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Some spectral properties of Sturm-Liouville problem with a spectral parameter in the boundary conditions

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Abstract. *In this paper we consider Sturm-Liouville problem with a spectral parameter in the boundary conditions. We associate this problem with a self-adjoint operator in a Pontryagin space. Using analytic methods, we investigate locations, multiplicities of eigenvalues and oscillatory properties of eigenfunctions of this problem.*

Keywords. spectral parameter, eigenvalue, eigenfunction, asymptotic formula

Mathematics Subject Classification (2010): 34B05, 34B08, 34B24, 34L10, 34L20

1 Introduction

We consider the following eigenvalue problem

$$-y''(x) + q(x)y(x) = \lambda y(x),\ 0 < x < 1, \tag{1.1}$$

$$y'(0) = a\lambda y(0), \tag{1.2}$$

$$y'(1) = b\lambda y(1), \tag{1.3}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, q is a continuous positive function on $[0, 1]$, a and b are real constants such that $a > 0$ and $b < 0$.

The well-known mathematical model describing small torsional vibrations of a rod (see [18, pp. 49-50]) consists of a wave equation for the angle of rotation of the rod and the corresponding boundary conditions. If there are pulleys at both ends of the bar, then the boundary conditions that simulate the forces contain second time derivatives. Solving the corresponding mathematical problem by the method of separation of variables, we obtain the spectral problem (1.1)-(1.3).

Spectral properties, including basis properties of root functions in L_p, $1 < p < \infty$, of Sturm-Liouville problems with a spectral parameter in the boundary condition were studied in [3, 6, 8, 10-12, 15]. In the case when both boundary conditions contain a spectral parameter, these properties of Sturm-Liouville problem were studied in [1, 2, 4, 9, 13] (it should be noted that in [1, 2, 9] the potential is identically equal to zero).

Note that the signs of the parameters a and b plays an important role. If $a < 0$ and $b > 0$, then problem (1.1)-(1.3) can be represented as an eigenvalue problem for a self-adjoint operator in a Hilbert space. In the case $a < 0$ and $b < 0$ or $a > 0$ and $b > 0$ this

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problem is treated as an eigenvalue problem for a $J-$ self-adjoint operator in the "one-dimensional" Pontryagin space. If $a > 0$ and $b < 0$, then problem (1.1)-(1.3) reduced to the spectral problem for a $J-$ self-adjoint operator in the "two-dimensional" Pontryagin space. In two last cases, problem (1.1)-(1.3) can have real multiple eigenvalues or non-real eigenvalues.

Problem (1.1)-(1.3) with $q \equiv 0$ was considered in [2], where, in particular, it was shown that the eigenvalues of this problem are real and simple, and except the case $b = -(a+1)$, where $\lambda = 0$ is a double eigenvalue, form an infinitely nondecreasing sequence. Moreover, the author investigated the oscillation properties of eigenfunctions and the locations of eigenvalues on the real axis, and also studied the basis property of the subsystem of root functions in the space $L_p, 1 < p < \infty$.

The purpose of this paper is to study the locations, multiplicities of eigenvalues and the oscillation properties of eigenfunctions of problem (1.1)-(1.3).

2 Operator interpretation of the spectral problem (1.1)-(1.3)

Let $H = L_2(0,1) \oplus \mathbb{C}^2$ be a Hilbert space with the scalar product

$$(\hat{y}_1, \hat{y}_2)_H = (\{y_1(x), m_1, n_1\}, \{y_2(x), m_2, n_2\})_H$$
$$= \int_0^1 y_1(x)\,\overline{y_2(x)}\,dx + |a|^{-1} m_1\,\overline{n_1} + |b|^{-1} m_2\,\overline{n_2}.$$

We define in H an operator

$$L\hat{y} = L\{y(x), m, n\} = \{-y''(x) + q(x)y(x), y'(0), y'(1)\}$$

with the domain

$$D(L) = \{\hat{y} = \{y, m, n\} \in H :$$
$$y,\, y' \in AC[0,1],\, -y'' + qy \in L_2(0,1), m = ay(0), n = by(1)\}.$$

