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On solvability of a boundary value problem for a second order elliptic differential-operator equations with a complex linear parameter

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Abstract. *In separable Hilbert space H, we study solvability of a boundary value problem for a second order elliptic differential-operator equation in the case when one and the same complex parameter linearly enters into the equation and into one of the boundary conditions.*

Keywords. differential -operator equations · boundary value problem · complex parameter · international space.

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1 Introduction

Solvability of boundary value problems for second order elliptic differential-operator equations when one the same parameter enters into the equation and into boundary conditions were studied in various aspects in [1-6] and others. In the present paper, in a separable Hilbert space H, we study solvability of the following boundary value problem

$$L(\lambda)\,u := \lambda u(x) - u''(x) + Au(x) = f(x),\ x \in (0,1)\,, \tag{1.1}$$

$$L_1(\lambda)\,u := u'(0) + \lambda u(1) = f_1,\ \ L_2 u := u(0) = f_2, \tag{1.2}$$

where λ is a complex parameter; A is φ-positive operator in H (definition of φ -positive operator will be given below). It is proved that for sufficiently large $|\lambda|$ at some angle $|\arg\lambda| \le \varphi < \pi$ boundary value problem (1.1), (1.2) is non-coercively solvable with respect to u in the space $L_p((0,1)\,;H)$.

In view of the trace theorem non-coerciveness of boundary value problem (1.1), (1.2) is characterized by the fact that when we look for the solution of problem (1.1), (1.2) belonging to $W_p^2((0,1)\,;\,H(A),H)$, the elements f_1 and f_2 can not be taken from natural interpolation spaces $(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}$ and $(H(A),H)_{\frac{1}{2p},p}$ respectively. Note that they are taken

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from narrower interpolation spaces $\left(H(A^2),\ H\right)_{\frac{1}{2}+\frac{1}{4p},p}$ and $\left(H(A^2),\ H\right)_{\frac{1}{4}+\frac{1}{4p},p}$ respectively. In this connection, the function $f(x)$ can not be taken from the space $L_p((0,1);H)$. This function has to be taken from a narrower space, more exactly from the space $L_p\left((0,1);H\left(A^{1/2}\right)\right),\, p\in(1,+\infty)$. As a result there is no isomorphism realized between the solution belonging to $W_p^2\left((0,1);H(A),H\right)$ and the right hand side of problem (1.1), (1.2). Moreover, some non-coercive estimation in the space $L_p\left((0,1);\ H\right),\ p\in(1,+\infty)$, is established for solving boundary value problem (1.1), (1.2).

A similar situation happened in [6], where solvability of the following boundary value problem were studied in H:

$$L\left(\lambda\right)u := \lambda u(x) - u''(x) + Au(x) = f(x),\ x\in(0,1)\,, \tag{1.3}$$

$$\begin{aligned} &L_1\left(\lambda\right)u := u'\left(1\right) + \lambda Bu\left(0\right) = f_1,\\ &L_2 u := u'\left(0\right) = f_2, \end{aligned} \tag{1.4}$$

where λ is a complex parameter; A is φ -positive operator in H; B is a linear bounded operator in spaces $H(A)$ and H. It was proved that for sufficiently large $|\lambda|$ at some angle $|\arg\lambda|\le\varphi<\pi$ the problem (1.3), (1.4) is non-coercively solvable in the space $L_p\left((0,1);H\right),\ p\in(1,\infty)$, with respect to u. As was noted in [6] non-coerciveness of the boundary value problem (1.3), (1.4) manifests itself mainly in two moments.

1) It follows from the trace theorem that the element f_2 can not belong to the interpolation space $\left(H(A),H\right)_{\frac{1}{2}+\frac{1}{2p},p}$. But this element should belong to narrower interpolation space, namely the space $\left(H(A),H\right)_{\frac{1}{2p},p}$, although at the same time the element f_1 may belong to the space $\left(H(A),H\right)_{\frac{1}{2}+\frac{1}{2p},p}$.

2) The vector-function $f(x)$ cannot be taken from the space $L_p\left((0,1);H\right)$. This function should be taken from narrower space, more exactly, from the space $L_p\left((0,1);H(A^{1/2})\right)$.

Note that in the case when boundary condition (1.4) does not contain the complex parameter λ and B is a linear bounded operator subjected to the operator $A^{1/2}$ in certain sense, then solvability of the obtained boundary value problems was studied in [7], where it was proved under certain conditions the considered problem is coercively solvable in the space $L_p\left((0,1);H\right),\ p\in(1,\infty)$. A boundary value problem for equation (1.1) with the following nonregular Birkhoff-Tamarkin boundary conditions also has non-coercively properties:

$$\begin{aligned} &L_1 u := \alpha u'\left(0\right) + \beta u'\left(1\right) + \gamma u(0) + \delta u(1) = f_1,\\ &L_2 u := \alpha u\left(0\right) - \beta u\left(1\right) = f_2, \end{aligned} \tag{1.5}$$

where λ is a complex parameter; $\alpha,\beta,\gamma,\delta$ are some complex numbers satisfying the condition $\alpha\delta+\beta\gamma\ne0$. Solvability of the boundary value problem (1.1), (1.5) in the Hilbert space H, was studied in the monograph [8, Chapter 5, Section 5.5], where it was shown that for sufficiently large $|\lambda|$ at some angle $|\arg\lambda|\le\varphi<\pi$, boundary value problem (1.1), (1.5) is non-coercively solvable in the space $L_p\left((0,1);H\right)$, $1<p<\infty$.

Some spectral properties of various boundary value problems for a second order elliptic differential-operator equation on a finite segment, with the some spectral parameter in the equation and in boundary conditions were studied in [9]-[19].

We also note the papers [20], [21], where some spectral properties of boundary value problems were studied for a second order ordinary differential equation with boundary conditions similar to boundary condition (1.2).

We introduce definitions and notions used in the paper.

Let E_1 and E_2 be Banach spaces. The set $E_1\dot{+}E_2$ of all vectors of the form (u,v), where $u\in E_1, v\in E_2$, with ordinary coordinate-linear operators and with the norm

$$\|(u,v)\|_{E_1\dot{+}E_2} :\ = \|u\|_{E_1} + \|v\|_{E_2}$$

is a Banach spaces and is called a direct sum of Banach spaces E_1 and E_2.

