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Criteria for componentwise uniform equiconvergence with trigonometric series of spectral expansions responding to discontinuous Dirac operator

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Abstract. In this paper we consider a discontinuous Dirac operator on the interval $(0,\,2\pi)$. It is assumed that the coefficient (potential) is a complex valued matrix-function summable on$(0,\,2\pi)$. In the case of a potential from $L_p\,(0,\,2\pi)\otimes C^{2\times 2},\ p>2$, was established necessary and sufficient conditions of componentwise equiconvergence on a compact with trigonometric series of expansions in biorthogonal series of an arbitrary vector-function $f\in\ L_2^2\,(0,\,2\pi)$ by the system of root vector-functions of the given operator.

Keywords. eigen vector-function, associated vector-function, componentwise equiconvergence.

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1 Main notions and formulation of results.

In this paper we study uniform equiconvergence on a compact with trigonometric series of spectral expansions in root functions of a discontinuous Dirac operator. The root vector-functions are understood in the generalized setting, i.e. regardless to boundary conditions (see [2]). With such a generalized understanding of them, V.A. Il'in [2-3] established necessary and sufficient conditions of uniform equiconvergence on a compact with trigonometric series of expansions in root functions of differential operators with smooth coefficients. Uniform equiconvergence and uniform equiconvergence rate for differential operators with non-smooth coefficients were thoroughly studied in [11-15], while equiconvergence in integral metrics (i.e. in the metrics $L_p,\ 1\le p<\infty$) was studied in [13-18].

Componentwise uniform equiconvergence on a compact for the Dirac operator was studied in [10], and a theorem on componentwise uniform equiconvergence for an arbitrary vector-function $f\in\ L_2^2\,(a,\,b)$ was proved, where $(a,\,b)$ is an arbitrary interval of a real

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straight line. Componentwise equiconvergence on the metrics $L_p,\ 1 \le p \le \infty$ and componentwise uniform equiconvergence rate on a compact were studied in [1,7], respectively.

Let the interval $(0,\ 2\pi)$ be divided by the points $\{\xi_i\}_{i=0}^m,\ 0 = \xi_0 < \xi_1 < ... < \xi_m = 2\pi$, into the intervals $G_l = (\xi_{l-1}, \xi_l),\ \ l = \overline{1,m}$. Denote by A_l a class of absolutely continuous two-component vector-functions on the segment $\overline{G_l}$. Define the class $A\ (0,\ 2\pi)$ as follows: if $f \in A\,(0,\ 2\pi)$, then for every $l,\ l = \overline{1,m}$ there exists such a vector-function $f_l(x) \in A_l$ that $f(x) = f_l(x)$ for $\xi_{l-1} < x < \xi_l$.

Let us consider the Dirac operator

$$Ly \equiv B\frac{dy}{dx} + P(x)y, \quad x \in \bigcup_{l=1}^{m} G_l = G,$$

where $B = (b_{ij})_{i,j=1}^2,\ b_{ii} = 0,\ b_{i,3-i} = (-1)^{i-1},\ y(x) = (y_1(x),\, y_2(x))^T,\ \ P(x) = = diag\,(p(x), q(x))$ and $p(x)$, $q(x)$ are summable complex-valued functions on $(0,\ 2\pi)$.

Following [3] we will understand root (i.e. eigen and associated) vector-functions of the operator L regardless to the form of boundary conditions and "transmission" conditions namely; under the eigen vector-function of the operator L, responding to the complex eigenvalue λ we will understand any not identically zero complex valued vector-function $\overset{0}{y}(x) \in A\,(0, 2\pi)$ satisfying almost everywhere in G the equation $L\overset{0}{y} = \lambda\overset{0}{y}$. Then, by induction: under the associated vector-function of order $r,\ r \ge 1$ responding to the same λ and the eigen-function $\overset{0}{y}(x)$, we will understand any complex valued vector-function $\overset{r}{y}(x) \in \in A\,(0, 2\pi)$ satisfying almost everywhere on G the equation

$$L\overset{r}{y} = \lambda\overset{r}{y} + \overset{r-1}{y}.$$

Let $\{u_k(x)\}_{k=1}^\infty$ be an arbitrary system composed of the root (eigen and associated) vector-functions of the operator L, while $\{\lambda_k\}_{k=1}^\infty$ be the corresponding system of eigenvalues. In what follows, we assume that each vector-function $u_k(x)$ enters into the system $\{u_k(x)\}_{k=1}^\infty$ together with all its lower order associated functions, and the lengths of the chains of the root vector-functions are uniformly bounded. This means that each vector-function $u_k(x)$ almost everywhere in G satisfies the equation

$$Lu_k = \lambda_k u_k + \theta_k u_{k-1}\,,$$

where θ_k equals either zero (in this case $u_k(x)$ is an eigen vector-function), or one (in this case $u_k(x)$ is an associated vector-function $\lambda_k = \lambda_{k-1}$).

Let $L_p^2(0, 2\pi),\ p \in [1, \infty]$, be a space of two-component vector-functions and $f(x) = (f_1(x),\ f_2(x))^T$ with the norm

$$\|f\|_p \equiv \|f\|_{p,[0,2\pi]} = \left(\int_0^{2\pi} |f(x)|^p\ dx\right)^{1/p},\ \ if\ \ p \ne \infty,$$

while in the case $p = \infty$ with the norm

$$\|f\|_\infty \equiv \|f\|_{\infty,[0,2\pi]} = \operatorname*{ess\ sup}_{x\in[0,2\pi]} |f(x)|.$$

Obviously, the "inner product"

$$(f, g) = \int_0^{2\pi} \sum_{j=1}^{2} f_j(x)\,\overline{g_j(x)}\ \ dx$$

was determined for the vector-functions $f \in L_p^2(0, 2\pi)$, $g \in L_q^2(0, 2\pi)$, $1/p + 1/q = 1$, $p \geq 1$.

