 &= \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{\complement_{(2\mathcal{E})}} \prod_{j=1}^{2} \big|b_i(y) - (b_i)_{\mathcal{E}}\big| \frac{|f(y)|dy}{\rho(x_0-y)^{|a|}} \\ &\approx \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{\complement_{(2\mathcal{E})}} \prod_{j=1}^{2} \big|b_i(y) - (b_i)_{\mathcal{E}}\big| \, |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{|a|+1}} dy \\ &\approx \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \int_{2r\le|x_0-y|\le t} \prod_{j=1}^{2} \big|b_i(y) - (b_i)_{\mathcal{E}}\big| \, |f(y)|dy \frac{dt}{t^{|a|+1}} \\ &\lesssim \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \int_{\mathcal{E}(x_0,t)} \prod_{j=1}^{2} \big|b_i(y) - (b_i)_{\mathcal{E}}\big| \, |f(y)|dy \, \frac{dt}{t^{|a|+1}}. \end{aligned}$$

Applying Hölder's inequality (3.1) and by Lemma 3.1, we get

$$\begin{aligned} I_1 &\lesssim \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \Big\| \prod_{j=1}^{2} \big|b_i(\cdot) - (b_i)_{\mathcal{E}}\big| \Big\|_{L^{p'(\cdot)}_{w^{-1}}(\mathcal{E}(x_0,t))} \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}} \\ &\lesssim \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \Big\| \prod_{j=1}^{2} \big|b_i(\cdot) - (b_i)_{\mathcal{E}(x_0,t)}\big| \Big\|_{L^{p'(\cdot)}_{w^{-1}}(\mathcal{E}(x_0,t))} \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}} \\ &+ \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \big|(b_1)_{\mathcal{E}(x_0,t)} - (b_1)_{\mathcal{E}}\big| \Big\| b_2(\cdot) - (b_2)_{\mathcal{E}(x_0,t)} \Big\|_{L^{p'(\cdot)}_{w^{-1}}(\mathcal{E}(x_0,t))} \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}} \\ &+ \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \big|(b_2)_{\mathcal{E}(x_0,t)} - (b_2)_{\mathcal{E}}\big| \Big\| b_1(\cdot) - (b_1)_{\mathcal{E}(x_0,t)} \Big\|_{L^{p'(\cdot)}_{w^{-1}}(\mathcal{E}(x_0,t))} \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}} \\ &+ \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \prod_{j=1}^{2} \Big|(b_i)_{\mathcal{E}(x_0,t)} - (b_i)_{\mathcal{E}}\Big| \|w^{-1}\|_{L^{p'(\cdot)}(\mathcal{E}(x_0,t))} \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}} \end{aligned}$$

$$\lesssim \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \prod_{j=1}^{2} \left\|b_i(\cdot) - (b_i)_{\mathcal{E}(x_0,t)}\right\|_{L^{2p'(\cdot)}_{w^{-1}}(\mathcal{E}(x_0,t))} \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}}$$

$$+ \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \ln\frac{t}{r} \|w^{-1}\|_{L^{p'(\cdot)}(\mathcal{E}(x_0,t))} \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}}$$

$$+ \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \prod_{j=1}^{2} \left|(b_i)_{\mathcal{E}(x_0,t)} - (b_i)_{\mathcal{E}}\right| \|w^{-1}\|_{L^{p'(\cdot)}(\mathcal{E}(x_0,t))} \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}}$$

$$\lesssim \prod_{j=1}^{m} \|b_j\|_{BMO} \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \ln^2\left(e+\frac{t}{r}\right) \|w^{-1}\|_{L^{p'}(\mathcal{E}(x_0,t))} \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}}$$

$$\lesssim \prod_{j=1}^{m} \|b_j\|_{BMO} \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \ln^2\left(e+\frac{t}{r}\right) \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t}.$$

Let us estimate I_2.

$$I_2 = \left\|b_1(\cdot) - (b_1)_{\mathcal{E}}\right\|_{L^{p(\cdot)}_w(\mathcal{E})} \int_{\complement_{(2\mathcal{E})}} \frac{\left|b_2(y) - (b_2)_{\mathcal{E}}\right|}{\rho(x_0-y)^{|a|}} |f(y)| dy$$

$$\approx \left\|b_1(\cdot) - (b_1)_{\mathcal{E}}\right\|_{L^{p(\cdot)}_w(\mathcal{E})} \int_{\complement_{(2\mathcal{E})}} \left|b_2(y) - (b_2)_{\mathcal{E}}\right| |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{|a|+1}} dy$$

$$= \left\|b_1(\cdot) - (b_1)_{\mathcal{E}}\right\|_{L^{p(\cdot)}_w(\mathcal{E})} \int_{2r}^{\infty} \int_{2r\le|x_0-y|\le t} \left|b_2(y) - (b_2)_{\mathcal{E}}\right| |f(y)| dy \frac{dt}{t^{|a|+1}}$$

$$\le \left\|b_1(\cdot) - (b_1)_{\mathcal{E}}\right\|_{L^{p(\cdot)}_w(\mathcal{E})} \int_{2r}^{\infty} \int_{\mathcal{E}(x_0,t)} \left|b_2(y) - (b_2)_{\mathcal{E}}\right| |f(y)| dy \frac{dt}{t^{|a|+1}}.$$