Let E_1 and E be two Banach space. Denote by $B(E_1, E)$ a Banach space of all linear bounded operators acting from E_1 to E with an ordinary operator norm. In particular case, $B(E) := B(E, E)$.

Definition 1.1 A linear closed operator A in Hilbert space H will be called φ- positive if its domain of definition $D(A)$ is dense in H and for some $\varphi \in [0, \pi)$, for all points from the angle $|\arg \mu| \leq \varphi$ (including $\mu = 0$) there exist operators $(A + \mu I)^{-1}$, for which these μ we have the estimation

$$\left\|(A + \mu I)^{-1}\right\|_{B(H)} \leq C (1 + |\mu|)^{-1},$$

where I is a unit operator in H, $C = const > 0$. For $\varphi = 0$ the operator A is called positive.

A simple example of a φ-positive operator is a self-adjoint positive-definite operator A acting in a Hilbert space H. Note that φ-positivity of the operator A follows from φ-positivity of the operator A^{α}, $\alpha \in (0, 1)$.

Let A be a φ-positive operator in H. Since the inverse operator A^{-1} is bounded in H, then

$$H(A^n) := \left\{u : \ u \in D(A^n), \ \|u\|_{H(A^n)} = \|A^n u\|_H\right\}, \ n \in N,$$

is a Hilbert space whose is equivalent to the norm of the graph of the operator A^n.

Let A be a φ-positive operator in a Hilbert space H. Then it is known that the operator A is a generating operator analytic for $t > 0$, the semigroups e^{-tA}, and this semigroup exponentially decreases, i.e. there exist two numbers $C > 0, \ \delta_0 > 0$ such that $\left\|e^{-tA}\right\| \leq Ce^{-\delta_0 t}$, $0 \ \leq t < +\infty$.

Definition 1.2 ([see.22, theorem 1.14.5]). Let A be a φ-positive operator in the space H. Then interpolation spaces $(H(A^n), H)_{\theta,p}$ of Hilbert spaces $H(A^n)$ and H are determined by the equality

$$(H(A^n), \ H)_{\theta,p} := \left\{u : u \in H, \ \|u\|_{(H(A^n),H)_{\theta,p}}\right.$$

$$\left. := \left(\int_0^{+\infty} t^{-1+n\theta p} \left\|A^n e^{-tA} u\right\|_H^p dt\right)^{\frac{1}{p}} < \infty\right\}, \ \theta \in (0, 1), \ p > 1, \ n \in N.$$

And this time, by definition $(H(A^n), \ H)_{0,p} := H(A^n), \ \ (H(A^n), \ H)_{1,p} := H$.

By $L_p((0, 1); H)$, $1 < p < \infty$, we denote a Banach space (for $p = 2$ a Hilbert space) vector-functions $x \to u(x) : [0, 1] \to H$, strongly measurable and summable in the p-th degree, with the norm

$$\|u\|_{L_p((0,1);H)} := \left(\int_0^1 \|u(x)\|_H^p dx\right)^{1/p} < \infty,$$

and by $W_p^n((0, 1); H(A^n), H) := \left\{u : A^n u, \ \ u^{(n)} \in L_p((0, 1); H)\right\}$ a Banach space of vector-functions with the norm

$$\|u\|_{W_p^n((0,1);H(A^n),H)} := \|A^n u\|_{L_p((0,1);H)} + \left\|u^{(n)}\right\|_{L_p((0,1);H)}.$$

It is known that [22, theorem 1.8.2] if $u \in W_p^n((0, 1); H(A^n), H)$, then

$$u^{(j)}(\cdot) \in (H(A^n), H)_{\frac{j+\frac{1}{p}}{n}, p}, \ \ j = 0, \ldots, n - 1.$$

By Ff and $F^{-1}f$ we denote Fourier transformation and inverse Fourier transformation of the function f from $L_p(R;H)$, where $R=(-\infty,+\infty)$.

$$Ff:=(Ff)(\sigma):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i\sigma x}f(x)dx,$$

$$F^{-1}f:=\left(F^{-1}f\right)(x):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{i\sigma x}f(\sigma)d\sigma\,.$$

The mapping $\sigma\to T(\sigma):R\to B(H)$ is called a Fourier multiplicator in the space $f\in L_p(R;H)$, if

$$\left\|F^{-1}TFf\right\|_{L_p(R;H)}\le C\left\|f\right\|_{L_p(R;H)},\ f\in L_p(R;H)\,.$$

It is known [23, p.346] that if the mapping $\sigma\to T(\sigma):R\to B(H)$ is continuously differentiable and

$$\|T(\sigma)\|\le C,\quad \left\|T'(\sigma)\right\|\le C\left|\sigma\right|^{-1},\ \ \sigma\in R,$$

then the function $T(\sigma)$ is a Fourier multiplicator (Mikhlin-Schwarts theorem).

2 Homogeneous equation

First, in a separable Hilbert space H, we consider the following boundary value problem

$$L(\lambda)u:=\lambda u(x)-u''(x)+Au(x)=0,x\in(0,\,1)\,, \tag{2.1}$$

$$\begin{array}{l}L_1(\lambda)u:=u'(0)+\lambda u(1)=f_1\,,\\ L_2u:=u(0)=f_2,\end{array} \tag{2.2}$$

where λ is a complex parameter; A is φ-positive operator in H.

Theorem 2.1 *Let A be φ-positive operator in H, where $\varphi\in[0,\frac{\pi}{2})$. Then for*

$$f_k\in\left(H(A^2),H\right)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p},\ p\in(1,\,+\infty)\,,\ k=1,\,2$$

and for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\le\varphi<\pi$, problem (2.1), (2.2) has a unique solution $u\in W_p^2\left((0,1)\,;H\left(A\right),H\right)$ and the following non-coercive estimation is valid for solving it

$$|\lambda|\left\|u\right\|_{L_p((0,1);H)}+\left\|u''\right\|_{L_p((0,1);H)}+\left\|Au\right\|_{L_p((0,1);H)}$$

$$\le C\sum_{k=1}^{2}\left(\left\|f_k\right\|_{\left(H(A^2),H\right)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}}+|\lambda|^{\frac{k+1}{2}-\frac{1}{2p}}\left\|f_k\right\|_H\right), \tag{2.3}$$

where $C>0$ is some constant independent of λ.

