

On the completeness of the system of Weber functions

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Abstract. Weber functions $D_{\frac{\lambda_n-1}{2}}(\sqrt{2}x)$, $n = 0, 1, 2, \dots$, are considered, where λ_n is the eigenvalue of the perturbed harmonic oscillator on the semi-axis with finite potential and with the Dirichlet boundary condition at zero. The completeness in the space $L_2(0, \infty)$ of a system of functions $\left\{D_{\frac{\lambda_n-1}{2}}(\sqrt{2}x)\right\}_{n=0}^{\infty}$ is proved.

Keywords. Weber function · perturbed harmonic oscillator · eigenvalues · completeness of a system of functions.

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1 Introduction and main result

In the space $L_2(0, \infty)$ we consider self-adjoint operator

$$L_0 = -\frac{d^2}{dx^2} + x^2,$$

generated by the left side of the equation

$$-y'' + x^2y = \lambda y, \quad 0 < x < \infty, \quad \lambda \in C, \quad (1.1)$$

and boundary condition

$$y(0) = 0. \quad (1.2)$$

The equation (1.1) has [1] the solution $f_0(x, \lambda)$ in the form

$$f_0(x, \lambda) = D_{\frac{\lambda-1}{2}}(\sqrt{2}x),$$

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where $D_\nu(x)$ is the Weber function. It is well known (see [1, 8]) that for each $x \in [0, \infty)$ the function $f_0(x, \lambda)$ is entire and the following asymptotic is fulfilled:

$$f_0(x, \lambda) = \left(\sqrt{2x}\right)^{\frac{\lambda-1}{2}} e^{-\frac{x^2}{2}} (1 + O(x^{-2})), \quad x \rightarrow \infty. \quad (1.3)$$

Since $q_0(x) = x^2 \rightarrow +\infty$ for $x \rightarrow +\infty$, then the spectrum of the operator L_0 consists of simple real eigenvalues. From (1.3) it follows that for each fixed λ from the complex plane, the relation $f_0(x, \lambda) \in L_2(0, \infty)$ holds. Therefore, the spectrum of the problem (1.1) - (1.2), i.e. of the operator L_0 , coincide with the zeros of the function $f_0(0, \lambda) = D_{\frac{\lambda-1}{2}}(0)$. It is shown in the work [8], that the spectrum of L_0 is purely discrete and consists of simple eigenvalues $\hat{\lambda}_n = 4n + 3$, $n = 0, 1, \dots$. The corresponding eigenfunctions $\left\{f_0(x, \hat{\lambda}_n)\right\}_{n=0}^{\infty}$ form an orthogonal basis in the space $L_2(0, \infty)$. Consequently, the following relation is true:

$$\sum_{n=0}^{\infty} \frac{f_0(x, \hat{\lambda}_n)}{\alpha_n^0} \frac{f_0(y, \hat{\lambda}_n)}{\alpha_n^0} = \delta(x - y), \quad (1.4)$$

where $(\alpha_n^0)^2 = (2n + 1)! \frac{\sqrt{\pi}}{2}$, $\delta(x)$ is Dirac's delta.

We now consider the self-adjoint operator

$$L = L_0 + q(x)$$

in space $L_2(0, \infty)$, where the potential $q(x)$ is real and satisfies the condition

$$\int_0^{\infty} (1 + x^3) |q(x)| dx < \infty.$$

It is well known that the equation

$$-y'' + x^2y + q(x)y = \lambda y, \quad 0 < x < \infty, \quad \lambda \in C \quad (1.5)$$

under the condition (1.6) has a solution $f(x, \lambda)$ such that

$$f(x, \lambda) = f_0(x, \lambda) (1 + o(1)), \quad x \rightarrow \infty.$$

The spectrum of the operator L , i.e. of the problem (1.5), (1.2) coincides with the roots of the function $f(0, \lambda)$. In the paper [8] the asymptotic behavior of the eigenvalues of the operator L is studied. It was also proved there that the spectrum of the operator L consists of a sequence of simple real eigenvalues $\lambda_n, n \geq 0$, and the following asymptotic formula is valid

$$\lambda_n = 4n + 3 + O\left(n^{-\frac{1}{2}}\right), \quad n \rightarrow \infty. \quad (1.6)$$

Of particular interest is the question of the completeness of the system of functions $\{f_0(x, \lambda_n)\}_{n=0}^{\infty}$. Studying the last question can be very useful in studying the inverse spectral problem for the operator T , since the completeness of the mentioned system of functions plays a key role in the unique solvability of the Gelfand-Levitan equation (see [2]- [5]).