It is obvious that the operator L is well defined in H. Problem (1.1)-(1.3) takes the form

$$L\hat{y} = \lambda\hat{y},\ \ \hat{y} \in D(L),$$

i.e., the eigenvalues λ_k of problem (1.1)-(1.3) and the operator L coincide (counting their multiplicity); moreover, there is a one-to-one correspondence between the root functions

$$y_k(x) \longleftrightarrow \hat{y}_k = \{y_k(x), m_k, n_k\}, m_k = ay_k(0),\ n_k = by_k(1).$$

If $a < 0$ and $b > 0$, then L is a self-adjoint discrete lower-semibounded operator in H. In the case $a < 0$ and $b < 0$ or $a > 0$ and $b > 0$ the operator L is not self-adjoint in H, but is $J-$ self-adjoint in the Pontryagin space $\Pi_1 = L_2(0,1) \oplus \mathbb{C}^2$ (see [1, 2]).

If $a > 0$ and $b < 0$ the operator L is closed (nonself-adjoint) in H with a compact resolvent. In this case we define the operator $J : H \to H$ by

$$J\hat{y} = J\{y, m, n\} = J\{y, -m, -n\}.$$

The operator J is unitary and symmetric in H with spectrum consisting of two eigenvalues: -1 with multiplicity 2 and $+1$ with infinite multiplicity. Hence, this operator generates the Pontryagin space $\Pi_2 = L_2(0,1) \oplus \mathbb{C}^2$ with inner product (J-metric) (see [7])

$$[\hat{y}_1, \hat{y}_2] = (\hat{y}_1, \hat{y}_2)_{\Pi_2} = (\{y_1(x), m_1, n_1\}, \{y_2(x), m_2, n_2\})_{\Pi_2}$$

$$= \int_0^1 y_1(x)\,\overline{y_2(x)}\,dx \; - |a|^{-1} m_1\,\overline{n_1} - |b|^{-1} m_2\,\overline{n_2}$$
$$= \int_0^1 y_1(x)\,\overline{y_2(x)}\,dx - a^{-1} m_1\,\overline{n_1} + b^{-1} m_2\,\overline{n_2}.$$

Lemma 2.1 *L is a $J-$ self-adjoint operator in Π_2.*

The proof of this lemma is similar to that of [6, Propostion 1].

Let λ be an eigenvalue of L of algebraic multiplicity $\chi(L)$. We put $\sigma(\lambda)$ to be equal to $\chi(L)$ if $\operatorname{Im}\lambda \neq 0$ and to the integer part $\left[\frac{\chi(L)}{2}\right]$ if $\operatorname{Im}\lambda = 0$.

Lemma 2.2 *The eigenvalues of the operator L are located symmetrically about the real axis, and*

$$\sum_{k=1}^{s} \sigma(\lambda_k) \leq 2$$

for any system $\{\lambda_k\}_{k=1}^{s}$ of eigenvalues of the operator L with nonnegative imaginary parts.

The proof of the first part of this lemma follows from [7, Theorem $2.2'$], and that of the second part, from [17, Theorem 3] in view of Lemma 2.1.

It follows from the second part of Lemma 2.2 that problem (1.1)-(1.3) can have either one or two pairs of non-real conjugate simple eigenvalues, or one pairs of non-real conjugate double eigenvalues, or two real multiple eigenvalues whose sum of multiplicities does not exceed 5, or one real multiple eigenvalue which multiplicity does not exceed 5. Despite this, below we will show that all eigenvalues of problem (1.1)-(1.3) are real and simple.

3 Preliminary and properties of the solution to the initial Sturm-Liouville problem

Along with problem (1.1)-(1.3), consider the following spectral problem

$$\begin{aligned} -y''(x) + q(x)y(x) = \lambda y(x),\ 0 < x < 1, \\ y'(0)\sin\alpha = y(0)\cos\alpha, \\ y'(1) = b\lambda y(1), \end{aligned} \tag{3.1}$$

where $\alpha \in [0, \pi/2]$.

