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Some weighted inequalities for Gegenbauer fractional integrals

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Abstract. In this paper we prove a some weighted inequality for G-fractional integrals $J_G^{\alpha,\lambda}$f associated with the Gegenbauer differential operator $G = \left(x^2-1\right)^{\frac{1}{2}-\lambda} \frac{d}{dx}\left(x^2-1\right)^{\lambda+\frac{1}{2}} \frac{d}{dx}$. This results is an analog of Heining's results [1]. As an application of this results, the Stein-Weiss inequality for the G -fractional integral is proved.

Keywords. Gegenbauer differential operator, G -fractional integrals, Stein-Weiss inequality.

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1 Introduction

Let $\mathbb{R}_+ = [0,\infty)$ and let w be a weight function on $\mathbb{R}_+$, i.e. w is nonnegative measurable function on $\mathbb{R}_+$. The weighted Lebesgue space $L_{p,w,\lambda} \equiv L_{p,w,\lambda}(\mathbb{R}_+), 1 \le p < \infty$, is the set of all classes of measurable functions f with finite norm

$$\|f\|_{p,w,\lambda} = \left(\int_{\mathbb{R}_+} |f(cht)|^p w(cht) sh^{2\lambda} t dt\right)^{\frac{1}{p}}.$$

If $p=\infty$, we assume

$$L_{\infty,w,\lambda}(\mathbb{R}_+) = L_\infty(\mathbb{R}_+) = \left\{ f : \|f\|_{\infty,w} = \underset{t\in\mathbb{R}_+}{ess\,\sup} |w(chtf)(cht)| < \infty \right\}$$

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The fractional integral operators play an important role in the harmonic analysis and his application. Many mathematicians have dealt with the fractional integrals and related topics associated with the Laplace-Bessel differential operator (see,for example [2-9,11,12,16,18]).

In this paper we consider G-fractional integrals in the weighted Lebesgue space $L_{p,w,\lambda}(\mathbb{R}_+)$ associated with the generalized shift operator [10]

$$A^{\lambda}_{cht} f(chx) = c_\lambda \int_0^\pi f(chxcht - shxsht\cos\varphi)(\sin\varphi)^{2\lambda-1} d\varphi,$$

where

$$c_\lambda = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} \left(\int_0^\pi (sh\varphi)^{2\lambda-1}\, d\varphi \right)^{-1}.$$

A^{λ}_{cht} generated the corresponding G-covolution [10]

$$(f\otimes g)(chx) = \int_{\mathbb{R}_+} f(cht) A^{\lambda}_{cht} f(chx) sh^{2\lambda} t dt.$$

The G-fractional integral is defined by [12]

$$J_G^{\alpha,\lambda} f(chx) = \int_{\mathbb{R}_+} A^{\lambda}_{cht} \left(sh\frac{x}{2}\right)^{\alpha-2\lambda-1} f(cht) sh^{2\lambda} t dt,\ \ 0<\alpha<2\lambda+1.$$

In the case $w\equiv 1$, for the $J_G^{\alpha,\lambda}$ in [11] is proved that $1<p<\infty$ if and $\alpha p<2\lambda+1$ then $J_G^{\alpha,\lambda}$ is an operator of strong type (p,q), where $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$, and if, $p=1$ then $J_G^{\alpha,\lambda}$ is an operator of weak type $(1,q)$, where $1-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$.

In the following we give the Heinig's results [1] for the boundedness of the $J_G^{\alpha,\lambda}$ in weighted Lebesgue space which is a generalization of the Hardy-Littlewood-Sobelov theorem for $J_G^{\alpha,\lambda}$.

2 Preliminaries

Let $1\le p\le\infty$.In the case $w=1$, if f is in $L_{p,\lambda}(R_+)$ and φ is in $L_{1,\lambda}(\mathbb{R}_+)$, then the function $f\otimes\varphi$ belongs $L_{p,\lambda}(\mathbb{R}_+)$ and (see [10], Lemma 4).

$$\|f\otimes\varphi\|_{p,\lambda} \le \|f\|_{p,\lambda}\|\varphi\|_{1,\lambda}.$$

Suppose f is a measurable function defined on $\mathbb{R}_+$ .For any measurable set $E\subset\mathbb{R}_+$, let $|E|_\lambda = \int\limits_E sh^{2\lambda} x dx$. The distribution function $f_{*,\lambda}$ of the function f is given by (see [12]).

$$f_{*,\lambda}(s) = |\{x\in\mathbb{R}_+ : |f(chx)|>s\}|_\lambda,\ \ for\ \ s\ge 0.$$

The distribution function $f_{*,\lambda}$ is nonnegative, decreasing, and continuous from the right (see [12]). With the distribution function we associate the decreasing rearrangement of f on $\mathbb{R}_+$ defined by (see [12])

$$f^{*,\lambda}(cht) = \inf\left\{ s\ge 0 : f_{*,\lambda}(s) \le sh\frac{t}{2} \right\},\ \ t\ge 0.$$

If $f \in L_{p,\lambda}(\mathbb{R}_+), \quad 1 \le p < \infty$, then (see [12], Proposition 5)

$$\left(\int_{\mathbb{R}_+} |f(chx)|^p sh^{2\lambda}xdx\right)^{\frac{1}{p}} = \left(p\int\limits_0^\infty \left(\int\limits_0^{|f(chx)|} s^{p-1}ds\right) sh^{2\lambda}xdx\right)^{\frac{1}{p}}$$

$$= \left(p\int\limits_0^\infty s^{p-1}\left(\int\limits_{\{x\in[0,\infty):|f(chx)|>s\}} sh^{2\lambda}xdx\right) ds\right)^{\frac{1}{p}}$$

$$= \left(p\int\limits_0^\infty s^{p-1}\left(|\{x \in [0,\infty) : |f(chx)| > s\}|_\lambda\right) ds\right)^{\frac{1}{p}}$$

$$= \left(p\int\limits_0^\infty s^{p-1} f_{*,\lambda}(s)ds\right)^{\frac{1}{p}} = \left(\int\limits_0^\infty f^{*,\lambda}(cht)^p dt\right)^{\frac{1}{p}}. \tag{2.1}$$

Lemma 2.1 *[12]. Let $f, g, f_n(n = 1, 2, ...)$ measurable functions on measurable set $E \subset \mathbb{R}_+$. Then (i) $f_{*,\lambda}$ is nonnegative, decreasing and continuous from the right:*

(ii) If $|f(chx)| \le |g(chx)|$ for any $x \in E$ $\mu-$ a.e. then

$$f_{*,\lambda}(u) \le g_{*,\lambda}(u) \text{ for } u \ge 0.$$

(iii) If $|f(chx)| \le \liminf\limits_{n\to\infty}|f_n(chx)|$ for $x \in E$ $\mu-$ a.e., then $f_{*,\lambda}(u) \le \lim\limits_{n\to\infty}(f_n)_{*,\lambda}(u)$ for $u \ge 0$.

Lemma 2.2 *Let $f, g, f_n(n = 1, 2, ...)$ measurable functions on measurable set $E \subset \mathbb{R}_+$.Then the decreasing rearrangement $f^{*,\lambda}$ is a nonnegative, decreasing, right-continuous function on $\mathbb{R}_+$. Furthermore*

(i) $(f + g)^{*,\lambda}(cht_1 + cht_2) \le f^{*,\lambda}(cht_1) + g^{*,\lambda}(cht_2), \ (t_1, t_2 \ge 0)$

(ii) $|f(cht)| \le \lim\limits_{n\to\infty} \inf |f_n(cht)|$ for $x \in E$ $\mu-$ a.e., then $f^{*,\lambda}(cht) \le \lim\limits_{n\to\infty} \inf f_n^{*,\lambda}(cht)$.

