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Calderón-Zygmund operators with kernels of Dini's type and their multilinear commutators on generalized Morrey spaces

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Abstract. *In this paper, we obtain the endpoint boundedness for the Calderón-Zygmund operators with kernels of Dini's type on generalized Morrey spaces. We also get similar results for the multilinear commutators of Calderón-Zygmund operators with kernels of Dini's type with BMO functions.*

Keywords. Generalized Morrey spaces; Calderón-Zygmund operator; commutator; BMO.

Mathematics Subject Classification (2010): 42B20, 42B35.

1 Introduction

The theory of Calderón-Zygmund operators has played very important roles in modern harmonic analysis with lots of extensive applications in the others fields of mathematics, which has been extensively studied (see [2,3,4,20,21,28]). In particular, Yabuta introduced certain ω-type Calderón-Zygmund operators to facilitate his study of certain classes of pseudodifferential operators (see [31]). Let ω be a non-negative and non-decreasing function on $\mathbb{R}_+ = (0,\infty)$. We say that ω satisfies the $Dini$ condition and wirte $\omega \in Dini$, if

$$\int_0^\infty \frac{\omega(t)}{t} dt < \infty. \tag{1.1}$$

A measurable function $K(\cdot,\cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be a ω-type Calderón-Zygmund kernel if it satisfies

$$|K(x,y)| \le C\,|x-y|^{-n} \tag{1.2}$$

and for all distinct $x, y \in \mathbb{R}^n$, and all z with $2|x-z| < |x-y|$, there exist positive constants C and γ such that

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \le C\omega\Big(\frac{|x-z|}{|x-y|}\Big)\,|x-y|^{-n}. \tag{1.3}$$

Definition 1.1 *Let T be a linear operator from $\mathcal{S}(\mathbb{R}^n)$ into its dual $\mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class. One can say that T is a ω-type Calderón-Zygmund operator if it satisfies the following conditions:*
i) T can be extended to be a bounded linear operator on $L_2(\mathbb{R}^n)$;

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ii) there is a ω-type Calderón-Zygmund kernel $K(x,y)$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \text{ as } f \in C_c^\infty \text{ and } x \notin \text{supp} f. \tag{1.4}$$

It is easy to see that the classical Calderón-Zygmund operator with standard kernel is a special case of ω-type operator T as $\omega(t) = t^\varepsilon$ with $0 < \varepsilon \leq 1$. Given a locally integrable function b, the commutator generated by T and b is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x) = \int_{\mathbb{R}^n} [b(x) - b(y)]K(x,y)f(y)dy. \tag{1.5}$$

Let $\mathbf{b} = (b_1, ..., b_m)$ and b_j, $1 \leq j \leq m$ be locally integrable functions when we consider multilinear commutators as defined by

$$T_{\mathbf{b}} f(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y))K(x,y)f(y)dy. \tag{1.6}$$

Furthermore, if we take $b_i = b$, , $i = 1, \ldots, m$, then we define the following integral equation

$$T_{\mathbf{b}} f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x,y)f(y)dy = [b,T]^m f(x).$$

It is well known that Calderón-Zygmund operators play an important role in harmonic analysis (see [28]).

The classical Morrey spaces were introduced by Morrey [23] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [8,24,25] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [1,5,6,9,10,13,14,15,16,17, 18,27]).

The main purpose of this paper is to establish a number of results concerning generalized Morrey boundedness of Calderón-Zygmund operators with kernels of mild regularity. Let T be a linear Calderón-Zygmund operator of type $\omega(t)$ with ω being nondecreasing and $\omega \in Dini$, but without assuming to be concave. We show that the ω-type Calderón-Zygmund operators T and their multlinear commutators $T_{\mathbf{b}}$ are bounded from one generalized Morrey space M_{p,φ_1} to another M_{p,φ_2}, $1 < p < \infty$. We find the sufficient conditions on the pair (φ_1, φ_2) with $\mathbf{b} \in BMO^m(\mathbb{R}^n)$ which ensures the boundedness of the operators T and $T_{\mathbf{b}}$ from M_{p,φ_1} to M_{p,φ_2} for $1 < p < \infty$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Generalized Morrey spaces

We define the generalized Morrey spaces as follows.