This problem under more general boundary conditions was considered in [3]. It follows from statement (i) of [3, Theorem 2.1] that for problem (3.1) the following result holds.

Theorem 3.1 *For each α the eigenvalues of problem* (3.1) *are real, simple and form an infinitely increasing sequence $\{\lambda_k(\alpha)\}_{k=1}^{\infty}$ such that $\lambda_1(\alpha) < 0$ and $\lambda_k(\alpha) > 0$ for $k \geq 2$. Moreover, the eigenfunctions $y_{1,\alpha}(x)$ and $y_{2,\alpha}(x)$ corresponding to the eigenvalues $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$, respectively, have no zeros in $(0,1)$, and the function $y_{k,\alpha}(x)$, corresponding to the eigenvalue $\lambda_k(\alpha)$, for $k \geq 3$ has precisely $k-2$ simple zeros in $(0,1)$.*

The following lemma holds.

Lemma 3.1 *For each $\lambda \in \mathbb{C}$, there is a unique solution $y(x,\lambda)$ to Eq.* (1.1) *satisfying the initial conditions*

$$y(1) = 1,\ y'(1) = b\lambda. \tag{3.2}$$

For each fixed $x \in [0,1]$ the function $y(x,\lambda)$ is an entire function of λ.

The proof of this lemma is similar to that of [14, Ch. 1, § 1, Theorem 1.1].

Let $\lambda_k(0) = \mu_k$ and $\lambda_k(\pi/2) = \nu_k$. By following the arguments in Theorem 2.3 of [5] we get the following relations

$$\mu_1 < \nu_1 < 0 < \nu_2 < \mu_2 < \nu_3 < \mu_3 < \ \ldots \ . \tag{3.3}$$

To study the spectral properties of problem (1.1)-(1.3), we need to introduce the function

$$F(\lambda) = \frac{y'(0,\lambda)}{y(0,\lambda)},$$

which is defined on the domain

$$\mathcal{C} = (\mathbb{C}\backslash\mathbb{R}) \cup \bigcup_{k=1}^{\infty} \mathcal{C}_k,$$

where

$$\mathcal{C}_k = (\mu_{k-1}, \mu_k), \ k \in \mathbb{N}, \text{ and } \mu_0 = -\infty.$$

Let $\lambda, \ \mu \in \mathcal{C}$. Then by (1.1) we have

$$-y''(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda), \tag{3.4}$$

$$-y''(x,\mu) + q(x)y(x,\mu) = \mu y(x,\mu).$$

Multiplying the first equation by $y(x,\mu)$, the second by $y(x,\lambda)$ and subtracting the first from the second we obtain

$$-y''(x,\mu)y(x,\lambda) + y''(x,\lambda)y(x,\mu) = (\mu-\lambda)y(x,\mu)y(x,\lambda).$$

Integrating this equality from 0 to 1, using integration by parts, and taking into account the initial conditions (3.2), we get

$$y'(0,\mu)y(0,\lambda) - y'(0,\lambda)y(0,\mu) = (\mu-\lambda)\left\{\int_0^1 y(x,\mu)y(x,\lambda)dx + b\right\}. \tag{3.5}$$

Dividing both sides of (3.5) by $(\mu-\lambda)y(0,\mu)y(0,\lambda)$ for $\mu, \lambda \in \mathcal{C}_k$, $k \in \mathbb{N}$, $\mu \neq \lambda$, we obtain

$$\frac{\frac{y'(0,\mu)}{y(0,\mu)} - \frac{y'(0,\lambda)}{y(0,\lambda)}}{\mu-\lambda} = \frac{\int_0^1 y(x,\mu)y(x,\lambda)dx + b}{y(0,\mu)y(0,\lambda)}$$

Next, passing to the limit as $\mu \to \lambda$ in this equality, we get

$$\frac{d}{d\lambda}\left(\frac{y'(0,\lambda)}{y(0,\lambda)}\right) = \frac{\int_0^1 y^2(x,\lambda)dx + b}{y^2(0,\lambda)}\,.$$

Thus, we have proved the following lemma.