Proof. By [8, lemma 5.4.2/6], for $|\arg\lambda| \le \varphi < \pi$, there exists an analytic for $x > 0$, and strongly continuous for $x \ge 0$, semigroup $e^{-x(A+\lambda I)^{1/2}}$. By [8, lemma 5.3.2/1], for the function $u(x)$ to be the solution of equation (2.1), belonging to $W_p^2((0,1); H(A), H)$, $p \in (1, +\infty)$, it is necessary and sufficient that for $|\arg\lambda| \le \varphi < \pi$,

$$u(x) = e^{-x(A+\lambda I)^{1/2}} g_1 + e^{-(1-x)(A+\lambda I)^{1/2}} g_2, \tag{2.4}$$

where $g_1, g_2 \in (H(A), H)_{\frac{1}{2p},p}$.

Require the function of the form (2.4) satisfy boundary conditions (2.2). Then with respect to the elements g_1 and g_2 we get the following system

$$\begin{aligned} &\left[-(A+\lambda I)^{1/2} + \lambda e^{-(A+\lambda I)^{1/2}}\right] g_1 + \left[(A+\lambda I)^{1/2} e^{-(A+\lambda I)^{1/2}} + \lambda I\right] g_2 = f_1, \\ &g_1 + e^{-(A+\lambda I)^{1/2}} g_2 = f_2. \end{aligned} \tag{2.5}$$

Define $v_1 := (A+\lambda I)^{1/2} g_1,\ \ v_2 := (A+\lambda I)^{1/2} g_2$. Then from (2.5) we get

$$\begin{aligned} &\left[-I + \lambda (A+\lambda I)^{-1/2} e^{-(A+\lambda I)^{1/2}}\right] v_1 + \left[e^{-(A+\lambda I)^{1/2}} + \lambda (A+\lambda I)^{-1/2}\right] v_2 = f_1, \\ &(A+\lambda I)^{-1/2} v_1 + (A+\lambda I)^{-1/2} e^{-(A+\lambda I)^{1/2}} v_2 = f_2. \end{aligned} \tag{2.6}$$

The coefficients of the system (2.5) are linear combinations of bounded, commutating operators I, $(A+\lambda I)^{-1/2}$, $e^{-(A+\lambda I)^{\frac{1}{2}}}$, $(A+\lambda I)^{-1/2} e^{-(A+\lambda I)^{1/2}}$. Therefore the system (2.6) may be solved as in the scalar case. The "determinant" of the system (2.6) is of the form

$$D(\lambda) = -\lambda (A+\lambda I)^{-1} \left[I - \lambda^{-1}(A+\lambda I) R(\lambda)\right],$$

where

$$R(\lambda) := \lambda (A+\lambda I)^{-1} e^{-2(A+\lambda I)^{1/2}} - 2(A+\lambda I)^{-1/2} e^{-(A+\lambda I)^{1/2}}.$$

By [8, lemma 5.4.2/6], $\left\|\lambda^{-1}(A+\lambda I)R(\lambda)\right\|_{B(H)} \to 0$, for $|\arg\lambda| \le \varphi < \pi$, $|\lambda| \to \infty$. Consequently, for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$, the system (2.6) has a unique solution with respect to $v_k,\ k = 1, 2$, and

$$v_1 = R_{11}(\lambda) f_1 + \left[(A+\lambda I)^{1/2} + R_{21}(\lambda)\right] f_2,$$

$$v_2 = \left[\lambda^{-1}(A+\lambda I)^{1/2} + R_{12}(\lambda)\right] f_1 + \left[\lambda^{-1}(A+\lambda I) + R_{22}(\lambda)\right] f_2,$$

where

$$\|R_{jk}(\lambda)\|_{B(H)} \to 0\ (j,k = 1,2), \text{ for } |\arg\lambda| \le \varphi,\ |\lambda| \to \infty.$$

Consequently

$$g_1 = (A+\lambda I)^{-1/2} R_{11}(\lambda) f_1 + \left[I + (A+\lambda I)^{-1/2} R_{21}(\lambda)\right] f_2,$$

$$\begin{aligned} g_2 = &\left[\lambda^{-1} I + (A+\lambda I)^{-1/2} R_{12}(\lambda)\right] f_1 \\ &+ \left[\lambda^{-1}(A+\lambda I)^{1/2} + (A+\lambda I)^{-1/2} R_{22}(\lambda)\right] f_2. \end{aligned} \tag{2.7}$$

Having substituted (2.7) in (2.2) we obtain

$$u(x) = e^{-x(A+\lambda I)^{1/2}} \left[(A+\lambda I)^{-1/2} R_{11}(\lambda) f_1 + \left(I + (A+\lambda I)^{-1/2} R_{21}(\lambda) \right) f_2 \right]$$

$$+ e^{-(1-x)(A+\lambda I)^{1/2}} \left[\left(\lambda^{-1} I + (A+\lambda I)^{-1/2} R_{12}(\lambda) \right) f_1 \right.$$

$$\left. + \left(\lambda^{-1}(A+\lambda I)^{1/2} + (A+\lambda I)^{-1/2} R_{22}(\lambda) \right) f_2 \right].$$