Let the considered system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy the condition B_2:
1) the system $\{u_k(x)\}_{k=1}^{\infty}$ is complete and minimal in $L_2^2\,(0,\,2\pi)$;
2) the system of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ satisfies the two inequalities

$$|Im\,\lambda_k| \leq\, C_1, \qquad k = 1, 2, ..., \tag{1.1}$$

$$\sum_{t \leq |\lambda_k| \leq t+1} 1 \leq C_2, \quad \forall t \geq 0; \tag{1.2}$$

3) the system $\{\upsilon_k\}_{k=1}^{\infty} \subset L_2^2\;(0,\,2\pi)$ biorthogonally conjugate to the system $\{u_k(x)\}_{k=1}^{\infty}$, consists of root vector-functions of a formally adjoint operator

$$L^* = B\frac{d}{dx} + \overline{P(x)}, \quad i.e.,\;\; L^*\upsilon_k = \overline{\lambda_k}\upsilon_k + \theta_k \upsilon_{k+1}.$$

Note that the second one of the conditions B_2 allows to assume that all the elements of the system $\{u_k(x)\}_{k=1}^{\infty}$ were numbered in non-decreasing order of the value $|\lambda_k|$. For an arbitrary vector-function $f \in\; L_2^2\,(0,\,2\pi)$ we make up $n-$th order partial sum of biorthogonal expansion by the system $\{u_k(x)\}_{k=1}^{\infty}$:

$$\sigma_n\,(x,\,f) = \sum_{k=1}^{n} (f,\,\upsilon_k)\;u_k(x), \qquad x \in G, \tag{1.3}$$

$$\sigma_n\,(x,\,f) = \left(\sigma_n^1\,(x,\,f)\,,\;\sigma_n^2\,(x, f)\right)^T,$$

$$\sigma_n^j\,(x, f) = \sum_{k=1}^{n} (f,\,\upsilon_k)\;u_k^j(x), \;\; j = 1, 2,$$

$$u_k(x) = \left(u_k^1(x)\,,\,u_k^2(x)\right)^T\;.$$

We will compare $\sigma_n^j\,(x,\,f)\,,\;\; j = 1, 2$, with a modified partial sum of trigonometric Fourier series corresponding to the $j-$th component $f_j(x)$ of the vector-function $f(x)$

$$S_\nu\,(x,\,f_j) = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin\,\nu\;(x-y)}{x-y}\;f_j\;(y)\;dy \tag{1.4}$$

of order $\nu = |\lambda_n|$.

Definition. *We say that the $j-$th component of expansion of the vector-function $f \in L_2^2(0,\,2\pi)$ in biorthogonal series by the system $\{u_k(x)\}_{k=1}^{\infty}$ uniformly equiconverges on any compact of the set $G = \bigcup\limits_{l=1}^{m} G_l$ with expansion corresponding to the $j-$th component $f_j(x)$ of the vector-function $f(x)$ in trigonometric Fourier series if on any compact $K \subset G$*

$$\lim_{n\to\infty}\,\left\|\,\sigma_n^j\,(\cdot\,,\,f) - S_{|\lambda_n|}\,(\cdot\,,\,f)\,\right\|_{C(K)} = 0\,. \tag{1.5}$$

The following results are proved in the present paper.

Theorem 1.1. *Let the potential $P(x)$ belong to the class $L_p\;(0,\,2\pi) \otimes C^{2\times 2},\; p > 2$, and the system of root vector-functions $\{u_k(x)\}_{k=1}^{\infty}$ satisfy the condition B_2. Then for* (1.5) *to be fulfilled for an arbitrary vector-function $f \in L_2^2\,(0,\,2\pi)$ on any compact $K \subset G$, it* *is necessary and sufficient that for any compact $K_0 \subset G$ there exist a constant $C(K_0)$, providing validity for all the numbers k of the inequality*

$$\|u_k\|_{L_2^2(K_0)} \; \|\upsilon_k\|_{L_2^2(0,\,2\pi)} \le C(K_0)\ . \tag{1.6}$$

Theorem 1.2. *If the potential $P(x)$ of the operator L and the system of the root vector functions $\{u_k(x)\}_{k=1}^{\infty}$ satisfy the same requirements as in theorem 1.1, then subject to condition* (1.6) *for biorthogonal expansion of an arbitrary vector-function $f \in L_2^2(0,\,2\pi)$ the component principle of localization in G is valid: convergence or divergence of the $j-th$ component of the mentioned biorthogonal expansion at the point $x_0 \in G$ depends on the behavior in small vicinity of the point x_0 only of the appropriate $j-th$ component $f_j(x)$ of the decomposable vector-function $f(x)$ (and is independent of the behavior of another component).*

2 Auxiliary statements.

Here are some necessary statements that will be used when proving the theorems formulated above.

Statement 2.1 [6]. *If the functions $p(x)$ and $q(x)$ belong to the class $L_1^{loc}(G_l)$ and the points $x-t,\ x,\ x+t$ belong to the interval G_l, then for the root vector-functions $u_k(x)$ we have the mean value formula:*

$$\frac{u_k(x-t)+u_k(x+t)}{2} = \sum_{i=0}^{n_k}(-1)^i\,\frac{t^i}{i!}\,\cos\left(\lambda_k t+\frac{\pi}{2}i\right)\,u_{k-i}(x)$$
$$+\frac{1}{2}\sum_{i=0}^{n_k}\frac{(-1)^i}{i!}\int_0^t (t-r)^i\left\{\sin\left(\lambda_k(t-r)+\frac{\pi}{2}\right)\right.$$
$$\times\left[P(x-r)\,u_{k-i}(x-r)+P(x+r)\,u_{k-r}(x+r)\right]$$
$$\left.+\,B\,\cos\left(\lambda_k(t-r)+\frac{\pi}{2}i\right)\,\left[P(x+r)\,u_{k-i}(x+r)-P(x-r)\,u_{k-i}(x-r)\right]\right\}\,dr\,, \tag{2.1}$$

where n_k is the order of the root vector-function $u_k(x)$.