The main result of this paper is the following theorem.

Theorem 1.1 *Let numbers $\lambda_n, \lambda_n \neq \lambda_k, (n \neq k)$ of the form (1.6). Then the system of functions $\{f_0(x, \lambda_n)\}_{n=0}^{\infty}$ is complete in $L_2(0, \infty)$.*

2 Proof of the theorem

Let $h(x) \in L_2(0, \infty)$ be such that

$$\int_0^{\infty} h(x) f_0(x, \lambda_n) dx = 0, \quad n \geq 0.$$

Consider a function $H(\lambda) = \int_0^{\infty} h(x) f_0(x, \lambda) dx$, which is obviously an entire function.

As shown in the work [6], the function $D_{\frac{\mu^2-1}{2}}(\mu t\sqrt{2})$ satisfies the following asymptotic equality uniformly in t

$$D_{\frac{\mu^2-1}{2}}(\mu t\sqrt{2}) \sim C(2e)^{-\frac{1}{4}\mu^2} \mu^{\frac{1}{2}\mu^2-\frac{1}{2}} \frac{e^{-\mu^2\xi}}{(t^2-1)^{\frac{1}{4}}},$$

where

$$\xi = \frac{\pi i}{4} + \int_0^t \sqrt{u^2-1} du$$

and the regular branch of the radical is chosen from the condition $\sqrt{u^2-1}\Big|_{u=2} > 0$. Setting $\mu t = x$, $\lambda = \mu^2$, we find that the function $f_0(x, \lambda)$ has the asymptotics uniformly for all real x :

$$f_0(x, \lambda) \sim C(2e)^{-\frac{\lambda}{4}} \frac{(\sqrt{\lambda})^{\frac{\lambda}{2}}}{(x^2-\lambda)^{\frac{1}{4}}} e^{-\left(\frac{\pi i \lambda}{4} + \int_0^x \sqrt{y^2-\lambda} dy\right)}, \quad \lambda \rightarrow \infty. \quad (2.1)$$

Using (2.1), we establish (see [6, 7]) the following estimate for the function $f_0(x, \lambda)$

$$|f_0(x, \lambda)| \leq C a(\lambda) \rho^{-1}(x, \lambda) \exp\{-\sigma(x, \lambda)\}, \quad (2.2)$$

where

$$a(\lambda) = \left| \frac{\lambda}{2e} \right|^{\frac{Re\lambda}{4}} e^{\frac{\pi-\varphi}{4} Im\lambda}, \quad \lambda = |\lambda| e^{i\varphi}, \quad \varphi \in [0, 2\pi), \quad (2.3)$$

$$\rho(x, \lambda) = 1 + |\lambda|^{\frac{1}{12}} + |x^2 - \lambda|^{\frac{1}{4}}, \quad (2.4)$$

$$\sigma(x, \lambda) = Re \int_0^x \sqrt{y^2 - \lambda} dy \quad (2.5)$$

and the regular branch of the radical is chosen from the condition $\sqrt{y^2 - \lambda}\Big|_{\lambda=y^2-1} = 1$. It should be noted that the last condition implies the inequality $Re \sqrt{y^2 - \lambda} \geq 0$ for all values of λ .

Now let $|\lambda| = R$. We represent the function $H(\lambda) = \int_0^{\infty} h(x) f_0(x, \lambda) dx$ in the form

$$H(\lambda) = \int_0^R h(x) f_0(x, \lambda) dx + \int_R^{\infty} h(x) f_0(x, \lambda) dx = H_1(\lambda) + H_2(\lambda).$$

Using (2.2) - (2.5) taking into account the relation

$$\begin{aligned} \sigma(x, \lambda) &= \frac{1}{2} Re \left[x\sqrt{x^2 - \lambda} - \lambda \ln \left(x^2 + \sqrt{x^2 - \lambda} \right) + \lambda \ln \sqrt{-\lambda} \right] \\ &= \frac{x^2}{2} - \frac{Re\lambda}{2} \ln x + \frac{Re\lambda}{4} + \ln \left| \frac{\lambda}{4e} \right| + \frac{\pi - \varphi}{4} Im\lambda + O(x^{-2}), \quad x \rightarrow +\infty \end{aligned}$$

we find that

$$\begin{aligned} |H_1(\lambda)| &\leq C \|h\| a(R) R^{\frac{1}{2}}, \\ |H_2(\lambda)| &\leq Ca(R) \int_R^\infty \left| h(x) e^{-\frac{x^2}{4}} \right| dx \leq Ca(R) \|h\| \int_R^\infty e^{-\frac{x^2}{2}} dx. \end{aligned}$$