Then, for sufficiently large $|\lambda|$ from the angle $|\arg \lambda| \le \varphi < \pi$, we have

$$|\lambda| \, \|u\|_{L_p((0,1);H)} + \left\| u'' \right\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)}$$

$$\le |\lambda| \left[\|R_{11}(\lambda)\| \left(\int_0^1 \left\| e^{-x(A+\lambda I)^{1/2}} (A+\lambda I)^{-1/2} f_1 \right\|_H^p dx \right)^{1/p} \right.$$

$$+ \left(\int_0^1 \left\| e^{-x(A+\lambda I)^{1/2}} f_2 \right\|_H^p dx \right)^{1/p} + \|R_{21}(\lambda)\| \left(\int_0^1 \left\| e^{-x(A+\lambda I)^{1/2}} (A+\lambda I)^{-1/2} f_2 \right\|_H^p dx \right)^{1/p}$$

$$+ \left(\int_0^1 \left\| e^{-(1-x)(A+\lambda I)^{1/2}} \lambda^{-1} f_1 \right\|_H^p dx \right)^{1/p}$$

$$+ \|R_{12}(\lambda)\| \left(\int_0^1 \left\| e^{-(1-x)(A+\lambda I)^{1/2}} (A+\lambda I)^{-1/2} f_1 \right\|_H^p dx \right)^{1/p}$$

$$+ \left(\int_0^1 \left\| e^{-(1-x)(A+\lambda I)^{1/2}} \lambda^{-1} (A+\lambda I)^{1/2} f_2 \right\|_H^p dx \right)^{1/p}$$

$$\left. + \|R_{22}(\lambda)\| \left(\int_0^1 \left\| e^{-(1-x)(A+\lambda I)^{1/2}} (A+\lambda I)^{-1/2} f_2 \right\|_H^p dx \right)^{1/p} \right]$$

$$+ (1 + \|A(A+\lambda I)^{-1}\|) \left[\|R_{11}(\lambda)\| \left(\int_0^1 \left\| (A+\lambda I)^{1/2} e^{-x(A+\lambda I)^{1/2}} f_1 \right\|_H^p dx \right)^{1/p} \right.$$

$$+ \left(\int_0^1 \left\| (A+\lambda I) e^{-x(A+\lambda I)^{1/2}} f_2 \right\|_H^p dx \right)^{1/p}$$

$$+ \|R_{21}(\lambda)\| \left(\int_0^1 \left\| (A+\lambda I)^{1/2} e^{-x(A+\lambda I)^{1/2}} f_2 \right\|_H^p dx \right)^{1/p}$$

$$+ \left(\int_0^1 \left\| (A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} \lambda^{-1} f_1 \right\|_H^p dx \right)^{1/p}$$

$$+\|R_{12}(\lambda)\|\left(\int_0^1\left\|(A+\lambda I)^{1/2}e^{-(1-x)(A+\lambda I)^{1/2}}f_1\right\|_H^p dx\right)^{1/2}$$

$$+\left(\int_0^1\left\|(A+\lambda I)e^{-(1-x)(A+\lambda I)^{1/2}}\lambda^{-1}(A+\lambda I)^{1/2}f_2\right\|_H^p dx\right)^{1/p}$$

$$\left.+\|R_{22}(\lambda)\|\left(\int_0^1\left\|(A+\lambda I)^{1/2}e^{-(1-x)(A+\lambda I)^{1/2}}f_2\right\|_H^p dx\right)^{1/p}\right]. \tag{2.8}$$

Let us estimate some integrals in the right side of inequality (2.8). First, we estimate the following integral

$$I_1=\left(\int_0^1\left\|(A+\lambda I)e^{-(1-x)(A+\lambda I)^{1/2}}\frac{1}{\lambda}(A+\lambda I)^{1/2}f_2\right\|_H^p dx\right)^{1/p}.$$

From the representation

$$\frac{1}{\lambda}(A+\lambda I)^{1/2}A^{-1/2}=A^{1/2}(A+\lambda I)^{-1/2}\frac{1}{\lambda}(A+\lambda I)A^{-1},$$

by [8, lemma 5.4.2 /6], it follows that for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\le\varphi<\pi$, the operator $\frac{1}{\lambda}(A+\lambda I)^{1/2}A^{-1/2}$ boundedly acts from H to H and we have the estimations

$$\left\|\frac{1}{\lambda}(A+\lambda I)^{1/2}A^{-1/2}\right\|_{B(H)}\le\left\|A^{1/2}(A+\lambda I)^{-1/2}\right\|_{B(H)}\left\|\frac{1}{\lambda}I+A^{-1}\right\|_{B(H)}$$

$$\le C\left(\frac{1}{|\lambda|}+\|A^{-1}\|_{B(H)}\right)\le C,\quad\exists C>0. \tag{2.9}$$

From estimation (2.9), it follows that for sufficiently large $|\lambda|$ from the angle $|arg\lambda|\le\varphi<\pi$ the operator $\frac{1}{\lambda}(A+\lambda I)^{1/2}\ A^{-1/2}$ boundedly acts from $H(A)$ to $H(A)$ and also from $H(A^2)$ to $H(A^2)$ and we have the estimations

$$\left\|\frac{1}{\lambda}(A+\lambda I)^{1/2}A^{-1/2}\right\|_{B(H(A))}=\left\|A\frac{1}{\lambda}(A+\lambda I)^{-1/2}A^{-1/2}A^{-1}\right\|_{B(H)}\le C. \tag{2.10}$$

$$\left\|\frac{1}{\lambda}(A+\lambda I)^{1/2}A^{-1/2}\right\|_{B(H(A^2))}=\left\|\frac{1}{\lambda}(A+\lambda I)^{1/2}A^{-1/2}\right\|_{B(H)}\le C. \tag{2.11}$$

From estimations (2.9), (2.10), and (2.9), (2.11) by the interpolation theorem [22, theorem 1.3.3/(a)] (see. also [8, section 1.7.9]), it follows that for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\le\varphi<\pi$, the operator $\frac{1}{\lambda}(A+\lambda I)^{1/2}A^{-1/2}$ boundedly acts from $(H(A),H)_{\theta,p}$ to $(H(A),H)_{\theta,p}$ and from $\left(H\left(A^2\right),H\right)_{\theta,p}$ to $\left(H\left(A^2\right),H\right)_{\theta,p}$ for any $\theta\in(0,1)$ and we have the estimations

$$\left\|\frac{1}{\lambda}(A+\lambda I)^{1/2}A^{-1/2}\right\|_{B\left((H(A),H)_{\theta,p}\right)}\le C,\ \exists C>0 \tag{2.12}$$

$$\left\| \frac{1}{\lambda} (A + \lambda I)^{1/2} A^{-1/2} \right\|_{B\left((H(A^2),H)_{\theta,p}\right)} \leq C, \ \exists C > 0. \tag{2.13}$$