We fix an arbitrary segment $K=[a,b]\subset G_l$ and such a segment $K_R=[a+R,\ b-R]$ contained in it that $R = dist\ (K_R,\,\partial K) < (mesK)/2,\ \ \partial K=\{a,b\}$.

Statement 2.2 [9]. *If the functions $p(x)$ and $q(x)$ belong to the class $L_1^{loc}(G_l)$, then for K and K_R and there exist such positive constants $C_i(K,\,n_k)$, $i=\overline{1,3}$; $C_i(K,\,K_R,\,n_k)$, $i=4,5$, independent of λ_k, that the following estimations hold:*

$$C_1\,\|u_k\|_{p,K} \le \left[1+|Im\,\lambda_k|\right]^{1/s-1/p}\|u_k\|_{s,K}\le C_2\,\|u_k\|_{p,K}\,,\quad 1\le p<s\le\infty\,; \tag{2.2}$$

$$\|\theta_k u_{k-1}\|_{p,K}\le C_3\left[1+|Im\,\lambda_k|\right]\,\|u_k\|_{p,K}\,,\quad p\ge 1\,; \tag{2.3}$$

$$C_4\left[1+|Im\,\lambda_k|\right]^{-n_k}\|u_k\|_{p,K}\le\|u_k\|_{p,K_R}\exp\left(R\,|Im\,\lambda_k|\right)$$
$$\le C_5\left[1+|Im\,\lambda_k|\right]^{n_k}\|u_k\|_{p,K}\ ,\ \ p\ge 1\,, \tag{2.4}$$

where $\|\cdot\|_{p,K}=\|\cdot\|_{L_p^2(K)}$.

Provided $p,\ q\in L_1(G_l)$ in the estimates (2.2)-(2.4) the segment K can be replaced by $\overline{G_l}$.

Note that for $p, q \in L_1(G_l)$, $l = \overline{1, m}$, there exist the limits $u_k(0+)$, $u_k(2\pi - 0)$, $u_k(\xi_l \pm 0)$, $l = \overline{1, m-1}$. Under $u_k(\xi_l)$, $l = \overline{0, m-1}$, and $u_k(2\pi)$ we will understand unilateral limits $u_k(\xi_l + 0)$, $l = \overline{0, m-1}$, and $u_k(2\pi - 0)$.

Statement 2.3. *Let the potential $P(x)$ belong to the class $L_1(0, 2\pi) \otimes C^{2\times 2}$, the system of the root vector-functions $\{u_k(x)\}_{k=1}^{\infty}$ satisfy the condition B_2. Then if inequality* (1.6) *is fulfilled for all the numbers k, then each of the systems $\{u_k(x)\}_{k=1}^{\infty}$ and $\{\upsilon_k(x)\}_{k=1}^{\infty}$ form an unconditional basis in $L_2^2(0, 2\pi)$. Herewith, the systems $\left\{ u_k(x) \parallel u_k(x) \parallel_2^{-1} \right\}_{k=1}^{\infty}$ and $\left\{ \upsilon_k(x) \parallel \upsilon_k(x) \parallel_2^{-1} \right\}_{k=1}^{\infty}$ are Riesz bases in this space.*

Proof. By theorem 2 and remark 1 of the paper [8], it is enough for us to prove that under the conditions of statement 2.3 the inequality

$$\|u_k\|_2 \, \|\upsilon_k\|_2 \leq const \tag{2.5}$$

is fulfilled for all the numbers k.

Let $K^{(l)} = [a_l, b_l] \subset G_l$, $l = \overline{1, m}$, $0 < R^{(l)} = dist\left(K^{(l)}, \partial G_l\right) < (mesG_l)/2$ and $K^0 = \bigcup_{l=1}^{m} K^{(l)}$. Then

$$\|u_k\|_{L_2^2(K^0)} \, \|\upsilon_k\|_2 \leq C\left(K^0\right), \quad k = 1, 2, ... \tag{2.6}$$

is fulfilled due to inequality (1.6).

Estimate from the below the factor $\|u_k\|_{L_2^2(K^0)}$:

$$\|u_k\|_{L_2^2(K^0)}^2 = \int_{K^0} |u_k(x)|^2 \, dx = \sum_{l=1}^{m} \int_{K^{(l)}} |u_k(x)|^2 \, dx = \sum_{l=1}^{m} \|u_k\|_{2, K^{(l)}}^2 .$$

Here we apply the left hand side of the estimate (2.4) for $p = 2$, $K = \overline{G_l}$, $K_R = K^{(l)}$, $R = R^{(l)}$, $l = \overline{1, m}$, and take into account the ratio $\sup\limits_k n_k = N_0 < \infty$, that follows from (1.2). As a result we get

$$\|u_k\|_{L_2^2(K^0)}^2 \geq \sum_{l=1}^{m} C_4^2\left(\overline{G_l}, K^{(l)}, n_k\right) [1 + |Im\,\lambda_k|]^{-2n_k} \cdot \exp\left(-2R^{(l)} |Im\,\lambda_k|\right) \|u_k\|_{L_2^2(G_l)}^2$$

$$\geq \sum_{l=1}^{m} C_4^2\left(\overline{G_l}, K^{(l)}, n_k\right) \cdot \frac{1}{(1 + |Im\,\lambda_k|)^{2N_0} \exp\left(2R^{(l)} |Im\,\lambda_k|\right)} \|u_k\|_{L_2^2(G_l)}^2 .$$

