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Correction to "Two-Weighted Inequalities for the Riesz Potential in p-Convex Weighted Modular Banach Function Spaces"

Rovshan A. Bandaliyev* · **Vagif S. Guliyev** · **Sabir G. Hasanov**

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Abstract. *In this note, we provide a corrigendum to the paper Two-weighted inequalities for the Riesz potential in p-convex weighted modular Banach function spaces (Ukr. Math. J., 69(2018), 1673-1688). The proof of Theorem 3.1 in [1] contained a mistake. In this note we give the details of the correct argument.*

Keywords. Two-weight inequality, Riesz potential, p-convex modular Banach function spaces

Mathematics Subject Classification (2010): 2010 Mathematics Subject Classification: 46B50

1 Introduction

This result appears as Theorem 3.1 in the above-mentioned paper. In other words, sufficient conditions for general weights ensuring the validity of the two-weight strong type inequalities for Riesz potential were found. In this theorem the inequality (3.6) is not true. We presume that the reader is familiar with the contents and notation of our original paper. At the heart of our correction is the following Theorem, which replaces Theorem 3.1 in [1]. The numbering in this note remains as in [1].

Let $\mathbb{Z}$ be the set of integers. For $k \in \mathbb{Z}$ we define $E_k = \left\{x \in \mathbb{R}^n : \ 2^k < |x| \le 2^{k+1}\right\}$, $E_{k,1} = \left\{x \in \mathbb{R}^n : \ |x| \le 2^{k-1}\right\}$, $E_{k,2} = \left\{x \in \mathbb{R}^n : \ 2^{k-1} < |x| \le 2^{k+2}\right\}$ and let $E_{k,3} = \left\{x \in R^n : \ |x| > 2^{k+2}\right\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $E_{k,2}$ is equal to 3. Suppose that A is a Lebesgue measurable subset of $\mathbb{R}^n$. Let us denote by χ_A the characteristic function of A. Let V_1 and V_2 be normed spaces. Let $[V_1, V_2]$ denotes the space of bounded linear operators from V_1 to V_2 endowed with the operator norm. Suppose that $1 < p < \infty$ and $p' = \dfrac{p}{p-1}$.

* Corresponding author

R.A. Bandaliyev
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku
E-mail: bandaliyevr@gmail.com

V.S. Guliyev
Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku
E-mail: vagif@guliyev.com

S.G. Hasanov
Ganja State University, Ganja
E-mail: sabhasanov@yahoo.com

We now consider the Riesz potential

$$I^s f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}}\, dy,$$

where $0 < s < n$.

Theorem 1.1 *Let v and w be weight functions on $\mathbb{R}^n$. Let X and Y be Banach function spaces of functions on $\mathbb{R}^n$ with the Lebesgue measure and norms $\|\cdot\|_{X(\mathbb{R}^n)}$ and $\|\cdot\|_{Y(\mathbb{R}^n)}$, respectively. Suppose that $X_v(\mathbb{R}^n)$ and $Y_w(\mathbb{R}^n)$ is corresponding weighted spaces and there exists $p>1$ such that $Y_w(\mathbb{R}^n)$ is a p-convex Banach function space. Let $I^s \in [L_p(\mathbb{R}^n);\, Y(\mathbb{R}^n)]$, $X_v(\mathbb{R}^n) \hookrightarrow L_{p,v}(\mathbb{R}^n)$ and let satisfy the following condition:*

1)

$$A = \sup_{t>0} \left(\int_{|y|<t} [v(y)]^{-p'}\, dy \right)^{\frac{\alpha}{p'}} \left\| \frac{\chi_{\{|x|>t\}}}{|x|^{n-s}} \left(\int_{|y|<|x|} [v(y)]^{-p'}\, dy \right)^{\frac{1-\alpha}{p'}} \right\|_{Y_w} < \infty;$$

2)

$$B = \sup_{t>0} \left(\int_{|y|>t} [v(y)|y|^{n-s}]^{-p'} dy \right)^{\frac{\beta}{p'}} \left\| \chi_{\{|x|<t\}} \left(\int_{|y|>|x|} [v(y)]^{-p'} dy \right)^{\frac{1-\beta}{p'}} \right\|_{Y_w} < \infty,$$

where $0<\alpha,\beta<1$;

3) $\exists C>0 \quad \operatorname*{ess\,sup}_{y\in E_k} w(y) \le C \operatorname*{ess\,inf}_{y\in E_{k,2}} v(y), \forall k \in \mathbb{Z}$;

4) $\exists C>0 \quad \left\| \sum_k |g_k \chi_{E_k}| \right\|^p_{Y_w(\mathbb{R}^n)} \le C \sum_k \left\| g_k \chi_{E_k} \right\|^p_{Y_w(\mathbb{R}^n)}$.