Proof. That $f^{*,\lambda}$ is nonnegative , decreasing, and right-continuous follows from Lemma 2.1 and the fact that $f^{*,\lambda}$ is itself a distribution function (see [12] Proposition 2)

$$f^{*,\lambda}(cht) = m_{f_{*,\lambda}}\left(sh\frac{t}{2}\right), \ t \ge 0,$$

where m is the Lebesgue measure. We prove (i). Let $f^{*,\lambda}(cht_1) + g^{*,\lambda}(cht_2) < \infty$, otherwise there is nothing to prove. Then for $u_1 = f^{*,\lambda}(cht_1)$ and $u_2 = g^{*,\lambda}(cht_2)$, we have $f_{*,\lambda}(u_1) = f_{*,\lambda}\left(f^{*,\lambda}(cht_1)\right) \le sh\frac{t_1}{2}$ and $g_{*,\lambda}(u_2) = g_{*,\lambda}\left(f^{*,\lambda}(cht_2)\right) \le sh\frac{t_2}{2}$.

Since

$$\{x \in \mathbb{R}_+ : |f(chx) + g(chx)| > u_1 + u_2\} \subset \{x \in \mathbb{R}_+ : |f(chx)| > u_1\} \cup \{x \in \mathbb{R}_+ : |g(chx)| > u_2\}$$

it follows that

$$(f + g)_{*,\lambda}(u_1 + u_2) \le f_{*,\lambda}(u_1) + g_{*,\lambda}(u_2) \le sh\frac{t_1}{2} + sh\frac{t_2}{2}.$$

By inequality $f^{*,\lambda}\left(f_{*,\lambda}(u)\right) \leq u$ and the fact that $(f+g)^{*,\lambda}$ is decreasing, we get

$$(f+g)^{*,\lambda}(cht_1+cht_2) \leq (f+g)^{*,\lambda}(f+g)_{*,\lambda}\left(sh\frac{t_1}{2}+sh\frac{t_2}{2}\right)$$

$$\leq (f+g)^{*,\lambda}\left(f_{*,\lambda}+g_{*,\lambda}\right)(u_1+u_2) \leq u_1+u_2 = f^{*,\lambda}(cht_1)+g^{*,\lambda}(cht_2).$$

Remain prove (ii). This are immediate consequences of their counterparts in Lemma 2.1 and the definition of the decreasing rearrangement.

In the following we give several inequalities which we will need in the proof of our main results.

Lemma 2.3 *(Hardy inequalities type [12]). Suppose ξ and θ are nonnegative locally integrable functions defined on $(0,\infty)$ and $1<p\leq q<\infty$. Then there exists a constant $C>0$ such that for all nonnegative Lebesgue measurable ψ on $(0,\infty)$ the inequality*

$$\left(\int_0^\infty\left(\int_0^t\psi(chs)ds\right)^q\xi(cht)dt\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty(\psi(cht))^p\,\theta(cht)dt\right)^{\frac{1}{p}} \tag{2.2}$$

is satisfied if and only if

$$\sup_{s>0}\left(\int_s^\infty\xi(cht)dt\right)^{\frac{1}{q}}\left(\int_0^s\theta(cht)^{1-p'}dt\right)^{\frac{1}{p'}}<\infty. \tag{2.3}$$

Similarly for the dual operator,

$$\left(\int_0^\infty\left(\int_t^\infty\psi(chs)ds\right)^q\xi(cht)dt\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty(\psi(cht))^p\,\theta(cht)dt\right)^{\frac{1}{p}} \tag{2.4}$$

is satisfied if and only if

$$\sup_{s>0}\left(\int_0^s\xi(cht)dt\right)^{\frac{1}{q}}\left(\int_s^\infty\theta(cht)^{1-p'}dt\right)^{\frac{1}{p'}}<\infty, \tag{2.5}$$

where $p+p\prime=pp\prime$.

Lemma 2.4 *Let f and g be nonnegative measurable functions on $\mathbb{R}_+$.Then*

$$\int_{\mathbb{R}_+}f(chx)g(chx)sh^{2\lambda}xdx \leq \int_{\mathbb{R}_+}f^{*,\lambda}(chx)g^{*,\lambda}(chx)dx$$

and

$$\int_{\mathbb{R}_+}f^{*,\lambda}(chx)\frac{1}{\left(\frac{1}{g}\right)^{*,\lambda}(cht)}dt \leq \int_{\mathbb{R}_+}f(chx)g(chx)sh^{2\lambda}xdx.$$

First from this inequality is the Proposition 3.7 on [12]. Second inequality is prove if as well as Theorem 2.6 on [14], taking into account the Proposition 3.8 in [12] and also in the Lemma 2.1.

By analogy with [15], we introduce the following notation.

Definition 2.1 *An operator T is of weak type (p,q), $1 \le p,\ q \le \infty$, if and only if for every $f \in L_{p,\lambda}(\mathbb{R}_+)$ if there exists a constant C such that*

$$\sup_{t>0} \left(sh\frac{t}{2}\right)^{\frac{1}{q}} (Tf)^{*,\lambda}(cht) \le C \left\|f\right\|_{p,\lambda},$$

and strong type (p,q) if

$$\left(\int\limits_{\mathbb{R}_+} |Tf(chx)|^q sh^{2\lambda} x dx\right)^{\frac{1}{q}} \le C \left(\int\limits_{\mathbb{R}_+} f|(chx)|^p sh^{2\lambda} x dx\right)^{\frac{1}{p}}.$$

Further $A \lesssim B$ denotes that exists the constant $c > 0$ such that $0 < A \le cB$,, moreover c can depend on some parameters. Symbol $A \approx B$ denote that $A \lesssim B$ and $B \lesssim A$.

Lemma 2.5 *[11].For $0 < \lambda <$ the following relations*

$$|H(0,r)|_\lambda \approx \begin{cases} \left(sh\frac{r}{2}\right)^{2\lambda+1}, & 0 < r < 2, \\ \left(sh\frac{r}{2}\right)^{4\lambda}, & 2 \le r < \infty \end{cases}$$

is just.

Lemma 2.6 *Let $1 \le p_1 < p_2 < \infty$ and $1 \le q_1 < q_2 \le \infty$. A quasilinear operator T is simultaneously of weak types (p_1,q_1) and (p_2,q_2) if only if*

$$(Tf)^{*,\lambda}(cht) \lesssim \left[\left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}} \int\limits_0^{t^{\frac{\sigma_1}{\sigma_2}}} \left(sh\frac{u}{2}\right)^{\frac{1}{p_1}-1} f^{*,\lambda}(chu)du\right.$$

$$\left.+\left(sh\frac{t}{2}\right)^{-\frac{1}{q_2}} \int\limits_{t^{\frac{\sigma_1}{\sigma_2}}}^{\infty} \left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1} f^{*,\lambda}(chu)du\right], \quad 0 < t < \infty,$$

where $\sigma_1 = \frac{1}{q_1} - \frac{1}{q_2}$, $\sigma_2 = \frac{1}{p_1} - \frac{1}{p_2}$ and $\sigma_1 \ge \sigma_2$.

Proof. Let (α_1,β_1) and (α_2,β_2) two different points in open square $(0 < \alpha < 1, 0 < \beta < 1)$.