Definition 2.1 *Let $1 \leq p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. We denote by $M_{p,\varphi}$ the generalized Morrey space, the space of all functions $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,t)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))},$$

where $L_p(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_p(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}$ we denote the weak generalized weighted Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,t)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\, |f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

Remark 2.1 If $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi} = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi} = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space; If $\varphi(x,r) \equiv |B(x,t)|^{-\frac{1}{p}}$, then $M_{p,\varphi} = L_p(\mathbb{R}^n)$ is the Lebesgue space.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s)\, w(s)\, ds, \quad H_w^* g(t) := \int_t^\infty \left(1+\ln\frac{s}{t}\right)^m g(s)\, w(s)\, ds, \quad 0 < t < \infty,$$

where w is a weight. The following theorem was proved in [12].

Theorem 2.1 *[12] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w g(t) \le C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)\, ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 2.2 *[11] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w^* g(t) \le C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \left(1+\ln\frac{s}{t}\right)^m \frac{w(s)\, ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

3 ω-type Calderón-Zygmund operators in the spaces $M_{p,\varphi}(\mathbb{R}^n)$

The following theorem was proved in [26].

Theorem 3.1 *[26] Let $1 \le p < \infty$ and T be ω-type Calderón-Zygmund operator defined by* (1.4) *with ω satisfies* (1.1). *Then, the operator T is bounded on $L_p(\mathbb{R}^n)$ for $p > 1$ and bounded from $L_1(\mathbb{R}^n)$ into $WL_1(\mathbb{R}^n)$ for $p = 1$.*

The following Guliyev local estimates are valid (see [10]).

Theorem 3.2 *Let $1 \le p < \infty$ and T be ω-type Calderón-Zygmund operator defined by* (1.4) *with ω satisfies* (1.1). *Then, for $p > 1$ the inequality*

$$\|Tf\|_{L_p(B)} \lesssim |B|^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-\frac{1}{p}} \, \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Moreover, for $p = 1$ the inequality

$$\|Tf\|_{WL_1(B)} \lesssim |B| \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \, |B(x_0,t)|^{-1} \, \frac{dt}{t} \qquad (3.1)$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $p \in (1, \infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0, 2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad r > 0. \qquad (3.2)$$

Then we have

$$\|Tf\|_{L_p(B)} \le \|Tf_1\|_{L_p(B)} + \|Tf_2\|_{L_p(B)}.$$

Since $f_1 \in L_p$, $Tf_1 \in L_p$ and from the boundedness of T in L_p (see Theorem (3.1)) it follows that

$$\|Tf_1\|_{L_p(B)} \le \|Tf_1\|_{L_p} \le C\|f_1\|_{L_p} = C\|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

It is clear that $x \in B$, $y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$. We get

$$|Tf_2(x)| \le 2^n c_0 \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}.$$

Applying Hölder's inequality, we get

$$\int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-\frac{1}{p}} \, \frac{dt}{t}. \qquad (3.3)$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$\|Tf_2\|_{L_p(B)} \lesssim |B|^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-\frac{1}{p}} \, \frac{dt}{t} \qquad (3.4)$$

is valid. Thus

$$\|Tf\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)} + |B|^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-\frac{1}{p}} \, \frac{dt}{t}.$$

$$\lesssim |B|^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-\frac{1}{p}} \, \frac{dt}{t}.$$

Let $p=1$. From the weak $(1,1)$ boundedness of T it follows that:

$$\begin{aligned}\|Tf_1\|_{WL_1(B)} &\le \|Tf_1\|_{WL_1} \lesssim \|f_1\|_{L_1} = \|f\|_{L_1(2B)} \\ &\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \, |B(x_0,t)|^{-1} \, \frac{dt}{t}.\end{aligned} \tag{3.5}$$

Then by (3.4) and (3.5) we get the inequality (3.1).

Theorem 3.3 *Let* $1 \le p < \infty$, T *be* ω*-type Calderón-Zygmund operator defined by* (1.4) *with* ω *satisfies* (1.1)*, and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)|B(x,s)|^{1/p}}{|B(x,t)|^{1/p}} \frac{dt}{t} \le C\varphi_2(x,r) \tag{3.6}$$

where C *does not depend on* x *and* r*. Then the operator* T *is bounded from* M_{p,φ_1} *to* M_{p,φ_2} *for* $p>1$ *and from* M_{1,φ_1} *to* WM_{1,φ_2} *for* $p=1$.