Lemma 3.2 *For each $\lambda \in \mathcal{C}$ the following formula holds:*

$$\frac{dF(\lambda)}{d\lambda} = \frac{\int_0^1 y^2(x,\lambda)dx + b}{y^2(0,\lambda)}\,. \tag{3.6}$$

Lemma 3.3 *One has the following relation*

$$F(\lambda) \to -\infty \text{ as } \lambda \to -\infty. \tag{3.7}$$

Proof. In Eq. (1.1), we put $\lambda = -\varrho^2$. As is known [16, Ch. II, Theorem 1], in each subdomain T of the complex ϱ-plane this equation has two linearly independent and regular in ϱ (for sufficiently large $|\varrho|$) solutions $\varphi_k(x,\varrho)$, $k = 1, 2$, which satisfy the following relations

$$\begin{aligned} &\varphi_1(x,\varrho) = e^{-i\varrho x}\left(1+O\left(\tfrac{1}{\varrho}\right)\right), \quad \varphi_2(x,\varrho) = e^{i\varrho x}\left(1+O\left(\tfrac{1}{\varrho}\right)\right), \\ &\varphi_1'(x,\varrho) = -i\varrho\, e^{-i\varrho x}\left(1+O\left(\tfrac{1}{\varrho}\right)\right), \ \varphi_2'(x,\varrho) = i\varrho\, e^{i\varrho x}\left(1+O\left(\tfrac{1}{\varrho}\right)\right). \end{aligned} \tag{3.8}$$

Then we rewrite the function $y(x,\lambda)$ in the form

$$y(x,\lambda) = C_1(\varrho)\,\varphi_1(x,\varrho) + C_2(\varrho)\,\varphi_2(x,\varrho). \tag{3.9}$$

Using (3.8) and taking into account (3.2) from (3.9) we obtain a linear system of algebraic equations

$$\begin{cases} C_1(\varrho)e^{-i\varrho}[1] \ + \ C_2(\rho)e^{i\varrho}[1] = 1, \\ -i\varrho\, C_1(\varrho)\,e^{-i\varrho}[1] + i\varrho\, C_2(\varrho)\,e^{i\varrho}[1] = b\varrho^2, \end{cases}$$

where $[1] = 1 + O\left(\frac{1}{\varrho}\right)$, whence implies that

$$C_1(\varrho) = \frac{1+bi\varrho}{2}\, e^{i\varrho}[1] \text{ and } C_1(\varrho) = \frac{1-bi\varrho}{2}\, e^{-i\varrho}[1].$$

Therefore, by (3.8) we have

$$y(x,\lambda) = \frac{1+bi\varrho}{2}\, e^{i\varrho(1-x)}[1] + \frac{1-bi\varrho}{2}\, e^{-i\varrho(1-x)}[1],$$

$$y'(x,\lambda) = i\varrho\left\{-\frac{1+bi\varrho}{2}\, e^{i\varrho(1-x)}[1] + \frac{1-bi\varrho}{2}\, e^{-i\varrho(1-x)}[1]\right\}.$$

Then, by the last relations, for $F(\lambda)$ we obtain the following representation

$$F(\lambda) = i\varrho\,\frac{-(1+bi\varrho)\,e^{i\varrho} + (1-bi\varrho)\,e^{-i\varrho}}{(1+bi\varrho)\,e^{i\varrho} + (1-bi\varrho)\,e^{-i\varrho}}\,[1]$$

$$= \varrho\,\frac{\sin\varrho + b\varrho\cos\varrho}{\cos\varrho - b\varrho\sin\varrho}\,[1]. \tag{3.10}$$