Suppose that $\frac{\sigma_1}{\sigma_2} = m = \frac{\beta_2-\beta_1}{\alpha_2-\alpha_1}$. Then we get

$$\sigma_1(\alpha_2-\alpha_1) = \sigma_2(\beta_2-\beta_1) \Leftrightarrow (\alpha_2-\alpha_1)\left(\frac{1}{q_1}-\frac{1}{q_2}\right)$$

$$= (\beta_2-\beta_1)\left(\frac{1}{p_1}-\frac{1}{p_2}\right) \Leftrightarrow \frac{1}{q_1}-\frac{1}{q_2} = \frac{\beta_2-\beta_1}{\alpha_2-\alpha_1}\left(\frac{1}{p_1}-\frac{1}{p_2}\right)$$

$$\Leftrightarrow \frac{\beta_2-\beta_1}{p_1(\alpha_2-\alpha_1)} - \frac{1}{q_1} = \frac{\beta_2-\beta_1}{p_2(\alpha_2-\alpha_1)} - \frac{1}{q_2} \Leftrightarrow \frac{m}{p_1} - \frac{1}{q_1} = \frac{m}{p_2} - \frac{1}{q_2}.$$

Further let $f_1 = f - f^{*,\lambda}(cht^m)$ if $f > f^{*,\lambda}(cht^m)$, $f_1 = f + f^{*,\lambda}(cht^m)$ if $f < -f^{*,\lambda}(cht^m)$ and $f_1 = 0$ otherwise.

Then clearly, if $f_2 = f - f_1$ then by Lemma 2.2 we have, $f^{*,\lambda} \leq f_1^{*,\lambda} + f_2^{*,\lambda}$ and $f_1^*(chu) = 0$ if $u > t^m$.

$$f_2^{*,\lambda}(chu) = f_1^{*,\lambda}(cht^m), \ \ \textit{if} \ \ u < t^m.$$

By Lemma 2.2, we have

$$(Tf)^{*,\lambda}(cht) \leq C(Tf_1 + Tf_2)^{*,\lambda}(cht) \leq C(Tf_1)^{*,\lambda}\left(\frac{1}{2}cht\right) + C(Tf_2)^{*,\lambda}\left(\frac{1}{2}cht\right) \tag{2.6}$$

and, assume that $p_1 < p_2$ we get

$$(Tf_1)^{*,\lambda}\left(\frac{1}{2}cht\right) \leq C_1\left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}} \|f_1\|_{p_1,\lambda}$$

$$\leq C_1\left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}} \int\limits_0^{t^m} f_1^{*,\lambda}(chu)\left(sh\frac{u}{2}\right)^{\frac{1}{p_1}-1} du, \tag{2.7}$$

$$(Tf_2)^{*,\lambda}\left(\frac{1}{2}cht\right) \leq C_2\left(sh\frac{t}{2}\right)^{-\frac{1}{q_2}} \|f\|_{p_2,\lambda}$$

$$\leq C_2\left(sh\frac{1}{2}\right)^{-\frac{t}{q_2}} \int\limits_0^{\infty} f_2^{*,\lambda}(chu)\left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1} du. \tag{2.8}$$

Let $0 < t < 2$. But

$$\begin{aligned}
&\left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}} \int\limits_0^{t^m} \left(sh\frac{u}{2}\right)^{\frac{1}{p_1}-1} du \approx \left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}} \int\limits_0^{t^m} \left(\frac{u}{2}\right)^{\frac{1}{p_1}-1} du \\
&= \frac{2}{p_1}\left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}} \left(\frac{t}{2}\right)^{\frac{m}{p_1}} \approx \left(sh\frac{t}{2}\right)^{\frac{m}{p_1}-\frac{1}{q_1}} = \left(sh\frac{t}{2}\right)^{\frac{m}{p_2}-\frac{1}{q_2}} \\
&\approx \left(sh\frac{t}{2}\right)^{-\frac{1}{q_2}} \left(\frac{t}{2}\right)^{\frac{m}{p_2}} = 2^{\frac{1-m}{p_2}} \frac{2}{p_1}\left(sh\frac{t}{2}\right)^{-\frac{1}{q_2}} \int\limits_0^{t^m} \left(\frac{u}{2}\right)^{\frac{1}{p_2}-1} du \\
&\approx \left(sh\frac{t}{2}\right)^{-\frac{1}{q_2}} \int\limits_0^{t^m} \left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1} du.
\end{aligned} \tag{2.9}$$

Now we consider the case $2 \leq t < \infty$. Then

$$\left(sh\frac{t}{2}\right)^{\frac{m}{p_2}} = \frac{1}{p_2} \int\limits_0^{\left(sh\frac{t}{2}\right)^m} u^{\frac{1}{p_2}-1} du \geq \frac{1}{p_2} \int\limits_0^{\left(\frac{t}{2}\right)^m} u^{\frac{1}{p_2}-1} du = \left[u = 2^{-m-1}x\right]$$

$$= \frac{2^{-\left(1+\frac{m}{p_2}\right)}}{p_2} \int\limits_0^{t^m} \left(\frac{u}{2}\right)^{\frac{1}{p_2}-1} du \geq \frac{2^{-\left(1+\frac{m}{p_2}\right)}}{p_2} \int\limits_0^{t^m} \left(sh\frac{t}{2}\right)^{\frac{1}{p_2}-1} du. \tag{2.10}$$

On the other hand we have

$$\int_0^{t^m}\left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1}du \geq \int_{t^m+\frac{1}{p_2}-1}^{t^m}(shu)^{\frac{1}{p_2}-1}du$$

$$=\int_{t^m+\frac{1}{p_2}-1}^{t^m}(shu)^{\frac{1}{p_2}-1}(ch^2u-sh^2u)du \geq \int_{t^m+\frac{1}{p_2}-1}^{t^m}(shu)^{\frac{1}{p_2}-1}(chu-shu)du$$

$$=\int_{t^m+\frac{1}{p_2}-1}^{t^m}(shu)^{\frac{1}{p_2}-1}chudu-\int_{t^m+\frac{1}{p_2}-1}^{t^m}(shu)^{\frac{1}{p_2}}du$$

$$\geq p_2(sht^m)^{\frac{1}{p_2}}-p_2\left[sh\left(t^m+\frac{1}{p_2}-1\right)\right]^{\frac{1}{p_2}}-\left(1-\frac{1}{p_2}\right)(sht^m)^{\frac{1}{p_2}}$$

$$=\left(p_2+\frac{1}{p_2}-1\right)(sht^m)^{\frac{1}{p_2}}-p_2\left[sh\left(t^m+\frac{1}{p_2}-1\right)\right]^{\frac{1}{p_2}}$$

$$=\left(p_2+\frac{1}{p_2}-1\right)\left[sh\left(t^m+\frac{1}{p_2}-1+1-\frac{1}{p_2}\right)\right]^{\frac{1}{p_2}}-p_2\left[sh\left(t^m+\frac{1}{p_2}-1\right)\right]^{\frac{1}{p_2}}$$

$$=\left(p_2+\frac{1}{p_2}-1\right)\left[sh\left(t^m+\frac{1}{p_2}-1\right)ch\left(1-\frac{1}{p_2}\right)\right.$$

$$\left.+ch\left(t^m+\frac{1}{p_2}-1\right)sh\left(1-\frac{1}{p_2}\right)\right]^{\frac{1}{p_2}}-p_2\left[sh\left(t^m+\frac{1}{p_2}-1\right)\right]^{\frac{1}{p_2}}$$