Taking into account conditions (1.1), and denoting

$$C_4^2\left(K^0\right) = \min_{\substack{1 \leq l \leq m \\ 0 \leq n_k \leq N_0}} \left\{C_4^2\left(\overline{G_l}, K^{(l)}, n_k\right)\right\} \frac{\exp(-2\pi\, C_1)}{(1 + C_1)^{2N_0}}$$

we arrive at the inequality

$$\|u_k\|_{L_2^2(K^0)}^2 \geq C_4^2\left(K^0\right) \sum_{l=1}^{m} \|u_k\|_{L_2^2(G_l)}^2 = C_4^2\left(K^0\right) \|u_k\|_2^2 .$$

Consequently, the following estimation is fulfilled

$$\|u_k\|_{L_2^2(K^0)} \geq C_4\left(K^0\right) \|u_k\|_2, \quad k = 1, 2, ...$$

The validity of the relation (2.5) for any number l follows from the last inequality and from (2.6). Statement 2.3 is proved.

Denote

$$\delta_n^k = \delta\left(|\lambda_n|\,,\,\lambda_k\right) = \frac{1}{2}\left[1 + sign\ \left(|\lambda_n| - |\rho_k|\right)\right]\,, \qquad \rho_k = Re\,\lambda_k;$$

$$B_i\ \left(|\lambda_n|\,,\,\lambda_k,\,R\right) = \int_0^R t^{i-1}\sin\left(|\lambda_n|\ t\right)\,\cos\left(\lambda_k t + \frac{\pi i}{2}\right)\,dt, \qquad i = \overline{1, n_k}.$$

Under conditions (1.1) and (1.2) we have the following relations (see [2-3], [10]).

$$\left|\frac{2}{\pi}\int_0^R t^{-1}\sin|\lambda_n|\ t\,\cos\lambda_k t\,dt - \delta_n^k\right| \le \frac{C\,(R)}{1 + ||\lambda_n| - |\rho_k||}, \tag{2.7}$$

$$|B_i\left(|\lambda_n|\,,\,\lambda_k,\,R\right)| \le \frac{C\,(R)}{1 + ||\lambda_n| - |\rho_k||}, \tag{2.8}$$

where $C\,(R)$ is some positive constant, $\rho_k = Re\,\lambda_k,\ \ i = \overline{1, n_k}$.

3 Proof of the main results.

Proof of Theorem 1.1. Necessity of condition (1.6) for componentwise uniform equiconvergence is justified by the same scheme as in the paper [5], where this scheme was shown for a Schrodinger operator with a matrix potential. In our case, condition (1.6) is necessary even for the operator L with the potential $P(x)$ from the class $L_1\,(0,\,2\pi) \otimes C^{2\times 2}$. Therefore it remains for us to prove the sufficiency part of theorem 1.1. Without loss of generality we fix an arbitrary connected compact $K \subset G = \bigcup_{l=1}^m G_l$. Then for some $l_0,\ 1 \le l_0 \le m,\ \ K \subset G_{l_0}$. Choose the number $R > 0$ satisfying the condition $R < \frac{1}{2}\,dist\ (K,\ \partial G_{l_0})$. We will compare partial sum $\sigma_n\,(x, f)$ with $\tilde{S}_{|\lambda_n|}\,(x, f) = \left(\tilde{S}_{|\lambda_n|}\,(x, f_1)\,,\ \tilde{S}_{|\lambda_n|}\,(x, f_2)\right)^T$, where $f(x) = (f_1(x),\ f_2(x))^T \in L_2^2\,(0,\,2\pi)$,

$$\tilde{S}_{|\lambda_n|}\,(x, f_j) = \frac{1}{\pi}\int_{|x-y|\le R}\frac{\sin\left(|\lambda_n|\,(x-y)\right)}{x-y}\,f_j\,(y)\ dy, \qquad x \in K, \quad j = 1, 2\,.$$

From the theory of trigonometric series it is known that the difference $S_{|\lambda_n|}\,(x,\ f_j) - \tilde{S}_{|\lambda_n|}\,(x,\ f_j)$ tends to zero with respect to $x \in K$ as $n \to \infty$. Therefore it suffices to set up relation (1.5) for $\tilde{S}_{|\lambda_n|}\,(x,\ f_j)\,,\ \ j = 1, 2,$ i.e.

$$\lim_{n\to\infty}\left\|\,\sigma_n^j\,(\,\cdot\,,\,f) - \tilde{S}_{|\lambda_n|}\,(\,\cdot\,,\,f_j)\,\right\|_{C(K)} = 0. \tag{3.1}$$

By virtue of statement 2.3 we can expand the arbitrary vector-function $f \in L_2^2\,(0,\,2\pi)$ into a biorthogonal series by the system $\{u_k(x)\}_{k=1}^\infty$

$$f(x) = \sum_{k=1}^\infty\,(f,\,\upsilon_k)\ u_k(x)\,.$$

With allowance for this expansion, represent the vector-function $\tilde{S}_{|\lambda_n|}\,(x, f)$ in the form

$$\tilde{S}_{|\lambda_n|}\,(x, f) = \frac{2}{\pi}\sum_{k=1}^\infty\,(f,\,\upsilon_k)\int_0^R \frac{u_k\,(x-t) + u_k\,(x+t)}{2}\cdot\frac{\sin|\lambda_n|\ t}{t}\,dt\,. \tag{3.2}$$