If $\lambda < 0$ and $|\lambda|$ is large enough, then ϱ can be taken as $\varrho = \sqrt{\lambda} = i\sqrt{|\lambda|}$. Then by (3.10) we have

$$F(\lambda) = i\sqrt{|\lambda|}\,\frac{\sin i\sqrt{|\lambda|} + bi\sqrt{|\lambda|}\cos i\sqrt{|\lambda|}}{\cos i\sqrt{|\lambda|} - bi\sqrt{|\lambda|}\sin i\sqrt{|\lambda|}}\,[1]$$

$$= i\sqrt{|\lambda|}\,\frac{i\sinh\sqrt{|\lambda|} + bi\sqrt{|\lambda|}\cosh\sqrt{|\lambda|}}{\cosh\sqrt{|\lambda|} + b\sqrt{|\lambda|}\sinh\sqrt{|\lambda|}}\,[1] = -\sqrt{|\lambda|}\,[1],$$

which implies (3.7). The proof of this lemma is complete.

In view of Theorem 3.1, by Lemmas 3.1-3.3 we have the following properties of the function $F(\lambda)$ for $\lambda \in \mathbb{R}$.

Lemma 3.4 *The following assertions hold:*

(i) $\lim\limits_{\lambda\to\mu_1-0} F(\lambda) = -\infty$, $\lim\limits_{\lambda\to\mu_1+0} F(\lambda) = +\infty$, $\lim\limits_{\lambda\to\mu_2-0} F(\lambda) = +\infty$;

(ii) *if* $\lambda \in (\mu_1, \nu_1) \cup (\nu_2, \mu_2)$, *then* $F(\lambda) > 0$, *and if* $\lambda \in (\nu_1, \nu_2)$, *then* $F(\lambda) < 0$;

(iii) $\lim\limits_{\lambda\to\mu_{k-1}+0} F(\lambda) = -\infty$, $\lim\limits_{\lambda\to\mu_k-0} F(\lambda) = +\infty$, $k \geq 3$.

Using Theorem 3.1 and Lemmas 3.2-3.4, and following the arguments in Proposition 4 of [6], we can prove the following lemma.

Lemma 3.5 *The function* $F(\lambda)$ *has the representation*

$$F(\lambda) = \sum_{k=1}^{\infty} \frac{\lambda c_k}{\mu_k(\lambda - \mu_k)},$$

where $c_k = \mathop{res}\limits_{\lambda=\mu_k} F(\lambda)$, $c_1 > 0$ *and* $c_k < 0$ *for* $k \geq 2$.

Now consider the following equation

$$y(x,\lambda) = 0, \ x \in [0,1], \ \lambda \in \mathbb{R}.$$

Obviously, the zeros of this equation are functions of λ.

Remark 3.1 By the first condition of (3.2) we have $y(1,\lambda) \neq 0$ for any $\lambda \in \mathbb{R}$. Hence every zero $x(\lambda) \in [0,1)$ of this equation is simple and is a continuously differentiable function of λ.

Remark 3.2 By the corollary to Lemma 3.1 of [13, Ch. 1, § 3] and Remark 3.1 as $\lambda \in \mathbb{R}$ varies, the function $y(x,\lambda)$ can lose or gain zeros only by these zeros leaving or entering the interval $[0,1]$ through it endpoint $x = 0$. Then, by [12, Lemma 2.2] as $\lambda \in \mathbb{R}$ increases the number of zeros of the function $y(x,\lambda)$ contained in $[0,1)$ does not decrease.

In view of Remark 3.2, by Theorem 2.1 we have the following result on the number of zeros contained in $(0,1)$ of the function $y(x,\lambda)$ for $\lambda \in \mathbb{R}$.

Lemma 3.6 *If* $\lambda \in (-\infty, \mu_2]$, *then* $y(x,\lambda)$ *has no zeros, and if* $\lambda \in (\mu_{k-1}, \mu_k]$ *for* $k \geq 3$, *then* $y(x,\lambda)$ *has exactly* $k-2$ *simple zeros in* $(0,1)$.