Proof. For $p>1$ from Theorem 2.1 and Theorem 3.2 we get

$$\begin{aligned}\|Tf\|_{M_{p,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_p(B(x_0,t))} \, |B(x,t)|^{-\frac{1}{p}} \, \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} |B|^{-\frac{1}{p}} \, \|f\|_{L_p(B)} = \|f\|_{M_{p,\varphi_1}}\end{aligned}$$

and for $p=1$

$$\begin{aligned}\|Tf\|_{WM_{1,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_1(B(x_0,t))} \, |B(x_0,t)|^{-1} \, \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} |B|^{-1} \|f\|_{L_1(B)} = \|f\|_{M_{1,\varphi_1}}.\end{aligned}$$

Remark 3.1 Let $0 \le \kappa < 1$. Assume that ψ is a positive increasing function defined in $(0,\infty)$ and satisfies the following $\mathcal{D}_\kappa$ condition :

$$\frac{\psi(t_2)}{t_2^{\kappa}} \le C \frac{\psi(t_1)}{t_1^{\kappa}}, \text{ for any } 0 < t_1 < t_2 < \infty,$$

where $C>0$ is a constant independent of t_1 and t_2. If $\varphi_1(x,r) = \varphi_2(x,r) = \psi(|B(x,r)|)$ and ψ satisfy the $\mathcal{D}_\kappa$ condition, Theorems 3.2 and 3.3 were proved in [29]. Also, in the case $\omega(t) = t^{\varepsilon}$ with $0 < \varepsilon \le 1$, Theorems 3.2 and 3.3 were proved in [10].

4 Commutators of ω-type Calderón-Zygmund operators in the spaces $M_{p,\varphi}(\mathbb{R}^n)$

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 4.1 *Suppose that* $b \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$*, and let*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y) dy.$$

Define

$$BMO(\mathbb{R}^n) = \{ b \in L_1^{\mathrm{loc}}(\mathbb{R}^n) \ : \ \|b\|_* < \infty \}.$$

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.
The following lemma is valid.

Lemma 4.1 *[19,28]* (1) *Let $b \in BMO(\mathbb{R}^n)$. Then*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy \right)^{\frac{1}{p}} \tag{4.1}$$

for $1 < p < \infty$.
(2) *Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that*

$$\left| b_{B(x,r)} - b_{B(x,\tau)} \right| \le C\|b\|_* \log \frac{\tau}{r} \ \textit{for} \ 0 < 2r < \tau, \tag{4.2}$$

where C is independent of f, x, r and τ.

Since linear commutator has a greater degree of singularity than the corresponding ω-type Calderón-Zygmund operator, we need a slightly stronger version of condition

$$\int_0^1 \frac{\omega(t)}{t} \left(1 + \log \frac{1}{t}\right)^m dt < \infty. \tag{4.3}$$

The following weighted endpoint estimate for commutator $T_{\mathbf{b}}$ of the ω-type Calderón-Zygmund operator was established in [30] under a stronger version of condition (4.3) assumed on ω, if $\mathbf{b} \in BMO^m(\mathbb{R}^n)$ (for the unweighted case, see [22]).
The following theorem was proved in [30].

Theorem 4.1 *[30] Let T be linear ω-CZO and $\mathbf{b} \in BMO^m(\mathbb{R}^n)$. If ω satisfies condition* (4.3) *and $1 < p < \infty$, then there exists a constant $C > 0$ such that*

$$\|T_{\mathbf{b}} f\|_{L_p} \le C \, \|\mathbf{b}\|_* \, \|f\|_{L_p}.$$

The following Guliyev local estimates are valid (see [10]).

Theorem 4.2 *Let T be linear ω-CZO and $\mathbf{b} \in BMO^m(\mathbb{R}^n)$. Let also ω satisfies condition* (4.3) *and $1 < p < \infty$. Then*

$$\|T_{\mathbf{b}} f\|_{L_p(B)} \le C \, \|\mathbf{b}\|_* \, |B|^{\frac{1}{p}} \int_{2r}^{\infty} \ln^m \left(e + \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-1/p} \, \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$, where C does not depend on f, $x_0 \in \mathbb{R}^n$ and $r > 0$.