Using the mean value formula (2.1) and introducing the notation

$$A^{\pm}\left(P,\,u_{k-i},\,x,\,r\right)=P\left(x+r\right)\,u_{k-r}\left(x+r\right)\pm P\left(x-r\right)\,u_{k-i}\left(x-r\right)$$

we transform the integral in the representation (3.2)

$$\frac{2}{\pi}\int_0^R\frac{u_k\left(x-t\right)+u_k\left(x+t\right)}{2}\cdot\frac{\sin\,|\lambda_n|\,t}{t}\,dt=\frac{2}{\pi}\,u_k(x)\,\int_0^R\frac{\sin\,|\lambda_n|\,t}{t}\cos\lambda_k t\,dt$$

$$+\frac{2}{\pi}\sum_{i=1}^{n_k}\frac{(-1)^i}{i!}u_{k-i}(x)\int_0^R t^{i-1}\sin\,|\lambda_n|\,t\cos\left(\lambda_k t+\frac{\pi}{2}i\right)\,dt$$

$$+\frac{1}{\pi}\sum_{i=0}^{n_k}\frac{(-1)^i}{i!}\int_0^R\frac{\sin\,|\lambda_n|\,t}{t}\int_0^t(t-r)^i\sin\left(\lambda_k\left(t-r\right)+\frac{\pi}{2}i\right)\,A^{+}\left(P,\,u_{k-i},\,x,\,r\right)\,dr\,dt$$

$$+\frac{1}{\pi}\sum_{i=0}^{n_k}\frac{(-1)^i}{i!}B\int_0^R\frac{\sin\,|\lambda_n|\,t}{t}\int_0^t(t-r)^i\cos\left(\lambda_k\left(t-r\right)+\frac{\pi}{2}i\right)\,A^{-}\left(P,\,u_{k-i},\,x,\,r\right)\,dr\,dt.$$

Having changed the order of integration in the repeated integrals, in the last two sums we get

$$\frac{2}{\pi}\int_0^R\frac{u_k\left(x-t\right)+u_k\left(x+t\right)}{2}\cdot\frac{\sin\,|\lambda_n|\,t}{t}\,dt=\delta_n^k u_k(x)+u_k(x)\left[\frac{2}{\pi}\int_0^R\frac{\sin\,|\lambda_n|\,t}{t}\cos\lambda_k t\,dt-\delta_n^k\right]$$

$$+\frac{2}{\pi}\sum_{i=1}^{n_k}\frac{(-1)^i}{i!}u_{k-i}(x)\,B_i\left(|\lambda_n|\,,\,\lambda_k,\,R\right)+\frac{1}{\pi}\sum_{i=0}^{n_k}\frac{(-1)^i}{i!}\left\{\int_0^R A^{+}\left(P,\,u_{k-i},\,x,\,r\right)\,\Phi_{k1}^i\left(r,\,R,\,|\lambda_n|\right)\,dr\right.$$

$$\left.+B\int_0^R A^{-}\left(P,\,u_{k-i},\,x,\,r\right)\,\Phi_{k2}^i\left(r,\,R,\,|\lambda_n|\right)\,dr\right\}\,, \tag{3.3}$$

where

$$\Phi_{k1}^i\left(r,\,R,\,|\lambda_n|\right)=\int_r^R(t-r)^i\,\frac{\sin\,|\lambda_n|\,t}{t}\,\sin\left(\lambda_k\left(t-r\right)+\frac{\pi}{2}i\right)\,dt\,,$$

$$\Phi_{k2}^i\left(r,\,R,\,|\lambda_n|\right)=\int_r^R(t-r)^i\,\frac{\sin\,|\lambda_n|\,t}{t}\,\cos\left(\lambda_k\left(t-r\right)+\frac{\pi}{2}i\right)\,dt,\qquad i=\overline{0,\,n_k}\,.$$

Considering representation (3.3) in the equality (3.2) and taking into account definition of the number δ_n^k for $\tilde{S}_{|\lambda_n|}\left(x,f\right),\,x\in K$, we get the equality:

$$\tilde{S}_{|\lambda_n|}\left(x,f\right)-\sigma_n\left(x,f\right)$$

$$=-\frac{1}{2}\sum_{|\rho_k|=|\lambda_n|}\left(f,\upsilon_k\right)\,u_k(x)+\sum_{k=1}^{\infty}\left(f,\upsilon_k\right)\left\{\left[\frac{2}{\pi}\int_0^R\frac{\sin\,|\lambda_n|\,t}{t}\cos\lambda_k t\,dt-\delta_n^k\right]u_k(x)\right.$$

$$+\frac{2}{\pi}\sum_{i=1}^{n_k}\frac{(-1)^i}{i!}B_i\left(|\lambda_n|\,,\,\lambda_k,\,R\right)\,u_{k-i}(x)+\frac{1}{\pi}\sum_{i=0}^{n_k}\frac{(-1)^i}{i!}\left[\int_0^R A^{+}\left(P,\,u_{k-i},x,\,r\right)\,\Phi_{k1}^i\,\left(r,\,R,\,|\lambda_n|\right)\,dr\right.$$

$$\left.\left.+\;B\int_0^R A^{-}\left(P,\,u_{k-i},\,x,\,r\,\right)\,\Phi_{k2}^i\left(r,\,R,\,|\lambda_n|\right)\,dr\right]\right\}.$$