4 Main properties of eigenvalues of problem (1.1)-(1.3)

Lemma 4.1 *All eigenvalues of problem* (1.1)-(1.3) *are real. They form at most a countable set without a finite limit point.*

Proof. Let λ be an eigenvalue of problem (1.1)-(1.3). Then λ is a root of the equation

$$y'(0,\lambda) = a\lambda y(0,\lambda), \tag{4.1}$$

and $y(x,\lambda)$ is the corresponding eigenfunction. If $\lambda \in \mathbb{C}\backslash\mathbb{R}$, then $\bar{\lambda}$ is also eigenvalue of (1.1)-(1.3) with corresponding eigenfunction $y(x,\bar{\lambda}) = \overline{y(x,\lambda)}$, since the coefficients q, a and b are real.

Putting $\mu = \bar{\lambda}$ in (3.5) we obtain

$$\overline{y'(0,\lambda)}\, y(0,\lambda) - y'(0,\lambda)\, \overline{y(0,\lambda)} = (\bar{\lambda} - \lambda) \left\{ \int_0^1 |y(x,\mu)|^2 dx + b \right\}, \tag{4.2}$$

whence, with regard to (4.1), we get

$$a(\bar{\lambda}-\lambda)|y(0,\lambda)|^2 = (\bar{\lambda}-\lambda)\left\{\int_0^1 |y(x,\lambda)|^2 dx + b\right\}, \tag{4.3}$$

Since $\bar{\lambda} \neq \lambda$ it follows from (4.3) that

$$\int_0^1 |y(x,\lambda)|^2 dx + b - a|y(0,\lambda)|^2 = 0. \tag{4.4}$$

Multiplying both sides of (3.4) by $\overline{y(x,\lambda)}$, integrating the resulting relation from 0 to 1, using integration by parts, and taking into account (3.2), we obtain

$$\int_0^1 |y'(x,\lambda)|^2 dx + \int_0^1 q(x)|y(x,\lambda)|^2 dx = \lambda\left\{\int_0^1 |y(x,\lambda)|^2 dx + b - a\,|y(0,\lambda)|^2\right\}. \tag{4.5}$$

In view of (4.4), from (4.5) we get

$$\int_0^1 |y'(x,\lambda)|^2 dx + \int_0^1 q(x)|y(x,\lambda)|^2 dx = 0.$$

Since $q > 0$ it follows from last relation that $y(x,\lambda) \equiv 0$ which contradicts the fact that $y(x,\lambda)$ is an eigenfunction. Therefore, λ must be real.

Eigenvalues of problem (1.1)-(1.3) are zeros of the entire function $y'(0,\lambda) - a\lambda y(0,\lambda)$. We have shown that this function does not vanish for real λ. Consequently, it cannot be an identically zero function and its zeros form a countable set without finite limit point. The proof of this lemma is complete.

Remark 4.1 Let λ be an eigenvalue of problem (1.1)-(1.3). Then by Lemma 4.1 it follows from (4.5) that

$$\int_0^1 y'^2(x,\lambda)dx + \int_0^1 q(x)y^2(x,\lambda)dx = \lambda\left\{\int_0^1 y^2(x,\lambda)dx + b - a\,y^2(0,\lambda)\right\}. \tag{4.6}$$

If $\lambda = 0$, then (4.6) implies that

$$\int_0^1 y'^2(x,\lambda)dx + \int_0^1 q(x)y^2(x,\lambda)dx = 0.$$

It follows from the last relation that $y(x,0) \equiv 0$ which shows that $\lambda = 0$ cannot be an eigenvalue of problem (1.1)-(1.3).