Then $I^s \in [X_v(\mathbb{R}^n);\, Y_w(\mathbb{R}^n)]$.

Proof. Given $f \in L_{p,v}(\mathbb{R}^n)$, we can write

$$|I^s f(x)| = \sum_{k\in\mathbb{Z}} |I^s f(x)|\, \chi_{E_k}(x)$$

$$\le \sum_{k\in\mathbb{Z}} |I^s f_{k,1}(x)|\, \chi_{E_k}(x) + \sum_{k\in\mathbb{Z}} |I^s f_{k,2}(x)|\, \chi_{E_k}(x) + \sum_{k\in\mathbb{Z}} |I^s f_{k,3}(x)|\, \chi_{E_k}(x)$$

$$= I_1^s f(x) + I_2^s f(x) + I_3^s f(x),$$

where $f_{k,i} = f\, \chi_{E_{k,i}}$.

The fact that $I_1^s \in [X_v(\mathbb{R}^n);\, Y_w(\mathbb{R}^n)]$ and $I_3^s \in [X_v(\mathbb{R}^n);\, Y_w(\mathbb{R}^n)]$ is prove precisely as in Theorem 3.1 in [1].

The inaccuracy was in the proof of $I_2^s \in [X_v(\mathbb{R}^n);\, Y_w(\mathbb{R}^n)]$. Now we reduce the correct variant of this fact.

By virtue of conditions 3) and 4) and taking into account that $I^s \in [L_p(\mathbb{R}^n);\, Y(\mathbb{R}^n)]$, we have

$$\left\| \sum_{k\in\mathbb{Z}} w(\cdot)|I^s f_{k,2}|\chi_{E_k}(\cdot) \right\|^p_{Y(\mathbb{R}^n)} \le C \sum_{k\in\mathbb{Z}} \|w(\cdot)|I^s f_{k,2}|\|^p_{Y(E_k)}$$

$$\leq C \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{y\in E_k} w^p(y)\ \|I^s f_{k,2}|\|^p_{Y(E_k)} \leq C \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{y\in E_k} w^p(y)\ \|I^s f_{k,2}\|^p_{Y(\mathbb{R}^n)}$$
$$\leq C \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{y\in E_k} w^p(y)\ \|f_{k,2}\|^p_{L_p(\mathbb{R}^n)} = C \sum_{k\in\mathbb{Z}} \operatorname*{ess\,sup}_{y\in E_k} w^p(y)\ \left\|f\,\chi_{E_{k,2}}\right\|^p_{L_p(\mathbb{R}^n)}$$
$$\leq C \sum_{k\in\mathbb{Z}} \operatorname*{ess\,inf}_{y\in E_{k,2}} v^p(y)\ \left\|f\chi_{E_{k,2}}\right\|^p_{L_p(\mathbb{R}^n)} \leq C \sum_{k\in\mathbb{Z}} \left\|v\,f\,\chi_{E_{k,2}}\right\|^p_{L_p(\mathbb{R}^n)}$$
$$\leq C\,\|f\|^p_{L_{p,v}(\mathbb{R}^n)} \leq C\,\|f\|^p_{X_v(\mathbb{R}^n)}. \tag{1.1}$$

We observe that the last inequality in (1.1) follows from embedding $X_v(\mathbb{R}^n) \hookrightarrow L_{p,v}(\mathbb{R}^n)$. Therefore, we have

$$\left\|\sum_{k\in\mathbb{Z}} w(\cdot)|I^s f_{k,2}|\chi_{E_k}(\cdot)\right\|_{Y(\mathbb{R}^n)} \leq C\,\|f\|_{L_{p,v}(\mathbb{R}^n)}.$$

Thus $I^s_2 \in [X_v(\mathbb{R}^n)\,;\ Y_w(\mathbb{R}^n)]$ and the proof of Theorem 1 is complete.

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References

1. Bandaliyev, R.A., Guliyev, V.S., Hasanov, S.G.: *Two-weighted inequalities for the Riesz potential in p-convex weighted modular Banach function spaces*, Ukr. Math. J. **69** (11), 1673-1688 (2018).