Proof. Let $p \in (1,\infty)$. For arbitrary $x_0 \in \mathbb{R}^n$ and $r > 0$, set $B = B(x_0, r)$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. For all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ we define

$$T_{\mathbf{b}} f(x) := T_{\mathbf{b}} f_1(x) + \int_{\mathbb{R}^n} \prod_{j=1}^{m} (b_j(x) - b_j(y)) K(x,y) f_2(y) dy, \tag{4.4}$$

here $T_{\mathbf{b}}$ denotes the commutator as a bounded linear operator on L_p with $1 \le p < \infty$ and $w \in A_p(\mathbb{R}^n)$ (see [30]). It is easy to check that the definition of $T_{\mathbf{b}} f(x)$ does not depend on the choice of the ball B.
Hence

$$\|T_{\mathbf{b}} f\|_{L_p(B)} \le \|T_{\mathbf{b}} f_1\|_{L_p(B)} + \|T_{\mathbf{b}} f_2\|_{L_p(B)}.$$

From the boundedness of $T_{\mathbf{b}}$ in $L_p(\mathbb{R}^n)$ (see Theorem 4.1) it follows that:

$$\|T_{\mathbf{b}} f_1\|_{L_p(B)} \le \|T_{\mathbf{b}} f_1\|_{L_p} \lesssim \|\mathbf{b}\|_* \|f_1\|_{L_p} = \|\mathbf{b}\|_* \|f\|_{L_p(2B)}.$$

For the term $\|T_{\mathbf{b}} f_2\|_{L_p(B)}$, without loss of generality, we can assume $m = 2$. Thus, the operator $T_{\mathbf{b}} f_2$ can be divided into four parts

$$\begin{aligned} T_{\mathbf{b}} f_2(x) &= \big(b_1(x) - \big(b_1\big)_B\big)\big(b_2(x) - \big(b_2\big)_B\big) \int_{\mathbb{R}^n} K(x,y) f_2(y) dy \\ &+ \int_{\mathbb{R}^n} K(x,y)\big(b_1(y) - \big(b_1\big)_B\big)\big(b_2(y) - \big(b_2\big)_B\big) f_2(y) dy \\ &- \big(b_1(x) - \big(b_1\big)_B\big) \int_{\mathbb{R}^n} K(x,y)\big(b_2(y) - \big(b_2\big)_B\big) f_2(y) dy \\ &- \big(b_2(x) - \big(b_2\big)_B\big) \int_{\mathbb{R}^n} K(x,y)\big(b_1(y) - \big(b_1\big)_B\big) f_2(y) dy \\ &= I_1(x) + I_2(x) + I_3(x) + I_4(x). \end{aligned}$$

For $x \in B$ we have

$$\begin{aligned} |T_{\mathbf{b}} f_2(x)| &\le |I_1(x) + |I_2(x)| + |I_3(x)| + |I_4(x)| \\ &\lesssim \big|b_1(x) - \big(b_1\big)_B\big|\big|b_2(x) - \big(b_2\big)_B\big| \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy \\ &+ \int_{\complement_{(2B)}} \big|b_1(y) - \big(b_1\big)_B\big|\big|b_2(y) - \big(b_2\big)_B\big| \frac{|f(y)|}{|x_0 - y|^n} dy \\ &+ \big|b_1(x) - \big(b_1\big)_B\big| \int_{\complement_{(2B)}} \big|b_2(y) - \big(b_2\big)_B\big| \frac{|f(y)|}{|x_0 - y|^n} dy \\ &+ \big|b_2(x) - \big(b_2\big)_B\big| \int_{\complement_{(2B)}} \big|b_1(y) - \big(b_1\big)_B\big| \frac{|f(y)|}{|x_0 - y|^n} dy. \end{aligned}$$