By [8, lemma 5.4.2/6 and theorem 5.4.2/1] and estimations (2.9), (2.13), for sufficiently large $|\lambda|$ from angle $|\arg \lambda| \leq \varphi < \pi$, we have

$$I_1 = \left(\int_0^1 \left\| A^{1/2}(A+\lambda I)^{-1/2}(A+\lambda I)^{3/2} e^{-(1-x)(A+\lambda I)^{1/2}} \frac{1}{\lambda}(A+\lambda I)^{1/2} A^{-1/2} f_2 \right\|_H^p dx \right)^{1/p}$$

$$\leq \left\| A^{1/2}(A+\lambda I)^{-1/2} \right\|_{B(H)}$$

$$\times \left(\int_0^1 \left\| (A+\lambda I)^{3/2} e^{-(1-x)(A+\lambda I)^{1/2}} \frac{1}{\lambda}(A+\lambda I)^{1/2} A^{-1/2} f_2 \right\|_H^p dx \right)^{1/p}$$

$$\leq C \left(\left\| \frac{1}{\lambda}(A+\lambda I)^{1/2} A^{-1/2} f_2 \right\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} + |\lambda|^{\frac{3}{2}-\frac{1}{2p}} \left\| \frac{1}{\lambda}(A+\lambda I)^{1/2} A^{-1/2} f_2 \right\|_H \right)$$

$$\leq C \left(\|f_2\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} + |\lambda|^{\frac{3}{2}-\frac{1}{2p}} \|f_2\|_H \right).$$

We now estimate the integral

$$I_2 = \left(\int_0^1 \left\| (A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} \lambda^{-1} f_1 \right\|_H^p dx \right)^{1/p}.$$

By [8, lemma 5.4.2 /6 and theorem 5.4.2./1] and estimations (2.9), (2.12) for sufficently large $|\lambda|$ from the angle $|\arg \lambda| \leq \varphi < \pi$, we get

$$I_2 = \left(\int_0^1 \left\| (A+\lambda I)^{1/2} A^{1/2} e^{-(1-x)(A+\lambda I)^{1/2}} \frac{1}{\lambda}(A+\lambda I)^{1/2} A^{-1/2} f_2 \right\|_H^p dx \right)^{1/p}$$

$$\leq \left\| A^{1/2}(A+\lambda^2 I)^{-1/2} \right\|_{B(H)}$$

$$\times \left(\int_0^1 \left\| (A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} \frac{1}{\lambda}(A+\lambda I)^{1/2} A^{-1/2} f_2 \right\|_H^p dx \right)^{1/p}$$

$$\leq C \left(\left\| \frac{1}{\lambda}(A+\lambda I)^{1/2} A^{-1/2} f_2 \right\|_{(H(A),H)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} \left\| \frac{1}{\lambda}(A+\lambda I)^{1/2} A^{-1/2} f_2 \right\|_H \right)$$

$$\leq C \left(\|f_1\|_{(H(A),H)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} \|f_1\|_H \right) = C \left(\|f_1\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} + |\lambda|^{1-\frac{1}{2p}} \|f_1\|_H \right).$$

The remaning addends of the right side of the inequality are estimated (2.8) in the same may and we get the requred estimation (2.3). The theorem is proved.

3 Inhomogeneous equation

We now consider a boundary value problem for an inhomogeneous equation with a quadratic complex parameter in H, i.e. the problem

$$L(\lambda)\,u := \lambda u(x) - u''(x) + Au(x) = f(x)\,, \quad x \in (0,1)\,, \tag{3.1}$$

$$\begin{aligned} &L_1(\lambda)\,u := u'(0) + \lambda\, u(1) = f_1, \\ &L_2 u := u(0) = f_2\,. \end{aligned} \tag{3.2}$$

Theorem 3.1 *Let A be φ-positive operator in H. Then for $f \in L_p\left((0,1);\ H\left(A^{1/2}\right)\right)$, $f_k \in \left(H\left(A^2\right), H\right)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}$, $p \in (1,+\infty)$, and for sufficently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$, problem* (3.1), (3.2) *has a unique solution $u \in W_p^2\left((0,1)\,; H(A)\,, H\right)$ and for solving it we have the following non-coercive estimation*

$$|\lambda|\,\|u\|_{L_p((0,1);H)} + \left\|u''\right\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)}$$
$$\le C\left[|\lambda|\;\|f\|_{L_p((0,1);H(A^{1/2}))} + \sum_{k=1}^{2}\left(\|f_k\|_{(H(A^2),H)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}} + |\lambda|^{\frac{k+1}{2}-\frac{1}{2p}}\,\|f_k\|_H\right)\right]. \tag{3.3}$$

Proof. Uniqueness follows from theorem 2.1. Replace the solution of problem (3.1), (3.2), belonging to $W_p^2\left((0,1)\,; H(A)\,, H\right)$, in the from of the sum $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is a contraction on the segment $[0,1]$ of the solution of the equation

$$L(\lambda)\,\tilde{u}_1(x) := \lambda\tilde{u}_1(x) - \tilde{u}_1''(x) + A\tilde{u}_1(x) = \tilde{f}(x)\,, x \in (-\infty,\,+\infty)\,, \tag{3.4}$$

where $\tilde{f}(x) := f(x)$, if $x \in [0,1]$ and $\tilde{f}(x) = 0$, if $x \notin [0,1]$, and $u_2(x)$ is the solution of the problem

$$\begin{gathered} L(\lambda)\,u_2 = 0,\ x \in (0,1)\,, \\ L_1(\lambda)\,u_2 = f_1 - L_1(\lambda)\,u_1, \\ L_2u_2 = f_2 - L_2u_1. \end{gathered} \tag{3.5}$$

As was shown in the proof [8, theorem 5.4.4], the solution of the equation (3.4) is given by the formulas

$$\tilde{u}_1(x) = \frac{1}{\sqrt{2\pi}}\int_R e^{i\mu x} L(\lambda,\, i\mu)^{-1}\, F\,\tilde{f}(\mu)\, d\mu, \tag{3.6}$$

where $F\,\tilde{f}$ is a Fourier transformation of the function $\tilde{f}(x)$, and $L(\lambda,\sigma)$ is a characteristic operator pensil of the equation (3.4), i.e.,

$$L(\lambda,\sigma) = -\sigma^2 I + A + \lambda I,\ \ |\arg\lambda| \le \varphi < \pi.$$