$$\geq\left(p_2+\frac{1}{p_2}-1\right)e^{1-\frac{1}{p_2}}\left[sh\left(t^m+\frac{1}{p_2}-1\right)\right]^{\frac{1}{p_2}}-p_2\left[sh\left(t^m+\frac{1}{p_2}-1\right)\right]^{\frac{1}{p_2}}$$

$$=\left(\left(p_2+\frac{1}{p_2}-1\right)e^{1-\frac{1}{p_2}}-p_2\right)e^{1-\frac{1}{p_2}}\Big)\left[sh\left(t^m+\frac{1}{p_2}-1\right)\right]^{\frac{1}{p_2}}-p_2\left[sh\left(t^m+\frac{1}{p_2}-1\right)\right]^{\frac{1}{p_2}}$$

$$\gtrsim\left[sh\left(t^m+\frac{1}{p_2}-1\right)\right]^{\frac{1}{p_2}}\gtrsim\left(sh\frac{t^m}{2}\right)^{\frac{1}{p_2}}\gtrsim\left(sh\frac{t^m}{2}\right)^{\frac{m}{p_2}},$$

since by $m\geq 1$ $sh\frac{t^m}{2}\gtrsim sh\left(\frac{t}{2}\right)^m\geq\left(sh\frac{t}{2}\right)^m$.

From (2.10) and (2.11) follows that by $2\leq t<\infty$

$$\left(sh\frac{t}{2}\right)^{\frac{m}{p_2}}\approx\int_0^{t^m}\left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1}du. \tag{2.11}$$

Combining (2.9) and (2.12) we have

$$\left(sh\frac{t}{2}\right)^{\frac{m}{p_2}}\approx\int_0^{t^m}\left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1}du,\ \ 0<t<\infty.$$

From this we get

$$\left(sh\frac{t}{2}\right)^{\frac{m}{p_2}-\frac{1}{q_2}} \approx \left(sh\frac{t}{2}\right)^{-\frac{1}{q_2}} \int\limits_0^{t^m} \left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1} du, \quad 0 < t < \infty. \tag{2.12}$$

Now from (2.9) and (2.13) we obtain

$$\left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}} \int\limits_0^{t^m} \left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1} du \approx \left(sh\frac{u}{2}\right)^{\frac{m}{p_1}-\frac{1}{q1}}$$

$$= \left(sh\frac{t}{2}\right)^{\frac{m}{p_2}-\frac{1}{q_2}} \approx \left(sh\frac{t}{2}\right)^{-\frac{1}{q_2}} \int\limits_0^{t^m} \left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1} du$$

and therefore, since $f_2^{*,\lambda}(cht)$ is constant for $u < t^m$, we have

$$\left(sh\frac{t}{2}\right)^{-\frac{1}{q_2}} \int\limits_0^{\infty} f_2^{*,\lambda}(chu) \left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1} du$$

$$\approx \left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}} \int\limits_0^{t^m} f_2^{*,\lambda}(chu) \left(sh\frac{u}{2}\right)^{\frac{1}{p_1}-1} du$$

$$+ \left(sh\frac{t}{2}\right)^{-\frac{1}{q_2}} \int\limits_{t^m}^{\infty} f_2^{*,\lambda}(chu) \left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1} du. \tag{2.13}$$

Taking into account (2.7),(2.8) and (2.14) in (2.6) we obtain the assertion of Lemma 2.6.

Theorem A. [1]. *Suppose T is a quasilinear operator defined on simple functions such that T is simultaneously of weak type $(p_i, q_i), i = 0, 1;\ 1 \le p_0 < p_1 < \infty,\ 1 \le q_0,\ q_1 \le \infty$. Let $\sigma_1 = \frac{1}{q_0} - \frac{1}{q_1} \neq 0, \quad \sigma_2 = \frac{1}{p_0} - \frac{1}{p_1}$. If $1 \le p \le q \le \infty$ and*

$$\sup_{s>0} \left(\int\limits_{s^{\frac{\sigma_1}{\sigma_2}}}^{\infty} \left[U(t)t^{-\frac{n}{q_0}}\right]^q t^{n-1} dt \right)^{\frac{1}{q}} \left(\int\limits_0^{s^{\frac{\sigma_1}{\sigma_2}}} \left[V(t)t^{n\left(\frac{1}{p}-\frac{1}{p_0}\right)}\right]^{-p'} \frac{dt}{t} \right)^{\frac{1}{p'}} < \infty,$$

$$\sup_{s>0} \left(\int\limits_0^{s^{\frac{\sigma_1}{\sigma_2}}} \left[U(t)t^{-\frac{n}{q_1}}\right]^q t^{n-1} dt \right)^{\frac{1}{q}} \left(\int\limits_{s^{\frac{\sigma_1}{\sigma_2}}}^{\infty} \left[V(t)t^{n\left(\frac{1}{p}-\frac{1}{p_1}\right)}\right]^{-p'} \frac{dt}{t} \right)^{\frac{1}{p'}} < \infty,$$

then

$$\left(\int\limits_0^{\infty} [U(|x|)(Tf)^*(|x|)]^q \, dx \right)^{\frac{1}{q}} \le C \left(\int\limits_0^{\infty} [V(|x|) f^*(|x|)]^p \, dx \right)^{\frac{1}{p}}.$$

From this theorem for the Reisz potential I_α defined by

$$(J_\alpha f)(x) = \int\limits_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)dy$$

follows next result.

Theorem B. [1]. *Suppose u and v are defined on $\mathbb{R}^n$ and $U = u^*$, $\frac{1}{V} = \left(\frac{1}{v}\right)^*$. If $1 \le p \le q \le \infty$, $p < \infty$ and for some $r, 1 < r < \frac{n}{\alpha}$*

$$\sup_{s>0} \left(\int\limits_s^\infty \left[U(t)t^{\alpha-n}\right]^q t^{n-1} dt \right)^{\frac{1}{q}} \left(\int\limits_0^s V(t)(t)^{-p'} t^{n-1} dt \right)^{\frac{1}{p'}} < \infty$$

and

$$\sup_{s>0} \left(\int\limits_0 \left[U(t)t^{\alpha-\frac{n}{r}}\right]^q t^{n-1} dt \right)^{\frac{1}{q}} \left(\int\limits_s^\infty V(t)(t)^{-p'} t^{n-1} dt \right)^{\frac{1}{p'}} < \infty,$$

then

$$\left(\int\limits_{\mathbb{R}^n} |u(x) J_\alpha f(x)|^q dx \right)^{\frac{1}{q}} \le C \left(\int\limits_{\mathbb{R}^n} [v(x) f(x)]^p dx \right)^{\frac{1}{p}}.$$

The following theorem is analogue of the Theorem A.