Hence, with allowance for inequality (2.7) we arrive at the inequality

$$\left|\tilde{S}_{|\lambda_n|}(x,f)-\sigma_n(x,f)\right| \le \frac{1}{2}\sum_{|\rho_k|=|\lambda_n|}|(f,\upsilon_k\|u_k\|_2)|\ |u_k(x)|\ \|u_k\|_2^{-1}$$

$$+C(R)\sum_{k=1}^{\infty}|(f,\upsilon_k\|u_k\|_2)|\ \|u_k\|_2^{1}\ |u_k(x)|\ (1+||\lambda_n|-|\rho_k||)^{-1}$$

$$+C(R)\sum_{k=1}^{\infty}|(f,\ \upsilon_k\|u_k\|_2)|\left(\sum_{i=1}^{n_k}\frac{1}{i!}|B_i(|\lambda_n|,\ \lambda_k,\ R)|\frac{|u_{k-i}(x)|}{\|u_k\|_2}\right)$$

$$+\frac{1}{\pi}\sum_{k=1}^{\infty}|(f,\ \upsilon_k\|u_k\|_2)|\left(\sum_{i=0}^{n_k}\frac{1}{i!\ \|u_k\|_2}\left|\int_0^R A^+(P,\ u_{k-i},\ x,\ r)\ \Phi_{k1}^i(r,\ R,\ |\lambda_n|)\ dr\right|\right)$$

$$+\frac{1}{\pi}\sum_{k=1}^{\infty}|(f,\ \upsilon_k\|u_k\|)|\left(\sum_{i=0}^{n_k}\frac{1}{i!\,\|u_k\|_2}\left|\int_0^R A^-(P,\ u_{k-i},\ x,\ r)\ \Phi_{k2}^i(r,\ R,\ |\lambda_n|)\ dr\right|\right)$$

$$=S_1(x)+S_2(x)+S_3(x)+S_4(x)+S_5(x)\,.$$

Prove that the series $S_l(x)\,,\ l=\overline{1,5},\ \ x\in K$, uniformly converge and their sum does not exceed the value $C(K)\,\|f\|_2$.

At first we note that by statement 2.3, the system $\{\,\upsilon_k(x)\ \|u_k\|_2\}_{k=1}^{\infty}$is also the Riesz basis in $L_2^2\,(0,\,2\pi)$. Consequently, this system is a Bessel system in this space, i.e. for the arbitrary vector-function$f\in L_2^2\,(0,\,2\pi)$ the following Bessel inequality holds

$$\left(\sum_{k=1}^{\infty}|(f,\ \upsilon_k\|u_k\|_2)|^2\right)^{1/2}\le M\|f\|_2\,, \tag{3.4}$$

where the constant $M>0$ is independent of $f(x)$.

We also note that the estimation

$$\sum_{t\le|\rho_k|\le t+1}1\le const\ \ ,\ \ \forall t\ge 0 \tag{3.5}$$

follows from conditions (1.1) and (1.2)

To estimate the sum $S_1(x)$ we apply the Bessel inequality (3.4), estimation (2.2) for $s=\infty$, $p=2$, $|Im\,\lambda_k|\le C_1$, and inequality (3.5)

$$S_1(x)=\frac{1}{2}\sum_{|\rho_k|=|\lambda_n|}|\,(f,\ \upsilon_k\|u_k\|_2)\,|\ \ |u_k(x)|\ \ \|u_k\|_2^{-1}$$

$$\le\frac{1}{2}\left(\sum_{|\rho_k|=|\lambda_n|}|\,(f,\ \upsilon_k\|u_k\|_2)\,|^2\right)^{1/2}\left(\sum_{|\rho_k|=|\lambda_n|}|u_k(x)|^2\ \|u_k\|_2^{-2}\right)^{1/2}$$

$$\le M\,\|f\|_2C_2(K)\ \left(\sum_{|\rho_k|=|\lambda_n|}||u_k(x)||_{2,\,G_{l_0}}^2\ \|u_k\|_2^{-2}\right)^{1/2}$$

$$\leq M\,C_2(K)\,\|f\|_2 \left(\sum_{|\rho_k|=|\lambda_n|} 1 \right)^{1/2} \leq C\,(K)\,\|f\|_2\,,$$

where $C(K) > 0$ is some constant.

To estimate the series $S_2(x),\ x \in K$, we also apply the Bessel inequality (3.4), estimation (2.2) for $s = 0,\ p = 2,\ |Im\,\lambda_k| < C_1$ and inequality (3.5). As a result we have:

$$S_2(x) \leq C\,(R) \left(\sum_{k=1}^{\infty} |(f,\, \upsilon_k\ \|u_k\|_2)|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |u_k(x)|^2\, \|u_k\|_2^{-2}\, (1 + ||\lambda_n| - |\rho_k||)^{-2} \right)$$

$$\leq C\,(R)\,M\|f\|_2 \cdot C_2(K) \left(\sum_{k=1}^{\infty} \|u_k\|^2_{2,\,G_{l_0}}\, \|u_k\|_2^{-2}\, (1 + ||\lambda_n| - |\rho_k||)^{-2} \right)^{1/2}$$

$$\leq C(K)\|f\|_2 \left(\sum_{k=1}^{\infty} (1 + ||\lambda_n| - |\rho_k||)^{-2} \right)^{1/2}$$

$$\leq C(K)\|f\|_2 \left(\sum_{j=0}^{\infty} (1+j)^{-2} \sum_{j \leq ||\lambda_n|-|\rho_k|| \leq j+1} 1 \right)^{1/2}$$

$$\leq C(K)\|f\|_2 \left(\sum_{i=1}^{\infty} i^{-2} \right)^{1/2} \leq C(K)\|f\|_2.$$

Inequalities (2.2), (2.3), (1.1) and (1.2) for $x \in K$ yield

$$\frac{|u_{k-i}(x)|}{\|u_k\|_2} \leq \|u_{k-i}\|_{\infty,\,K}\, \|u_k\|_2^{-1} \leq C_2\, \|u_{k-i}\|_{2,K}\, \|u_k\|_2^{-1}\, (1 + |Im\,\lambda_k|)^{1/2}$$

$$\leq C_2 C_3^{n_k}\, (1 + |Im\,\lambda_k|)^{n_k + \frac{1}{2}}\, \|u_k\|_{2,K}\, \|u_k\|_2^{-1} \leq C_2 C_3^{N_0}\, (1 + C_1)^{N_0 + \frac{1}{2}} = C(K)\,,$$

i.e. it is fulfilled the estimation

$$\|u_{k-i}\|_{\infty,\,K} \leq C(K)\ \|u_k\|_2\,, \tag{3.6}$$

where $C(K) > 0$ is some constant.