Lemma 4.2 *All eigenvalues of problem* (1.1)-(1.3) *are simple.*

Proof. If λ is an eigenvalue of (1.1)-(1.3), then by Remark 4.1 it follows from (4.1) that $y(0,\lambda) \neq 0$. Then the eigenvalues of (1.1)-(1.3) are also roots of the equation

$$F(\lambda) = a\lambda. \tag{4.7}$$

Let λ be an eigenvalue of (1.1)-(1.3) with algebraic multiplicity 2. Then by (4.6) we have the following relations

$$F(\lambda) = a\lambda,\ \ F'(\lambda) = a.$$

Hence in view of (3.6) we get

$$\int_0^1 y^2(x,\lambda)dx + b - ay^2(0,\lambda) = 0. \tag{4.8}$$

By (4.8) it follows from (4.6) that

$$\int_0^1 y'^2(x,\lambda)dx + \int_0^1 q(x)y^2(x,\lambda)dx = 0,$$

whence implies that $y(x,\lambda) \equiv 0$ which is impossible by condition $y(x,\lambda) \not\equiv 0$. The proof of this lemma is complete.

5 Location of eigenvalues on the real axis and oscillatory properties of eigenfunctions of problem (1.1)-(1.3)

The following main result holds for problem (1.1)-(1.3).

Theorem 5.1 *There is an unboundedly increasing sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *of eigenvalues of problem* (1.1)-(1.3) *such that*

$$\lambda_1 \in \mathcal{C}_1,\ \lambda_2,\ \lambda_3 \in \mathcal{C}_2, \lambda_k \in \mathcal{C}_{k-1},\ k = 4,\ 5,\ \dots\ . \tag{5.1}$$

The eigenfunctions $y_1(x)$, $y_2(x)$ *and* $y_3(x)$, *corresponding to the eigenvalues* λ_1, λ_2 *and* λ_3, *respectively, have no zeros, the eigenfunction* $y_k(x)$, *corresponding to the eigenvalue* λ_k, *for* $k \geq 4$ *has exactly* $k-3$ *simple zeros in* $(0,1)$.

Proof. Since $a > 0$, by Lemma 3.4, it follows that Eq. (4.7) has at least one solution in each interval $\mathcal{C}_k$, $k \in \mathbb{N}$.

Let us show that Eq. (4.7) cannot have more than one root in each of the intervals $\mathcal{C}_k$ for $k = 1,\ 3,\ 4,\ \dots$. Indeed, if $\lambda^* \in \mathcal{C}_1$ is a root of this equation, then by Lemma 4.1 we have

$$F'(\lambda^*) - a \neq 0.$$

Hence by (3.6) we obtain

$$\int_0^1 y^2(x,\lambda^*)dx + b - ay^2(0,\lambda^*) \neq 0.$$

Since $q > 0$ and $\lambda^* < 0$ it follows from (4.6) that

$$\int_0^1 y^2(x,\lambda^*)dx + b - ay^2(0,\lambda^*) < 0,$$

and consequently,

$$F'(\lambda^*) - a < 0.$$

Therefore, the function $F(\lambda) - a\lambda$ in $\mathcal{C}_1$ can take zero value only strictly decreasing. Then this function has a unique root λ_1 in the interval $\mathcal{C}_1$. Similarly, it is proved that Eq. (4.7) has a unique root in each interval $\mathcal{C}_k$ for $k = 3,\ 4,\ \dots$.

By Lemma 3.5 we have

$$F''(\lambda) = 2\sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \mu_k)^3},$$

which implies that

$$F''(\lambda) > 0 \text{ for } \lambda \in \mathcal{C}_2,$$

i.e., the function is convex in $\mathcal{C}_2$. Then according to statements (i) and (ii) of Lemma 3.4 Eq. (4.7) has two roots λ_1 and λ_2 such that $\lambda_1 \in (\nu_1, 0)$ and $\lambda_2 \in (\nu_2, \mu_2)$.

By the above arguments Eq. (4.7) in the interval $\mathcal{C}_k$ for $k \geq 3$ has a unique root λ_{k+1}.

Next, the oscillatory properties of eigenfunctions of problem (1.1)-(1.3) follows (5.1) in view of Lemma 3.6. The proof of this theorem is complete.

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