Theorem 2.1 *Suppose T is a quasilinear operator defined on simple functions such that T is simultaneously of weak type $(p_i, q_i), i = 1, 2$. Let $1 < p_1 < p_2 < \infty$, and let $1 \le q_1, q_2 \le \infty$. Let $\sigma_1 = \frac{1}{q_1} - \frac{1}{q_2} \ne 0, \sigma_2 = \frac{1}{p_1} - \frac{1}{p_2}$. If $1 < p < q < \infty$ and*

$$\sup_{s>0} \left\{ \int\limits_{s^{\frac{\sigma_1}{\sigma_2}}}^\infty \left[U(cht) \left(sh\frac{t}{2} \right)^{-\frac{1}{q_1}} \right]^q dt \right\}^{\frac{1}{q}} \left\{ \int\limits_0^{s^{\frac{\sigma_1}{\sigma_2}}} \left[V(cht) \left(sh\frac{t}{2} \right)^{1-\frac{1}{p_1}} \right]^{-p'} dt \right\}^{\frac{1}{p'}} < \infty, \tag{2.14}$$

$$\sup_{s>0} \left\{ \int\limits_0^{s^{\frac{\sigma_1}{\sigma_2}}} \left[U(cht) \left(sh\frac{t}{2} \right)^{-\frac{1}{q_2}} \right]^q dt \right\}^{\frac{1}{q}} \left\{ \int\limits_{s^{\frac{\sigma_1}{\sigma_2}}}^\infty \left[V(cht) \left(sh\frac{t}{2} \right)^{1-\frac{1}{p_2}} \right]^{-p'} dt \right\}^{\frac{1}{p'}} < \infty, \tag{2.15}$$

then

$$\left\{ \int\limits_0^\infty \left[U(cht) \left(Tf\right)^{*,\lambda}(cht) \right]^q dt \right\}^{\frac{1}{q}} \lesssim \left\{ \int\limits_0^\infty \left[V(cht) f^{*,\lambda}(cht) \right]^p dt \right\}^{\frac{1}{p}}.$$

Proof. By Lemma 2.6 a quasilinear operator T satisfies the weak type hypothesis if and only if

$$(Tf)^{*\lambda}(cht) \lesssim \left[\left(sh\frac{t}{2} \right)^{-\frac{1}{q_1}} \int\limits_0^{t^{\frac{\sigma_1}{\sigma_2}}} \left(sh\frac{u}{2} \right)^{\frac{1}{p_1}-1} f^{*,\lambda}(chu) du \right.$$

$$+\left(sh\frac{t}{2}\right)^{-\frac{1}{q_2}}\int\limits_{t^{\frac{\sigma_1}{\sigma_2}}}^{\infty}\left(sh\frac{u}{2}\right)^{\frac{1}{p_2}-1}f^{*,\lambda}(chu)du\Bigg].$$

On multiplying $U(cht)$, integrating and applying the Minkowski inequality one obtains

$$\left\{\int\limits_0^{\infty}\left[U(cht)(Tf)^{*,\lambda}(cht)\right]^q dt\right\}^{\frac{1}{q}}$$

$$\lesssim\left\{\int\limits_0^{\infty}\left[U(cht)\left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}}\right]^q\left[\int\limits_0^{t^{\frac{\sigma_1}{\sigma_2}}}\left(sh\frac{s}{2}\right)^{\frac{1}{p_1}-1}f^{*,\lambda}(chs)ds\right]^q dt\right\}^{\frac{1}{q}}$$

$$+\left\{\int\limits_0^{\infty}\left[U(cht)\left(sh\frac{t}{2}\right)^{-\frac{1}{q_2}}\right]^q\left[\int\limits_{t^{\frac{\sigma_1}{\sigma_2}}}^{\infty}\left(sh\frac{s}{2}\right)^{\frac{1}{p_2}-1}f^{*,\lambda}(chs)ds\right]^q dt\right\}^{\frac{1}{q}}=\{J_1+J_2\},\tag{2.16}$$

respectively.

Take $\left[U(cht)\left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}}\right]^q=\xi(cht)$, $\left[V(cht)\left(sh\frac{t}{2}\right)^{1-\frac{1}{p_1}}\right]^p=\theta(cht)$, $\left(sh\frac{t}{2}\right)^{\frac{1}{p_1}-1}f^{*,\lambda}(cht)=\psi(cht)$ and using the first part Lemma 2.3 by $\sigma_1=\sigma_2$ we have

$$J_1\lesssim\left\{\int\limits_0^{\infty}\left[V(cht)f^{*,\lambda}(cht)\right]^p dt\right\}^{\frac{1}{p}},\tag{2.17}$$

is and only if

$$\left\{\int\limits_s^{\infty}\left[U(cht)\left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}}\right]^q dt\right\}^{\frac{1}{q}}\left\{\int\limits_0^{s}\left[V(cht)\left(sh\frac{t}{2}\right)^{1-\frac{1}{p_1}}\right]^{-p'}dt\right\}^{\frac{1}{p'}}\lesssim 1,$$

for all $s>0$.

A similar argument applies to J_2 only now one utilizes the second part of Lemma 2.3.

$$J_2\lesssim\left\{\int\limits_0^{\infty}\left[V(cht)f^{*,\lambda}(cht)\right]^p dt\right\}^{\frac{1}{p}},\tag{2.18}$$

if and only if

$$\left\{\int\limits_0^{s}\left[U(cht)\left(sh\frac{t}{2}\right)^{-\frac{1}{q_1}}\right]^q dt\right\}^{\frac{1}{q}}\left\{\int\limits_s^{\infty}\left[V(cht)\left(sh\frac{t}{2}\right)^{1-\frac{1}{p_2}}\right]^{-p'}dt\right\}^{\frac{1}{p'}}\lesssim 1.$$

Taking into account (2.18) and (2.19) on (2.17) we obtain the assertion of theorem.

Corollary 1. If T is an operator and U, V are functions on $\mathbb{R}_+$ satisfying the conditions of Theorem 2.2, then for $1 < p \leq q < \infty$,

$$\left\|(Tf)^{*,\lambda}\right\|_{q,U} \lesssim \left\|f^{*,\lambda}\right\|_{p,V}.$$

Furthermore, if u and v are such that $u^{*,\lambda} = U$ and $\left(\frac{1}{v}\right)^{*,\lambda} = \frac{1}{V}$, then $\|Tf\|_{q,u,\lambda} \lesssim \|f\|_{p,v,\lambda}$. If $p < \infty$ the operator T extends uniquely to all of $L_{p,v,\lambda}$ preserving the norm inequality with the same bound.

Proof. The first part of the corollary as a restatement of Theorem 2.2. Also, by (2.1)

$$\|Tf\|_{q,u,\lambda} \lesssim \left(\int\limits_0^\infty \left[u^{*,\lambda}(cht)(Tf)^{*,\lambda}(cht)\right]^q dt\right)^{\frac{1}{q}},$$

then by definition of u, U, v, V and Theorem 2.2 and Lemma 2.4 this integral is dominated by

$$\left(\int\limits_0^\infty \left[U(cht)(Tf)^{*,\lambda}(cht)\right]^q dt\right)^{\frac{1}{q}} \lesssim \left(\int\limits_0^\infty \left[V(cht)f^{*,\lambda}(cht)\right]^p dt\right)^{\frac{1}{p}}$$

$$\lesssim \left(\int\limits_0^\infty \left[\frac{f^{*,\lambda}(cht)}{\left(\frac{1}{v}\right)^{*,\lambda}(cht)}\right]^p dt\right)^{\frac{1}{p}} \lesssim \left(\int\limits_0^\infty |v(cht)f(cht)|^p \, sh^{2\lambda}t dt\right)^{\frac{1}{p}}.$$

Since $J_G^{\alpha,\lambda}$ is linear operator, we can suppose $T = J_G^{\alpha,\lambda}$, then we will obtain next results, which is analogue of the Theorem B.