From (3.6) it follows that

$$\|\tilde{u}_1\|_{W_p^2\left(R;H(A^{3/2}),H(A^{1/2})\right)} = \|\tilde{u}_1\|_{L_p\left(R;H(A^{3/2})\right)} + \left\|\tilde{u}_1''\right\|_{L_p\left(R;H(A^{1/2})\right)}$$
$$= \left\|\left(F^{-1}L(\lambda,i\mu)^{-1}\,F\tilde{f}(\mu)\right)(\cdot)\right\|_{L_p\left(R;H(A^{3/2})\right)}$$
$$+ \left\|\left(F^{-1}(i\mu)^2\,L(\lambda,i\mu)^{-1}\,F\tilde{f}(\mu)\right)(\cdot)\right\|_{L_p\left(R;H(A^{1/2})\right)}$$

$$\leq \left\| \left(F^{-1} AL\left(\lambda, i\mu\right)^{-1} F\tilde{f}\left(\mu\right)\right)(\cdot)\right\|_{L_p\left(R;H(A^{1/2})\right)}$$

$$+\left\| \left(F^{-1}\left(i\mu\right)^2 L\left(\lambda, i\mu\right)^{-1} F\tilde{f}\left(\mu\right)\right)(\cdot)\right\|_{L_p\left(R;H(A^{1/2})\right)}. \tag{3.7}$$

Show that the operator-function (with respect to μ)

$$T_{k+1}(\lambda,\mu) := (i\mu)^{2k} A^{1-k} L(\lambda, i\mu)^{-1},\ k=0,1, \tag{3.8}$$

are Fourier multiplications in the space $L_p(R;H(A^{1/2}))$. Since for $|\arg\lambda| \leq \varphi < \pi$ and $\mu \in R$, $|\arg(\lambda^2+\mu^2)| \leq \varphi < \pi$, then by the condition of theorem 3.1 we have the estimations

$$\left\|L(\lambda,i\mu)^{-1}\right\|_{B(H)} = \left\|\left(A+\left(\lambda+\mu^2\right)I\right)^{-1}\right\|_{B(H)} \leq \frac{C}{1+|\lambda+\mu^2|} \leq \frac{C}{\mu^2}; \tag{3.9}$$

$$\left\|AL(\lambda,i\mu)^{-1}\right\|_{B(H)} = \left\|A\left(A+\left(\lambda+\mu^2\right)I\right)^{-1}\right\|_{B(H)}$$

$$= \left\|I-\left(\lambda+\mu^2\right)\left(A+\left(\lambda+\mu^2\right)I\right)^{-1}\right\|_{B(H)} \leq 1+\left|\lambda+\mu^2\right|\frac{C}{1+|\lambda+\mu^2|} \leq C, \tag{3.10}$$

uniformly with respect to λ in the angle $|\arg\lambda| \leq \varphi < \pi$.

From estimates (3.9), (3.10), for $|\arg\lambda| \leq \varphi < \pi$ and $\mu \in R$, we get

$$\|T_1(\lambda,\mu)\|_{B(H(A^{1/2}))} = \left\|AL\left(\lambda,i\mu\right)^{-1}\right\|_{B\left(H\left(A^{1/2}\right)\right)} = \left\|AL\left(\lambda,i\mu\right)^{-1}\right\|_{B(H)} \leq C;$$

$$\|T_2(\lambda,\mu)\|_{B(H(A^{1/2}))} = \left\|\left(i\mu\right)^2 L\left(\lambda,i\mu\right)^{-1}\right\|_{B\left(H\left(A^{1/2}\right)\right)} = \left\|\left(i\mu\right)^2 L\left(\lambda,i\mu\right)^{-1}\right\|_{B(H)}$$

$$= |\mu|^2 \cdot \left\|L\left(\lambda,i\mu\right)^{-1}\right\|_{B(H)} \leq C \tag{3.11}$$

uniformly with respect to λ is the angle $|\arg\lambda| \leq \varphi < \pi$.

Since

$$\frac{\partial}{\partial\mu}T_1\left(\lambda,\mu\right) = A\frac{\partial}{\partial\mu}L\left(\lambda,i\mu\right)^{-1} = -AL\left(\lambda,i\mu\right)^{-1}\cdot 2\mu\, L\left(\lambda,i\mu\right)^{-1};$$

$$\frac{\partial}{\partial\mu}T_2\left(\lambda,\mu\right) = -2\mu\, L\left(\lambda,i\mu\right)^{-1} + \mu^2\, L\left(\lambda,i\mu\right)^{-1} 2\mu\, L\left(\lambda,i\mu\right)^{-1},$$

it follows from (3.9) and (3.10) that

$$\left\|\frac{\partial}{\partial\mu}T_1\left(\lambda,\mu\right)\right\|_{B(H(A^{1/2}))} = \left\|AL\left(\lambda,i\mu\right)^{-1} 2\mu\, L\left(\lambda,i\mu\right)^{-1}\right\|_{B(H(A^{1/2}))}$$

$$= \left\|AL\left(\lambda,i\mu\right)^{-1} 2\mu\, L\left(\lambda,i\mu\right)^{-1}\right\|_{B(H)} \leq C\left|\mu\right|; \tag{3.12}$$

$$\left\|\frac{\partial}{\partial\mu}T_2\left(\lambda,\mu\right)\right\|_{B(H(A^{1/2}))} \leq 2\left|\mu\right| \left\|L\left(\lambda,i\mu\right)^{-1}\right\|_{B(H(A^{1/2}))}$$

$$+\left|\mu\right|^2 \left\|L\left(\lambda,i\mu\right)^{-1}\right\|_{B(H(A^{1/2}))} 2\left|\mu\right| \left\|L\left(\lambda,i\mu\right)^{-1}\right\|_{B(H(A^{1/2}))} = 2\left|\mu\right| \left\|L\left(\lambda,i\mu\right)^{-1}\right\|_{B(H)}$$

$$+\left|\mu\right|^2 \left\|L\left(\lambda,i\mu\right)^{-1}\right\|_{B(H)} 2\left|\mu\right| \left\|L\left(\lambda,i\mu\right)^{-1}\right\|_{B(H)} \leq C\left|\mu\right|^{-1}. \tag{3.13}$$

By [8, theorem 1.3.7/1], estimations (3.11)-(3.13) imply that the operator-functions $\mu \to T_{k+1}(\lambda, \mu)$, $k = 0, 1$, determined by the equalities (3.8), are Fourier multiplications in the space $L_p\left((0,1); H(A^{1/2})\right)$. Then by (3.7) we have the following estimation

$$\|\tilde{u}_1\|_{W_p^2\left(R; H(A^{3/2}), H(A^{1/2})\right)} \leq C \left\|\tilde{f}\right\|_{L_p\left(R; H(A^{1/2})\right)} \tag{3.14}$$

uniformly with respect to λ.