We now estimate the series $S_3(x),\ x \in K$. By inequalities (3.6) and (2.8)

$$S_3(x) = C\,(R) \sum_{k=2}^{\infty} |(f,\, \upsilon_k\ \|u_k\|_2)| \left(\sum_{i=1}^{n_k} \frac{1}{i!}\ |B_i\,(|\lambda_n|\,,\, \lambda_k,\, R)|\ \|u_k\|_{\infty,\,K}\, \|u_k\|_2^{-1} \right)$$

$$\leq C(K)\,C\,(R) \sum_{k=1}^{\infty} |(f,\, \upsilon_k\, \|u_k\|_2)| \left(\sum_{i=1}^{N_0} \frac{1}{i!}\, C_i\,(R) \right) (1 + ||\lambda_n| - |\rho_k||)^{-1}$$

$$\leq C(K) \sum_{k=1}^{\infty} |(f,\, \upsilon_k\, \|u_k\|_2)|\ (1 + ||\lambda_n| - |\rho_k||)^{-1}\,.$$

Hence, by virtue of the Bessel inequality and estimate (3.5) we get (see estimation for $\mathrm{S}_2(x)$)

$$\mathrm{S}_3(x) \leq C(K)\,\|f\|_2.$$

We now estimate the series $S_4(x)$ and $S_5(x)$. Since they are estimated by a unique scheme, we estimate only the series $S_4(x)$. The expression $A^+(P, u_{n-i}, x, r)$, $x \in K$, $0 \le r \le R$ and estimation (2.2), (2.3) imply the inequality

$$\left|A^+(P, u_{n-i}, x, r)\right| \le C(G_{l_0})\, A(x,r)\cdot\|u_k\|_{2, G_{l_0}} \le C(G_{l_0})\, A(x,r)\, \|u_k\|_2 \le const\, A(x,r)\, \|u_k\|_2 ,$$

where

$$A(x,r) = |p(x-r)| + |q(x-r)| + |p(x+r)| + |q(x+r)| .$$

The estimation

$$\|A(x,\cdot)\|_{p,[0,R]} = \left(\int_0^R A^p(x,r)\, dr\right)^{1/p} \le const\, \left(\|p\|_p + \|q\|_p\right)$$

is fulfilled at each fixed $x \in K$ for $A(x,r)$.

Therefore, by the Holder inequality we get

$$\left|\int_0^R A^+(P, u_{k-i}, x, r)\, \Phi^i_{k1}(r, R, |\lambda_n|)\, dr\right| \le const\, \|u_k\|_2 \int_0^R A(x,r)\, \left|\Phi^i_{k1}(r, R, |\lambda_n|)\right| dr$$

$$\le const\, \|A(x,\cdot)\|_{p,[0,R]}\, \left\|\Phi^i_{k1}(\cdot, R, |\lambda_n|)\right\|_{q,[0,R]} \|u_k\|_2 .$$

Taking into account the obtained inequalities in the series $S_4(x)$ we get

$$S_4(x) = \frac{1}{\pi}\sum_{k=1}^{\infty} |(f, \upsilon_k \|u_k\|_2)| \left(\sum_{i=0}^{n_k} \frac{1}{i!\, \|u_k\|_2} \left|\int_0^R A^+(P, u_{k-i}, x, r)\, \Phi^i_{k1}(r, R, |\lambda_n|)\, dr\right|\right)$$

$$\le const\, \left(\|p\|_p + \|q\|_p\right) \sum_{k=1}^{\infty} |(f, \upsilon_k \|u_k\|)| \sum_{i=0}^{n_k} \left\|\Phi^i_{k1}(\cdot, R, |\lambda_n|)\right\|_{q,[0,R]} .$$

Hence, by the Bessel property it follows

$$S_4(x) \le const\, \|f\|_2 \left\{\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n_k} \left\|\Phi^i_{k_1}(\cdot, R, |\lambda_n|)\right\|_{q,[0,R]}\right)^2\right\}^{1/2}$$

$$\le const \|f\|_2 \left\{\sum_{k=1}^{\infty}\left(\sum_{i=1}^{N_0} \left\|\Phi^i_{k1}(\cdot, R, |\lambda_n|)\right\|_{q,[0,R]}\right)^2\right\}^{1/2} . \qquad (3.7)$$

Prove that the series in curly brackets converges, and estimate its sum. For the integrals $\Phi^i_{kj}\cdot(r, R, |\lambda_n|)$, $j = 1, 2$ the following estimation is valid (see [10])

$$\left|\Phi^i_{kj}\right| \le C_i(R, \alpha) \begin{cases} ||\lambda_n| - |\rho_k||^{-\alpha}\, r^{-\alpha} & for\ \ ||\lambda_n| - |\rho_k|| \ge 1,\ \ i = 0, \\ \max\{|\ln r|, |\ln R|\} & for\ \ ||\lambda_n| - |\rho_k|| < 1,\ \ i = 0, \\ ||\lambda_n| - |\rho_k||^{-1} & for\ \ ||\lambda_n| - |\rho_k|| \ge 1,\ \ i \ne 0, \\ (R-r)^i & for\ \ ||\lambda_n| - |\rho_k|| < 1,\ \ i \ne 0, \end{cases}$$

where $\alpha \in (0,1]$.