Theorem 2.2 *Suppose u and v are defined on $\mathbb{R}_+$ and $U = u^{*,\lambda}$, $\frac{1}{V} = (\frac{1}{v})^{*,\lambda}$. If $1 < p \leq q < \infty$, and for some r, $1 < r < \frac{2\lambda+1}{\alpha}$*

$$\sup_{s>0}\left\{\int\limits_s^\infty \left[U(cht)\left(sh\frac{t}{2}\right)^{\frac{\alpha}{2\lambda+1}-1}\right]^q dt\right\}^{\frac{1}{q}} \left\{\int\limits_0^s [V(cht)]^{-p'} dt\right\}^{\frac{1}{p'}} < \infty\,, \tag{2.19}$$

and

$$\sup_{s>0}\left\{\int\limits_0^s \left[U(cht)\left(sh\frac{t}{2}\right)^{\frac{\alpha}{2\lambda+1}-\frac{1}{r}}\right]^q dt\right\}^{\frac{1}{q}} \left\{\int\limits_s^\infty \left[V(cht)\left(sh\frac{t}{2}\right)^{\frac{1}{r'}}\right]^{-p'} dt\right\}^{\frac{1}{p'}} < \infty, \tag{2.20}$$

then $J_G^{\alpha,\lambda} : L_{p,v,\lambda}(\mathbb{R}_+) \longmapsto L_{q,u,\lambda}(\mathbb{R}_+)$ is bounded.

Proof. In the notation of Theorem 2.1, $p_1 = 1, p_2 = r,\quad q_1 = \frac{1}{1-\frac{\alpha}{2\lambda+1}}$ and $q_2 = \frac{1}{\frac{1}{r}-\frac{\alpha}{2\lambda+1}}$, so that $\sigma_1 = \sigma_2 = \frac{1}{r'}$. Therefore, condition (2.15) and (2.16) are equivalent to (2.20) respectively (2.21) and the result follows from Corollary 1.

Note that if $1 < p < \frac{2\lambda+1}{\alpha}, \frac{1}{p} - \frac{1}{r} = \frac{\alpha}{2\lambda+1}$ choose r, so that $1 < p < r$, then $u = v \equiv 1$ satisfy (2.20) and (2.21). Therefore Theorem 2.2 reduces to the Hardy-Littlewood- Sobolew theorem for the G -fractional integral $J_G^{\alpha,\lambda} f$ (see [12], Theorem 5.4).

3 Two weighted inequalities for G- fractional integrals.

In this section we prove an analog of Heinig's results for the G-fractional integral $J_G^{\alpha,\lambda} f$. Further, the Stein-Weiss inequality for $J_G^{\alpha,\lambda} f$ is proved as an application of this result.

Theorem 3.1 *Let* $0<\alpha<2\lambda+1, 1<r<\frac{2\lambda+1}{\alpha}, 1<p\le q<\infty$ *.Suppose that* u *and* v *are non-negative locally integrable functions on* $\mathbb{R}_+$ *with conditions*

$$\sup_{s>0}\left(\int\limits_s^\infty u^{*,\lambda}(cht)\left(sh\frac{t}{2}\right)^{-q\left(1-\frac{\alpha}{2\lambda+1}\right)}dt\right)^{\frac{1}{q}}\left(\int\limits_0^s\left(\left(\frac{1}{v}\right)^{*,\lambda}(cht)^{p'-1}dt\right)\right)^{\frac{1}{p'}}<\infty \tag{3.1}$$

and

$$\sup_{s>0}\left(\int\limits_0^s u^{*,\lambda}(cht)\left(sh\frac{t}{2}\right)^{-q\left(1-\frac{\alpha}{2\lambda+1}\right)}dt\right)^{\frac{1}{q}}\left(\int\limits_s^\infty\left(\left(\frac{1}{v}\right)^{*,\lambda}(cht)^{p'\left(\frac{1}{r}-1\right)}dt\right)\right)^{\frac{1}{p'}}<\infty. \tag{3.2}$$

Then $J_G^{\alpha,\lambda}$ *is a bounded operator from* $L_{p,v,\lambda}(\mathbb{R}_+)$ *to* $L_{q,u,\lambda}(\mathbb{R}_+)$*, that is for any* $f\in L_{p,v,\lambda}(\mathbb{R}_+)$

$$\left\|J_G^{\alpha,\lambda} f\right\|_{q,u,\lambda}\lesssim\|f\|_{p,v,\lambda}.$$

Proof. It is known that (see [12]) $J_G^{\alpha,\lambda} f$ is an operator of weak type $(1,\frac{1}{1-\frac{\alpha}{2\lambda+1}})$ and is an operator of strong type $\left(r,\frac{1}{\frac{1}{r}-\frac{q}{2\lambda+1}}\right)$, where $1<r<\infty$. Refer to $J_G^{\alpha,\lambda} f$, Lemma 2.7, taking $p_1=1,\ q_1=\frac{1}{1-\frac{\alpha}{2\lambda+1}},\ p_2=r,\ q_2=\frac{1}{\frac{1}{r}-\frac{\alpha}{2\lambda+1}}$.

Then

$$\sigma_1=\frac{1}{q_1}-\frac{1}{q_2}=1-\frac{1}{r},\ \ \sigma_2=\frac{1}{p_1}-\frac{1}{p_2}=1-\frac{1}{r},\ \ \frac{\sigma_1}{\sigma_2}=1,$$

and

$$\left\{\int\limits_0^\infty\left[\left(J_G^{\alpha,\lambda} f\right)^{*,\lambda}(cht)\right]^q u^{*,\lambda}(cht)dt\right\}^{\frac{1}{q}}\lesssim\left\{\int\limits_0^\infty\left[\left(sh\frac{t}{2}\right)^{\frac{\alpha}{2\lambda+1}-1}\int\limits_0^t f^{*,\lambda}(chs)ds\right.\right.$$

$$\left.\left.+\left(sh\frac{t}{2}\right)^{\frac{\alpha}{2\lambda+1}-\frac{1}{r}}\int\limits_t^\infty\left(sh\frac{s}{2}\right)^{\frac{1}{q}-1}f^{*,\lambda}(chs)ds\right]^q u^{*,\lambda}(cht)dt\right\}^{\frac{1}{q}}.$$

Applying the Minkowski inequality we get

$$\left\{\int_0^\infty \left[\left(J_G^{\alpha,\lambda} f\right)^{*,\lambda}(cht)\right]^q u^{*,\lambda}(cht)dt\right\}^{\frac{1}{q}}$$
$$\lesssim \left[\int_0^\infty u^{*,\lambda}(cht)\left(sh\frac{t}{2}\right)^{\left(\frac{\alpha}{2\lambda+1}-1\right)q}\left(\int_0^t f^{*,\lambda}(chs)ds\right)^q dt\right]^{\frac{1}{q}} \tag{3.3}$$
$$+\left[\int_0^\infty u^{*,\lambda}(cht)\left(sh\frac{t}{2}\right)^{\left(\frac{\alpha}{2\lambda+1}-\frac{1}{r}\right)q}\left(\int_t^\infty f^{*,\lambda}(chs)\left(sh\frac{s}{2}\right)^{\frac{1}{r}-1}ds\right)^q dt\right]^{\frac{1}{q}}.$$

If we take the notation $\xi(cht) = u^{*,\lambda}(cht)\left(sh\frac{t}{2}\right)^{\left(\frac{\alpha}{2\lambda+1}-1\right)q}$, $\psi(cht) = f^{*,\lambda}(cht)$, $\theta(cht) = \frac{1}{\left(\frac{1}{v}\right)^{*,\lambda}(cht)}$, then we have (2.3) from (3.1) and applying (2.2)

$$\left[\int_0^\infty u^{*,\lambda}(cht)\left(sh\frac{t}{2}\right)^{\left(\frac{\alpha}{2\lambda+1}-1\right)q}\left(\int_0^t f^{*,\lambda}(chs)ds\right)^q dt\right]^{\frac{1}{q}}$$
$$\lesssim \left[\int_0^\infty \frac{1}{\left(\frac{1}{v}\right)^{*,\lambda}(cht)}\left(f^{*,\lambda}(cht)\right)^p dt\right]^{\frac{1}{p}}. \tag{3.4}$$