Then

$$\begin{aligned} \|T_{\mathbf{b}} f_2\|_{L_p(B)} &\lesssim \Big(\int_B \Big(\int_{\complement_{(2B)}} \frac{\prod_{j=1}^{2} \big|b_i(y) - \big(b_i\big)_B\big|}{|x_0 - y|^n} |f(y)| dy \Big)^p dx \Big)^{\frac{1}{p}} \\ &+ \left(\int_B \big|b_1(x) - \big(b_1\big)_B\big| \left(\int_{\complement_{(2B)}} \frac{\big|b_2(y) - \big(b_2\big)_B\big|}{|x_0 - y|^n} |f(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\ &+ \left(\int_B \big|b_2(x) - \big(b_2\big)_B\big| \left(\int_{\complement_{(2B)}} \frac{\big|b_1(y) - \big(b_1\big)_B\big|}{|x_0 - y|^n} |f(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\ &+ \Big(\int_B \Big(\int_{\complement_{(2B)}} \frac{\prod_{j=1}^{2} \big|b_i(x) - \big(b_i\big)_B\big|}{|x_0 - y|^n} |f(y)| dy \Big)^p dx \Big)^{\frac{1}{p}} \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Let us estimate I_1.

$$
\begin{aligned}
I_1 &= |B|^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{\prod\limits_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_B\right|}{|x_0 - y|^n} |f(y)| dy \\
&\approx |B|^{\frac{1}{p}} \int_{\complement_{(2B)}} \prod_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_B\right| |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
&\approx |B|^{\frac{1}{p}} \int_{2r}^{\infty} \int_{2r \le |x_0 - y| \le t} \prod_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_B\right| |f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim |B|^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} \prod_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_B\right| |f(y)| dy \frac{dt}{t^{n+1}}.
\end{aligned}
$$

Applying Hölder's inequality and by Lemma 4.1, we get

$$
\begin{aligned}
I_1 &\lesssim |B|^{\frac{1}{p}} \int_{2r}^{\infty} \prod_{j=1}^{2} \left(\int_{B(x_0,t)} \left|b_i(y) - \left(b_i\right)_B\right|^{2p'} w(y)^{1-2p'} dy \right)^{\frac{1}{2p'}} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim \prod_{j=1}^{2} \|b_j\|_* |B|^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \|1\|_{L_{p'}(B(x_0,t))} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim \|\mathbf{b}\|_* |B|^{\frac{1}{p}} \int_{2r}^{\infty} \ln^2 \left(e + \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} |B(x_0,t)|^{-1/p} \frac{dt}{t}.
\end{aligned}
$$

Let us estimate I_2.

$$
\begin{aligned}
I_2 &= \left(\int_B \left|b_1(x) - \left(b_1\right)_B\right|^p dx \right)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{\left|b_2(y) - \left(b_2\right)_B\right|}{|x_0 - y|^n} |f(y)| dy \\
&\lesssim \|b_1\|_* |B|^{\frac{1}{p}} \int_{\complement_{(2B)}} \left|b_2(y) - \left(b_2\right)_B\right| |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
&\approx \|b_1\|_* |B|^{\frac{1}{p}} \int_{2r}^{\infty} \int_{2r \le |x_0-y| \le t} \left|b_2(y) - \left(b_2\right)_B\right| |f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim \|b_1\|_* |B|^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} \left|b_2(y) - \left(b_2\right)_B\right| |f(y)| dy \frac{dt}{t^{n+1}}.
\end{aligned}
$$

Applying Hölder's inequality and by Lemma 4.1, we get

$$
\begin{aligned}
I_2 &\lesssim \|b_1\|_* |B|^{\frac{1}{p}} \int_{2r}^{\infty} \Big(\int_{B(x_0,t)} \left|b_2(y) - \left(b_2\right)_B\right|^{p'} dy \Big)^{\frac{1}{p'}} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim \prod_{j=1}^{2} \|b_j\|_* |B|^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|1\|_{L_{p'}(B(x_0,t))} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim \|\mathbf{b}\|_* |B|^{\frac{1}{p}} \int_{2r}^{\infty} \ln^2 \left(e + \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} |B(x_0,t)|^{-1/p} \frac{dt}{t}.
\end{aligned}
$$

In the same way, we shall get the result of I_3

$$I_3 \lesssim \|\mathbf{b}\|_* \, |B|^{\frac{1}{p}} \int_{2r}^{\infty} \ln^2\left(e+\frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-1/p} \, \frac{dt}{t}.$$