It follows from the last estimates that $H(\lambda)$ is an entire function of order $\rho_1 = 1$ and for sufficiently large $|\lambda|$ the estimate

$$|H(\lambda)| \leq C \|h\| a(|\lambda|) |\lambda|^{\frac{1}{2}}$$

is valid. Moreover, for $\lambda \rightarrow \infty$, the following relation holds (see [3], [6], [7])

$$|f_0^{-1}(0, \lambda)| \leq Ca^{-1}(\lambda) |\lambda|^{-\frac{1}{4}}.$$

Whence it follows that for all $\lambda \rightarrow \infty$ the function $f_0^{-1}(0, \lambda) \int_0^\infty h(x) f_0(x, \lambda) dx$ admits the following estimate

$$\left| f_0^{-1}(0, \lambda) \int_0^\infty h(x) f_0(x, \lambda) dx \right| \leq C \|h\| |\lambda|^{\frac{1}{4}}.$$

Since $f_0(0, \lambda)$ is an entire function of order $\rho = 1$, then consider the Hadamard factorization of this function:

$$f_0(0, \lambda) = C_1 e^{p\lambda} \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\hat{\lambda}_n} \right) e^{\frac{\lambda}{\hat{\lambda}_n}},$$

where, $C_1 = f_0(0, 0)$, $p = \dot{f}_0(0, 0)$ where us agree to denote differentiation with respect to λ a dot. We also introduce the function

$$G(\lambda) = C_2 e^{p\lambda} \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n} \right) e^{\frac{\lambda}{\lambda_n}}, \quad (2.6)$$

the set of roots of which coincides with the sequence λ_n , here $C_2 = C_1 \prod_{n=0}^{\infty} \frac{\lambda_n}{\hat{\lambda}_n}$. It follows from (1.6), (2.6) that $G(\lambda)$ is an entire function of order $\rho = 1$. This implies that $G^{-1}(\lambda) \int_0^\infty h(x) f_0(x, \lambda) dx$ is an entire function of order $\rho = 1$.

On the other hand, inside the corner $\delta \leq \arg \lambda \leq 2\pi - \delta$, $\delta > 0$, the relation

$$\left| \frac{\lambda_n - \hat{\lambda}_n}{\lambda - \lambda_n} \right| \leq \frac{Cn^{-\frac{1}{2}}}{|\lambda_n \sin \delta|} \leq \frac{C}{n^{\frac{3}{2}}}$$

is satisfy. Then from the formula

$$\frac{f_0(0, \lambda)}{G(\lambda)} = \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n - \hat{\lambda}_n}{\lambda - \lambda_n} \right)$$

follows that

$$\left| \frac{f_0(0, \lambda)}{G(\lambda)} \right| \leq C.$$

Using the last relations, we obtain

$$\left| G^{-1}(\lambda) \int_0^{\infty} h(x) f_0(x, \lambda) dx \right| \leq M \|h\| R^{\frac{1}{2}}, \quad (2.7)$$

where $R = |\lambda|$, $\delta \leq \arg \lambda \leq 2\pi - \delta$, $R = |\lambda| > R_0$, $\delta \leq \arg \lambda \leq 2\pi - \delta$. Now let $\delta > 0$ be such that the sector angle $-\delta \leq \arg \lambda \leq \delta$ is smaller π . Applying the Phragmen-Lindelof theorem [9] to the function $(1 + \lambda)^{-\frac{1}{2}} G^{-1}(\lambda) \int_0^{\infty} h(x) f_0(x, \lambda) dx$, we find that estimate (2.7) also holds in the sector $-\delta \leq \arg \lambda \leq \delta$. From this, using the Liouville's theorem [9] we conclude that $G^{-1}(\lambda) \int_0^{\infty} h(x) f_0(x, \lambda) dx \equiv 0$, i.e.

$$H(\lambda) = \int_0^{\infty} h(x) f_0(x, \lambda) dx \equiv 0.$$

Setting $\lambda = \hat{\lambda}_n = 4n + 3$ in the last equality and taking into account the expansion formula (1.4), we obtain $h(x) = 0$. This completes the proof of the theorem.

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