Now if we take $\xi(cht) = u^{*,\lambda}(cht)\left(sh\frac{t}{2}\right)^{\left(\frac{\alpha}{2\lambda+1}-\frac{1}{r}\right)q}$, $\psi(cht) = \left(sh\frac{t}{2}\right)^{\frac{1}{r}-1} f^{*,\lambda}(cht)$, $\theta(cht) = \frac{1}{\left(\frac{1}{v}\right)^{*,\lambda}(cht)}\left(sh\frac{t}{2}\right)^{p\left(\frac{1}{r}-1\right)}$, then we have (1.5) from (2.2) and applying (2.4) we can assert that

$$\left[\int_0^\infty u^{*,\lambda}(cht)\left(sh\frac{t}{2}\right)^{\left(\frac{\alpha}{2\lambda+1}-\frac{1}{r}\right)q}\left(\int_t^\infty \left(sh\frac{s}{2}\right)^{\frac{1}{r}-1} f^{*,\lambda}(chs)ds\right)^q dt\right]^{\frac{1}{q}}$$
$$\le \left\{\int_0^\infty \left[\left(sh\frac{s}{2}\right)^{\frac{1}{r}-1} f^{*,\lambda}(cht)\right]^p \frac{1}{\left(\frac{1}{v}\right)^{*,\lambda}(cht)}\left(sh\frac{t}{2}\right)^{p\left(\frac{1}{r}-1\right)} dt\right\}^{\frac{1}{p}} \tag{3.5}$$
$$= \left[\int_0^\infty \frac{1}{\left(\frac{1}{v}\right)^{*,\lambda}(cht)}\left(f^{*,\lambda}(cht)\right)^p dt\right]^{\frac{1}{p}}.$$

Combining (3.3),(3.4),(3.5) yields

$$\left\{\int_0^\infty \left[\left(J_G^{\alpha,\lambda} f\right)^{*,\lambda}(cht)\right]^q u^{*,\lambda}(cht)dt\right\}^{\frac{1}{q}}$$

$$\lesssim \left[\int_0^\infty \frac{1}{\left(\frac{1}{v}\right)^{*,\lambda}(cht)} \left(f^{*,\lambda}(cht)\right)^p dt \right]^{\frac{1}{p}}. \tag{3.6}$$

Applying Lemma 2.4 and (3.6) we have

$$\begin{aligned} \left\{ \int_{\mathbb{R}_+} \left[J_G^{\alpha,\lambda} f(cht) \right]^q u(chx) sh^{2\lambda} x dx \right\}^{\frac{1}{q}} &\lesssim \left[\int_0^\infty \frac{1}{\left(\frac{1}{v}\right)^{*,\lambda}(cht)} \left(f^{*,\lambda}(cht)\right)^p dt \right]^{\frac{1}{p}} \\ &\lesssim \left[\int_{\mathbb{R}_+} (f(chx))^p v(chx) sh^{2\lambda} x dx \right]^{\frac{1}{p}}. \end{aligned} \tag{3.7}$$

In the following theorem we prove the Stein- Weiss inequality for G- fractional integrals by using Theorem 3.1.

Note that the Stein-Weiss inequality for classical Riesz potentials was given in [18]. For B-fractional integrals, this inequality was proved in [8] and [9].

Let

$$u(chx) = \begin{cases} \left(sh\frac{x}{2}\right)^\beta, & 0 < x < 2, \\ \left(sh\frac{x}{2}\right)^{\frac{2\lambda+1}{4\lambda\beta}}, & 2 \le x < \infty, \end{cases}$$

$$v(chx) = \begin{cases} \left(sh\frac{x}{2}\right)^{\beta+\alpha p}, & 0 < x < 2, \\ \left(sh\frac{x}{2}\right)^{\frac{(2\lambda+1)(\beta+\alpha p)}{4\lambda}}, & 2 \le x < \infty. \end{cases}$$

Theorem 3.2 *Let* $0 < \alpha < 2\lambda+1,\ 1 < p < \frac{2\lambda+1}{\alpha},\ \beta < 0,\ 0 < \beta+\alpha p < (2\lambda+1)(p-1)$. *Then* $J_G^{\alpha,\lambda} f$ *is a bounded operator from* $J_{q,u,\lambda}(\mathbb{R}_+)$ *to* $J_{p,v,\lambda}(\mathbb{R}_+)$*,i.e. .*

$$\left(\int_{\mathbb{R}_+} \left| J_G^{\alpha,\lambda} f(chx) \right|^q u(chx) sh^{2\lambda} x dx \right)^{\frac{1}{q}} \lesssim \left(\int_{\mathbb{R}_+} |f(chx)|^p v(chx) sh^{2\lambda} x dx \right)^{\frac{1}{p}}.$$

Proof. Let $0 < x < 2$. By Lemma 2.6 since $\beta < 0$ we have

$$\begin{aligned} u_{*,\lambda}(s) &= \left| \left\{ x \in (0,2) : sh^\beta \frac{x}{2} > s \right\} \right|_\lambda = \left| \left\{ x \in (0,2) : sh\frac{x}{2} < s^{\frac{1}{\beta}} \right\} \right|_\lambda \\ &= \left| H\left(0, s^{\frac{1}{\beta}}\right) \right|_\lambda \approx \left(sh\frac{s}{2}\right)^{\frac{2\lambda+1}{\beta}}. \end{aligned}$$

Now let $2 \le x \le \infty$. Again by Lemma 2.6 we get

$$\begin{aligned} u_{*,\lambda}(s) &= \left| \left\{ x \in [2,\infty) : \left(sh\frac{x}{2}\right)^{\frac{4\lambda\beta}{2\lambda+1}} > s \right\} \right|_\lambda = \left| \left\{ x \in [2,\infty) : sh\frac{x}{2} < s^{\frac{2\lambda+1}{4\lambda\beta}} \right\} \right|_\lambda \\ &= \left| H\left(0, s^{\frac{2\lambda+1}{4\lambda\beta}}\right) \right|_\lambda \approx \left(sh\frac{s}{2}\right)^{\frac{2\lambda+1}{\beta}}, \end{aligned}$$

and

$$u^{*,\lambda}(cht) = \inf\left\{ s > 0 : u_{*,\lambda}(s) \le sh\frac{t}{2} \right\} = \inf\left\{ s > 0 : \left(sh\frac{s}{2}\right)^{\frac{2\lambda+1}{\beta}} \le sh\frac{t}{2} \right\}$$

$$= \inf\left\{ s > 0 : sh\frac{s}{2} \geq \left(sh\frac{t}{2}\right)^{\frac{\beta}{2\lambda+1}} \right\} = \left(sh\frac{t}{2}\right)^{\frac{\beta}{2\lambda+1}}.$$

Since $\beta + \alpha p > 0$ we can write

$$\left(\frac{1}{v}\right)_{*,\lambda}(s) = \left|\left\{ x \in (0,2) : \left(sh\frac{x}{2}\right)^{-(\beta+\alpha p)} > s \right\}\right|_{\lambda} = \left|\left\{ x \in (0,2) : sh\frac{x}{2} > s^{-\frac{1}{\beta+\alpha p}} \right\}\right|_{\lambda}$$
$$= \left| H\left(0, s^{-\frac{1}{\beta+\alpha p}}\right)\right|_{\lambda} \approx \left(sh\frac{s}{2}\right)^{-\frac{2\lambda+1}{\beta+\alpha p}}$$

and also

$$\left(\frac{1}{v}\right)_{*,\lambda}(s) = \left|\left\{ x \in [2,\infty) : \left(sh\frac{x}{2}\right)^{-\frac{(\beta+\alpha p)4\lambda}{2\lambda+1}} > s \right\}\right|_{\lambda}$$
$$= \left|\left\{ x \in [2,\infty) : sh\frac{x}{2} < s^{-\frac{2\lambda+1}{(\beta+\alpha p)4\lambda}} \right\}\right|_{\lambda} = \left| H\left(0, s^{-\frac{2\lambda+1}{(\beta+\alpha p)4\lambda}}\right)\right|_{\lambda} \approx \left(sh\frac{s}{2}\right)^{-\frac{2\lambda+1}{(\beta+\alpha p)4\lambda}}.$$