It follows from (3.14) that $u_1 \in W_p^2\left((0,1); H\left(A^{3/2}\right), H\left(A^{1/2}\right)\right)$ and we have the estimation

$$\|u_1\|_{W_p^2\left((0,1); H(A^{3/2}), H(A^{1/2})\right)} \leq C \|f\|_{L_p\left((0,1); H(A^{1/2})\right)}. \tag{3.15}$$

By continuity of the embedding $W_p^2\left((0,1); H\left(A^{3/2}\right), H(A^1 1/2)\right) \subset$

$\subset W_p^2\left((0,1);\ H(A), H\right)$, from (3.15) we get

$$\|u_1\|_{W_p^2((0,1); H(A), H)} \leq C \|f\|_{L_p\left((0,1); H(A^{1/2})\right)}. \tag{3.16}$$

Then form (3.4) and (3.17), for λ from the angle $|\arg \lambda| \leq \varphi < \pi$, we obtain

$$\begin{aligned} &|\lambda| \|u_1\|_{L_p((0,1);H)} + \left\|u_1''\right\|_{L_p((0,1);H)} + \|Au_1\|_{L_p((0,1);H)} \\ &\qquad \leq C \|f\|_{L_p\left((0,1); H\left(A^{1/2}\right)\right)}. \end{aligned} \tag{3.17}$$

Indeed, by (3.4) for $u_1(x)$ we have

$$\lambda u_1(x) = f(x) + u_1''(x) - Au_1(x), \ \ x \in (0,1).$$

Hence, in view of (3.16), we get

$$\begin{aligned} |\lambda| \|u_1\|_{L_p((0,1);H)} &\leq \|f\|_{L_p((0,1);H)} + \left\|u_1''\right\|_{L_p((0,1);H)} + \|Au_1\|_{L_p((0,1);H)} \\ &\leq \|f\|_{L_p((0,1);H)} + C \|f\|_{L_p\left((0,1); H\left(A^{1/2}\right)\right)} \leq C \|f\|_{L_p\left((0,1); H\left(A^{1/2}\right)\right)}. \end{aligned} \tag{3.18}$$

Then inequality (3.17) follows from inequalities (3.16) and (3.18). By [8, theorem 1.7.7/1], for any fixed $x_0 \in [0,1]$ and $s = 0, 1$, we have

$$u_1^{(s)}(x_0) \in \left(H\left(A^{3/2}\right), H(A^{1/2})\right)_{\frac{s}{2} + \frac{1}{2p}, p}.$$

Moreover, by [8, lemma 1.7.3/1, 1.7.3/6 and 1.7.3/5], for $s = 0, 1$, we have

$$\left(H\left(A^{3/2}\right), H(A^{1/2})\right)_{\frac{1+sp}{2p}, p} = \left(H\left(A^2\right), H\right)_{\frac{1}{4} + \frac{1+2p}{4p}, p}.$$

Consequently, for any fixed $x_0 \in [0,1]$,

$$u_1'(x_0) \in \left(H\left(A^2\right), H\right)_{\frac{1}{2} + \frac{1}{4p}, p} = \left(H(A), H\right)_{\frac{1}{2p}, p},$$

$$u_1(x_0) \in \left(H\left(A^2\right), H\right)_{\frac{1}{4} + \frac{1}{4p}, p}.$$

Obviously, for λ from the angle $|\arg \lambda| \leq \varphi < \pi$, $\lambda u_1(1) \in \left(H\left(A^2\right), H\right)_{\frac{1}{2} + \frac{1}{4p}, p}$, because the embedding $\left(H\left(A^2\right), H\right)_{\frac{1}{4} + \frac{1}{4p}, p} \subset \left(H\left(A^2\right), H\right)_{\frac{1}{2} + \frac{1}{4p}, p}$ is continuous. Consequently, $L_1(\lambda)\ u_1 \in \left(H\left(A^2\right), H\right)_{\frac{1}{2} + \frac{1}{4p}, p}$, $L_2 u_1 \in \left(H\left(A^2\right), H\right)_{\frac{1}{4} + \frac{1}{4p}, p}$.

Then by virtue of Theorem 2.1, for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$, for solving problem (3.5) we have

$$|\lambda| \|u_2\|_{L_p((0,1);H)} + \left\|u_2''\right\|_{L_p((0,1);H)} + \|Au_2\|_{L_p((0,1);H)}$$

$$\le C \Bigg(\|f_1\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} + \|f_2\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} + |\lambda|^{1-\frac{1}{2p}} \|f_1\|_H$$

$$+ |\lambda|^{\frac{3}{2}} \|f_2\|_H + \left\|u_1'(0)\right\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} + |\lambda| \|u_1(1)\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}}$$

$$+ \|u_1(0)\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} + |\lambda|^{1-\frac{1}{2p}} \left\|u_1'(0)\right\|_H + |\lambda|^{2-\frac{1}{2p}} \|u_1(1)\|_H + |\lambda|^{\frac{3}{2}-\frac{1}{2p}} \|u_1(0)\|_H \Bigg) . \tag{3.19}$$