Applying Hölder's inequality (3.1) and by Lemma 3.1, we get

$$I_2 \lesssim \|b_1\|_* \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \left\|b_2(\cdot) - (b_2)_{\mathcal{E}}\right\|_{L^{p'(\cdot)}_{w^{-1}}(\mathcal{E}(x_0,t))} \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}}$$

$$\lesssim \prod_{j=1}^{2} \|b_j\|_* \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \|w^{-1}\|_{L^{p'(\cdot)}(\mathcal{E}(x_0,t))} \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}}$$

$$\lesssim \prod_{j=1}^{2} \|b_j\|_{BMO} \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \ln^2\left(e+\frac{t}{r}\right) \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t}.$$

In the same way, we will get the result of I_3

$$I_3 \lesssim \prod_{j=1}^{m} \|b_j\|_{BMO} \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \ln^2\left(e+\frac{t}{r}\right) \|f\|_{L^{p(\cdot)}_w(\mathcal{E}(x_0,t))} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t}.$$

In order to estimate I_4 note that

$$I_4 = \Big\| \prod_{j=1}^{2} \left|b_i(\cdot) - (b_i)_{\mathcal{E}}\right| \Big\|_{L^{p(\cdot)}_w(\mathcal{E})} \int_{\complement_{(2\mathcal{E})}} \frac{|f(y)|}{\rho(x_0-y)^{|a|}} dy$$

$$\le \prod_{j=1}^{2} \left\|b_i(\cdot) - (b_i)_{\mathcal{E}}\right\|_{L^{2p(\cdot)}_w(\mathcal{E})} \int_{\complement_{(2\mathcal{E})}} \frac{|f(y)|}{\rho(x_0-y)^{|a|}} dy.$$

By Theorem 3.1, we get

$$I_4 \lesssim \prod_{j=1}^{m} \|b_j\|_{BMO} \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{\complement_{(2\mathcal{E})}} \frac{|f(y)|}{\rho(x_0-y)^{|a|}} dy.$$

Applying Hölder's inequality, we get

$$\begin{aligned} \int_{\complement_{(2\mathcal{E})}} \frac{|f(y)|}{\rho(x_0-y)^{|a|}} dy &\lesssim \int_{2r}^{\infty} \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x_0,t))} \|w^{-1}\|_{L^{p'(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}} \\ &\lesssim \int_{2r}^{\infty} \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x_0,t))} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}))(x_0,t))} \frac{dt}{t}. \end{aligned} \tag{4.1}$$

Thus, by (4.1)

$$I_4 \lesssim \prod_{j=1}^{m} \|b_j\|_{BMO} \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x_0,t))} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t}.$$

Summing up I_1 and I_4, for all $p \in [1,\infty)$ we get

$$\begin{aligned} &\|T_{\mathbf{b}} f_2\|_{L_w^{p(\cdot)}(\mathcal{E})} \\ &\lesssim \prod_{j=1}^{m} \|b_j\|_{BMO} \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int\limits_{2r}^{\infty} \ln^2\left(e+\frac{t}{r}\right) \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x_0,t))} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t}. \end{aligned} \tag{4.2}$$

On the other hand

$$\begin{aligned} \|f\|_{L_w^{p(\cdot)}(2\mathcal{E})} &\approx |B| \|f\|_{L_w^{p(\cdot)}(2\mathcal{E})} \int_{2r}^{\infty} \frac{dt}{t^{|a|+1}} \\ &\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}} \\ &\le \|w\|_{L^{p(\cdot)}(\mathcal{E})} \|w^{-1}\|_{L^{p'(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}} \\ &\le \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x_0,t))} \|w^{-1}\|_{L^{p'(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t^{|a|+1}} \\ &\lesssim \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x_0,t))} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t}. \end{aligned} \tag{4.3}$$

Finally,

$$\begin{aligned} \|T_{\mathbf{b}} f\|_{L_w^{p(\cdot)}(\mathcal{E})} &\lesssim \prod_{j=1}^{m} \|b_j\|_{BMO} \|f\|_{L_w^{p(\cdot)}(2\mathcal{E})} \\ &+ \prod_{j=1}^{m} \|b_j\|_{BMO} \|w\|_{L^{p(\cdot)}(\mathcal{E})} \int_{2r}^{\infty} \ln^m\left(e+\frac{t}{r}\right) \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x_0,t))} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x_0,t))} \frac{dt}{t}, \end{aligned}$$

and the statement of Theorem 2.2 follows by (4.3).

Proof of Theorem 2.3. By Theorem 2.2 and Theorem 3.3 we have

$$
\begin{aligned}
&\|T_{\mathbf{b}}f\|_{M_w^{p(\cdot),\varphi_2}} \\
&\lesssim \prod_{j=1}^{m} \|b_j\|_{BMO} \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \ln^m\left(e+\frac{t}{r}\right) \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x,t))} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x,t))} \frac{dt}{t} \\
&= \prod_{j=1}^{m} \|b_j\|_{BMO} \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \|w\|^{-1}_{L^{p(\cdot)}(\mathcal{E}(x,r))} \|f\|_{L_w^{p(\cdot)}(\mathcal{E}(x,r))} \\
&= \prod_{j=1}^{m} \|b_j\|_{BMO} \|f\|_{M_w^{p(\cdot),\varphi_1}}.
\end{aligned}
$$

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