On the other hand

$$\left(\frac{1}{v}\right)^{*,\lambda}(cht) = \inf\left\{ s > 0 : \left(\frac{1}{v}\right)_{*,\lambda}(s) \leq sh\frac{t}{2} \right\}$$
$$= \inf\left\{ s > 0 : \left(sh\frac{s}{2}\right)^{-\frac{2\lambda+1}{\beta+\alpha p}} \leq sh\frac{t}{2} \right\} = \inf\left\{ s > 0 : sh\frac{s}{2} \geq \left(sh\frac{t}{2}\right)^{-\frac{\beta+\alpha p}{2\lambda+1}} \right\} = \left(sh\frac{t}{2}\right)^{-\frac{\beta+\alpha p}{2\lambda+1}}.$$

Take $p = q = r$ and examine (3.1) and (3.2) since $\beta + \alpha p < (2\lambda + 1)(p - 1)$ we have $\frac{\beta+\alpha p}{2\lambda+1} - p < -1$ and $-\frac{\beta+\alpha p}{2\lambda+1}(p' - 1) > -1$. Then

$$\sup_{s>0}\left(\int_s^\infty u^{*,\lambda}(cht)\left(sh\frac{t}{2}\right)^{-q\left(1-\frac{\alpha}{2\lambda+1}\right)} dt\right)^{\frac{1}{q}} \left(\int_0^s \left(\left(\frac{1}{v}\right)^{*,\lambda}(cht)^{p'-1} dt\right)\right)^{\frac{1}{p'}}$$

$$\approx \sup_{s>0}\left(\int_s^\infty \left(sh\frac{t}{2}\right)^{\frac{\beta}{2\lambda+1}}\left(sh\frac{t}{2}\right)^{-p\left(1-\frac{\alpha}{2\lambda+1}\right)} dt\right)^{\frac{1}{p}} \left(\int_0^s \left(\left(sh\frac{t}{2}\right)^{-\frac{\beta+\alpha p}{2\lambda+1}}\right)^{p'-1}\right)^{\frac{1}{p'}}$$

$$\lesssim \sup_{s>0}\left(\int_s^\infty \left(sh\frac{t}{2}\right)^{\frac{\beta+\alpha p}{2\lambda+1}-p} d\left(sh\frac{t}{2}\right)\right)^{\frac{1}{p}} \left(\int_0^s \left(\left(sh\frac{t}{2}\right)^{-\frac{\beta+\alpha p}{2\lambda+1}}\right)^{p'-1} d\left(sh\frac{t}{2}\right)\right)^{\frac{1}{p'}}$$

$$\lesssim \sup_{s>0}\left(\frac{1}{\frac{\beta+\alpha p}{2\lambda+1} - p + 1}\left(sh\frac{t}{2}\right)^{\frac{\beta+\alpha p}{2\lambda+1}-p+1}\Bigg|_s^\infty\right)^{\frac{1}{p}} \left(\frac{1}{-\frac{\beta+\alpha p}{2\lambda+1}(p'-1)+1}\left(sh\frac{t}{2}\right)^{-\frac{\beta+\alpha p}{2\lambda+1}(p'-1)+1}\Bigg|_0^s\right)^{\frac{1}{p'}}$$

$$\lesssim \left(\frac{-1}{\frac{\beta+\alpha p}{2\lambda+1} - p + 1}\right)^{\frac{1}{p}} \left(\frac{1}{-\frac{\beta+\alpha p}{2\lambda+1}(p'-1)+1}\right)^{\frac{1}{p'}} \sup_{s>0}\left(sh\frac{s}{2}\right)^{\left(\frac{\beta+\alpha p}{2\lambda+1}-p+1\right)\frac{1}{p}+\left(-\frac{\beta+\alpha p}{2\lambda+1}(p'-1)+1\right)\frac{1}{p'}}$$

$$= \left(\frac{-1}{\frac{\beta+\alpha p}{2\lambda+1} - p + 1}\right)^{\frac{1}{p}} \left(\frac{1}{-\frac{\beta+\alpha p}{2\lambda+1}(p'-1)+1}\right)^{\frac{1}{p'}} < \infty.$$

Now examine (3.2). Since $\beta+\alpha p > 0$ we have $\frac{\beta+\alpha p}{2\lambda+1} - 1 > -1$ and $-\frac{\beta+\alpha p}{2\lambda+1}\cdot(p'-1) - 1 < 1$. Then

$$\sup_{s>0} \left(\int_0^s u^{*,\lambda}(cht)\left(sh\frac{t}{2}\right)^{-q\left(\frac{1}{r}-\frac{\alpha}{2\lambda+1}\right)} dt\right)^{\frac{1}{q}}$$

$$\times \left(\int_s^\infty \left(\left(\frac{1}{v}\right)^{*,\lambda}(cht)^{p'-1}\left(sh\frac{t}{2}\right)^{p'\left(\frac{1}{r}-1\right)} dt\right)\right)^{\frac{1}{p'}}$$

$$\approx \sup_{s>0} \left(\int_0^s \left(sh\frac{t}{2}\right)^{\frac{\beta}{2\lambda+1}} \left(sh\frac{t}{2}\right)^{-p\left(\frac{1}{p}-\frac{\alpha}{2\lambda+1}\right)} dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_s^\infty \left(\left(sh\frac{t}{2}\right)^{-\frac{\beta+\alpha p}{2\lambda+1}}\right)^{p'-1} \left(sh\frac{t}{2}\right)^{p'\left(\frac{1}{p}-1\right)} dt\right)^{\frac{1}{p'}}$$

$$\lesssim \sup_{s>0} \left(\frac{2\lambda+1}{\beta+\alpha p}\left(sh\frac{t}{2}\right)^{\frac{\beta+\alpha p}{2\lambda+1}}\Bigg|_0^s\right)^{\frac{1}{p}} \left(\frac{2\lambda+1}{(\beta+\alpha p)(1-p')}\left(sh\frac{t}{2}\right)^{-\frac{\beta+\alpha p}{2\lambda+1}(p'-1)}\Bigg|_s^\infty\right)^{\frac{1}{p'}}$$

$$= \frac{2\lambda+1}{\beta+\alpha p}\left(\frac{1}{p'-1}\right)^{\frac{1}{p'}} \sup_{s>0}\left(sh\frac{t}{2}\right)^{\frac{\beta+\alpha p}{2\lambda+1}\cdot\frac{1}{p} - \frac{\beta+\alpha p}{2\lambda+1}\cdot\frac{p'-1}{p'}} = \frac{2\lambda+1}{\beta+\alpha p}\left(\frac{1}{p'-1}\right)^{\frac{1}{p'}} < \infty.$$

Therefore (3.2) and (3.3) are satisfied and from Theorem 2.3 we have the result of the corollary.

Remark. *We note that similar result for B- fractional integrals is obtained in [16].*

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