We estimate the norm $|\lambda| \|u_1(1)\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}}$. Taking into account continuity of the embedding $\left(H\left(A^2\right), H\right)_{\frac{1}{4}+\frac{1}{4p},p} \subset \left(H\left(A^2\right), H\right)_{\frac{1}{2}+\frac{1}{4p},p}$, by [22, theorem 1.8.2] (see also [8, theorem 1.7.7/1]) and estimation (3.15), for λ from the angle $|\arg\lambda| \le \varphi < \pi$, we have

$$\begin{aligned} &|\lambda| \|u_1(1)\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} \le C|\lambda| \|u_1(1)\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} \\ &\le C|\lambda| \|u_1\|_{W_p^2\left((0,1);H(A^{3/2}),H(A^{1/2})\right)} \le C|\lambda| \|f\|_{L_p((0,1);H(A^{1/2}))} . \end{aligned} \tag{3.20}$$

Furthermore, by [22, theorem 1.8.2] and estimation (3.15), we get the estimations

$$\left\|u_1'(0)\right\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} \le C\|u_1\|_{W_p^2\left((0,1);H(A^{3/2}),H(A^{1/2})\right)} \le C\|f\|_{L_p((0,1);H(A^{1/2}))} ; \tag{3.21}$$

$$\|u_1(0)\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} \le C\|u_1\|_{W_p^2\left((0,1);H(A^{3/2}),H(A^{1/2})\right)} \le C\|f\|_{L_p((0,1);H(A^{1/2}))} . \tag{3.22}$$

We now estimate the norms $|\lambda|^{1-\frac{1}{2p}}\|u_1'(0)\|_H$, $|\lambda|^{2-\frac{1}{2p}}\|u_1(1)\|_H$ and $|\lambda|^{\frac{3}{2}-\frac{1}{2p}}\|u_1(0)\|_H$.

By [8, theorem 1.7.7/2], for $\lambda \in \mathbb{C}$ and $u_1 \in W_p^2((0,1);H)$, the following inequality is valid

$$|\mu|^k \left\|u_1^{(2-k)}(x_0)\right\|_H \le C\left(|\mu|^{\frac{1}{p}} \|u_1\|_{W_p^2((0,1);H)} + |\mu|^{2+\frac{1}{p}} \|u_1\|_{L_p((0,1);H)}\right), \tag{3.23}$$

where $x_0 \in [0,1]$, $s = \{0,1\}$, $p \in (1,+\infty)$.

Having divided (3.23) by $|\mu|^{\frac{1}{p}}$, and denoting $\lambda = \mu^2$ for $\lambda \in \mathbb{C}$, $|\lambda| \to \infty$, we get

$$|\lambda|^{\frac{k}{2}-\frac{1}{2p}} \left\|u_1^{(2-k)}(x_0)\right\|_H \le C\left(\|u_1\|_{W_p^2((0,1);H)} + |\lambda| \|u_1\|_{L_p((0,1);H)}\right). \tag{3.24}$$

In inequality (3.24) we take $k = 1,\ x_0 = 0$ and multiply the obtained inequality by $|\lambda|^{1/2}$. Then by inequality (3.17), for λ from the angle $|\arg\lambda| \le \varphi < \pi$, we have

$$|\lambda|^{1-\frac{1}{2p}} \left\|u_1'(0)\right\|_H \le C|\lambda|^{1/2}\left(\|u_1\|_{W_p^2((0,1);H)} + |\lambda| \|u_1\|_{L_p((0,1);H)}\right)$$

$$\le C|\lambda|^{1/2}\left(\|u_1\|_{W_p^2((0,1);H(A),H)} + |\lambda| \|u_1\|_{L_p((0,1);H)}\right) \le C|\lambda|^{1/2} \|f\|_{L_p\left((0,1);H(A^{1/2})\right)} . \tag{3.25}$$

Now in inequality (3.24), we take $k = 2$ and $x_0 = 1$ and multiply the obtained inequality by $|\lambda|$. Then by virtue of the inequality (3.17), for λ from the angle $|\arg\lambda| \le \varphi < \pi$ we get

$$|\lambda|^{2-\frac{1}{p}} \|u_1(1)\|_H \le C|\lambda| \left(\|u_1\|_{W_p^2((0,1),H)} + |\lambda| \|u_1\|_{L_p((0,1);H)}\right)$$

$$\leq C\,|\lambda|\left(\|u_1\|_{W_p^2((0,1);H(A),H)}+|\lambda|\,\|u_1\|_{L_p((0,1);H)}\right)$$
$$\leq C\,|\lambda|\,\|f\|_{L_p\left((0,1);H(A^{1/2})\right)}\,. \tag{3.26}$$

We now take in inequality (3.24) , $k=2$ and $x_0=0$ and multiply the obtained inequality by $|\lambda|^{1/2}$. Then by virtue of inequality (3.17), for λ from the angle $|\arg\lambda|\leq\varphi<\pi$ we have

$$|\lambda|^{\frac{3}{2}}\,\|u_1(0)\|_H=C\,|\lambda|^{\frac{1}{2}}\left(\|u_1\|_{W_p^2((0,1);H)}+|\lambda|\,\|u_1\|_{L_p((0,1);H)}\right)$$
$$\leq C\,|\lambda|^{\frac{1}{2}}\left(\|u_1\|_{W_p^2((0,1);H(A),H)}+|\lambda|\,\|u_1\|_{L_p((0,1);H)}\right)\leq C\,|\lambda|^{\frac{1}{2}}\,\|f\|_{L_p\left((0,1);H(A^{1/2})\right)}\,. \tag{3.27}$$

Taking into account estimations (3.20), (3.21), (3.22), (3.25), (3.26) and (3.27) in (3.19), for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\leq\varphi<\pi$, we get

$$|\lambda|\,\|u_2\|_{L_p((0,1);H)}+\left\|u_2''\right\|_{L_p((0,1);H)}+\|Au_2\|_{L_p((0,1);H)}$$

$$\leq C\left[|\lambda|\,\|f\|_{L_p\left((0,1);H(A^{1/2})\right)}+\sum_{k=1}^{2}\left(\|f_k\|_{(H(A^2),H)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}}+|\lambda|^{\frac{k+1}{2}-\frac{1}{2p}}\,\|f_k\|_H\right)\right]. \tag{3.28}$$

Now the estimation (3.3) follows from (3.17) and (3.28). The proof of Theorem 3.1 is complete.

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