In order to estimate I_4 note that

$$\begin{aligned} I_4 &= \left(\int_B \prod_{j=1}^{2} \left| b_i(x) - \left(b_i\right)_B \right|^p dx \right)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n} dy \\ &\le \prod_{j=1}^{2} \left(\int_B \left| b_i(x) - \left(b_i\right)_B \right|^{2p} dx \right)^{\frac{1}{2p}} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n} dy. \end{aligned}$$

By Lemma 4.1, we get

$$I_4 \lesssim \|\mathbf{b}\|_* \, |B|^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n} dy.$$

Applying Hölder's inequality, we get

$$\begin{aligned} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n} dy &\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, \|1\|_{L_{p'}(B(x_0,t))} \, \frac{dt}{t^{n+1}} \\ &\le \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-1/p} \, \frac{dt}{t}. \end{aligned} \tag{4.5}$$

Thus, by (4.5)

$$I_4 \lesssim \|\mathbf{b}\|_* \, |B|^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-1/p} \, \frac{dt}{t}.$$

Summing up I_1 and I_4, for all $p \in [1,\infty)$ we get

$$\|T_{\mathbf{b}} f_2\|_{L_p(B)} \lesssim \|\mathbf{b}\|_* \, |B|^{\frac{1}{p}} \int_{2r}^{\infty} \ln^2\left(e+\frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-1/p} \, \frac{dt}{t}. \tag{4.6}$$

On the other hand,

$$\begin{aligned} \|f\|_{L_p(2B)} &\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\le |B|^{\frac{1}{p}} \|1\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\le |B|^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, \|1\|_{L_{p'}(B(x_0,t))} \, \frac{dt}{t^{n+1}} \\ &\le |B|^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-1/p} \, \frac{dt}{t}. \end{aligned} \tag{4.7}$$

Finally,

$$\begin{aligned} \|T_{\mathbf{b}} f\|_{L_p(B)} &\lesssim \|\mathbf{b}\|_* \, \|f\|_{L_p(2B)} \\ &\quad + \|\mathbf{b}\|_* \, |B|^{\frac{1}{p}} \int_{2r}^{\infty} \ln^m\left(e+\frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{-1/p} \, \frac{dt}{t}, \end{aligned}$$

and the statement of Theorem 4.2 follows by (4.7).

Theorem 4.3 *Let T be linear ω-CZO and $\mathbf{b} \in BMO^m(\mathbb{R}^n)$. Let also ω satisfies condition* (4.3), $1 < p < \infty$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \ln^m \Big(e + \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, |B(x,s)|^{1/p}}{|B(x,t)|^{1/p}} \frac{dt}{t} \le C\varphi_2(x,r) \tag{4.8}$$

where C does not depend on x and r. Then the operator $T_{\mathbf{b}}$ is bounded from M_{p,φ_1} to M_{p,φ_2}. Moreover,

$$\|T_{\mathbf{b}} f\|_{M_{q,\varphi_2}} \lesssim \|\mathbf{b}\|_* \, \|f\|_{M_{p,\varphi_1}}.$$

Proof. Using the Theorem 2.2 and the Theorem 4.2 we have

$$\|T_{\mathbf{b}} f\|_{M_{p,\varphi_2}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} |B(x,t)|^{-\frac{1}{p}} \|T_{\mathbf{b}} f\|_{L_p B(x,r)}$$

$$\lesssim \|\mathbf{b}\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \ln^m \Big(e + \frac{t}{r}\Big) \|f\|_{L_p(B(x,t))} \, |B(x,t)|^{-1/p} \frac{dt}{t}$$

$$\lesssim \|\mathbf{b}\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} |B(x,t)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))} = \|\mathbf{b}\|_* \, \|f\|_{M_{p,\varphi_1}}.$$

Remark 4.1 Note that, if $\varphi_1(x,r) = \varphi_2(x,r) = \psi(w(x,r))$ and ψ satisfy the $\mathcal{D}_\kappa$ condition, Theorems 4.2 and 4.3 were proved in [29]. Also, in the case $m = 1$ and $\omega(t) = t^\varepsilon$ with $0 < \varepsilon \le 1$, Theorems 4.2 and 4.3 were proved in [11].

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