Apply these estimations for $p > 2$, $\alpha \in \left(\frac{1}{2}, \frac{p-1}{p}\right)$,

$$\sum_{k=1}^{\infty}\left(\sum_{i=0}^{N_0} \left\|\Phi^i_{k_1}(\cdot, R, |\lambda_n|)\right\|_{q,[0,R]}\right)^2$$

$$\le C\,(N_0,\,R,\,\alpha)\left\{\sum_{||\lambda_n|-|\rho_k||<1}\left(\int_0^R(\max\,\{|\ln\,r|\,,\,|\ln\,R|\})^q\,dr\right)^{2/q}\right.$$

$$\left.+\sum_{||\lambda_n|-|\rho_k||\ge 1}||\lambda_n|-|\rho_k||^{-2\alpha}\left\|r^{-\alpha}\right\|^2_{q,\,[0,\,R]}\right\}$$

$$=C\,(N_0,\,R,\,\alpha)\left\{\left(\int_0^R(\max\,\{|\ln\,r|\,,\,|\ln\,R|\})^q\,dr\right)^{2/q}\sum_{||\lambda_n|-|\rho_k||<1}1\right.$$

$$\left.+\left\|r^{-\alpha}\right\|^2_{q,\,[0,\,R]}\sum_{||\lambda_n|-|\rho_k||\ge 1}||\lambda_n|-|\rho_k||^{-2\alpha}\right\}\,,\,1/p+1/q=1.$$

Since $q\alpha<1$, then by the condition (1.2) we get

$$\sum_{k=1}^{\infty}\left(\sum_{i=0}^{N_0}\left\|\varPhi^i_{k_1}\,(\cdot,\,R,\,|\lambda_n|)\right\|_{q,\,[0,\,R]}\right)^2\le C_1\,(N_0,\,R,\,\alpha)\left\{1+\sum_{||\lambda_n|-|\rho_k||\ge 1}||\lambda_n|-|\rho_k||^{-2\alpha}\right\}$$

$$\le C_2\,(N_0,\,R,\,\alpha)\left\{1+\sum_{l=1}^{\infty}l^{-2\alpha}\left(\sum_{l\le||\lambda_n|-|\rho_k||\le l+1}1\right)\right\}$$

$$\le C_2\,(N_0,\,R,\,\alpha)\;C_2\left\{1+\sum_{l=1}^{\infty}l^{-2\alpha}\right\}<\infty\,.$$

Consequently, the last relation and (3.7) imply the inequality

$$S_4(x)\;\le C(K)\,\|f\|_2. \tag{3.8}$$

The series $S_5(x)$ is estimated in the same way, and estimation (3.8) for it is fulfilled as well.

From the estimations obtained for $S_j(x)\,,\,j\;=\;\overline{1,5}\,,$ it follows that for an arbitrary vector-function $f\in L^2_2\,(0,\,2\pi)$ the following estimation is valid:

$$\left\|\tilde{S}_{|\lambda_n|}\,(\cdot\,,f)-\sigma_n\,(\cdot\,,f)\right\|_{C(K)}\le C_1(K)\,\|f\|_2, \tag{3.9}$$

where $C_1(K)>0$ is a constant independent of f.

Now from estimation (3.9) we derive relation (3.1). From the completeness of the system $\{u_k(x)\}_{k=1}^{\infty}$ in the space $L^2_2\,(G)$ it follows that for an arbitrary $f\in L^2_2\,(G)$ and for any $\varepsilon>0$ there exist such constants $\alpha_l,\;l=\overline{1,\,n\;(\varepsilon,f)}$ that

$$\|f-g\|_2<\frac{\varepsilon}{(2C_1(K))}\,,\qquad g(x)=\sum_{l=1}^{n(\varepsilon,f)}\alpha_l u_l(x),$$

where $C_1(K)$ is a constant from the inequality (3.9).

Obviously, for sufficiently large *n* we have the equality $\sigma_n\,(x,g)=g(x)$. Therefore, for large *n*

$$\left\|\tilde{S}_{|\lambda_n|}\,(\cdot\,,f)-\sigma_n\,(\cdot\,,f)\right\|_{C(K)}\le\left\|\tilde{S}_{|\lambda_n|}\,(\cdot\,,f-g)-\sigma_n\,(\cdot\,,f-g)\right\|_{C(K)}+\left\|g-\tilde{S}_{\nu}\,(\cdot\,,g)\right\|_{C(K)}.$$

The estimation (3.9) and the last relation imply that for sufficiently large *n*

$$\left\|\tilde{S}_{|\lambda_n|}(\cdot\,,f)-\sigma_n(\cdot\,,f)\right\|_{C(K)}$$

$$\leq C_1(K)\,\|f-g\|_2+\left\|\tilde{S}_{|\lambda_n|}(\cdot\,,g)-g\right\|_{C(K)}<\frac{\varepsilon}{2}+\left\|\tilde{S}_{|\lambda_n|}(\cdot\,,g)-g\right\|_{C(K)}.$$

The value $\left\|S_{|\lambda_n|}(\cdot\,,g_j)-g_j(x)\right\|_{C(K)}$, where $g(x)=(g_1(x)\,,g_2(x))^T$, tends to zero as $n\to\infty$, because $g_j(x)\in W_p^1(G_{l_0})\,,\ p>2$.

Consequently for sufficiently large *n*

$$\left\|\tilde{S}_{|\lambda_n|}(\cdot\,,f)-\sigma_n(\cdot\,,f)\right\|_{C(K)}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Relation (3.1) is proved. Theorem 1.1 is proved.

The statement of Theorem 1.2 follows from the statement of Theorem 1.1 and localization principle for trigonometric series.

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