Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **41** (4), 125-130 (2021).

Cantor functions associated with generalized expansions

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Received: 02.01.2021 / Revised: 28.06.2021 / Accepted: 27.07.2021

Abstract. *The goal of this note is to propose an expansion of real numbers, which generalizes the binary and ternary expansions. As an application of this expansion, the ternary Cantor function is generalized to the one adapted to the expansion. This generalized Cantor function can be adjusted to have the set of discontinuity of any Hausdorff dimension.*

Keywords. Cantor function, expansion, discontinuity, Hausdorff dimension.

Mathematics Subject Classification (2010): 2010 Mathematics Subject Classification: 26A33, 42B35

1 Introduction

The aim of this paper is to show that the following expansion theorem is useful to construct various Hölder continuous functions, where $[\cdot]$ denotes the Gauss function.

Theorem 1.1 *Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $h_n \geq 2$ for all $n \in \mathbb{N}$. Then for any $x \in [0,1]$, there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of integers such that*

$$x = \sum_{n=1}^{\infty} \frac{a_n}{h_1 h_2 \cdots h_n}, \quad 0 \leq a_n \leq h_n - 1 \quad (n = 1, 2, \ldots).$$

More precisely, we have the following algorithm to find each a_n: If $x = 1$, then simply set $a_n = h_n - 1$ for all $n \in \mathbb{N}$. If $x \in [0,1)$ instead, define

$$a_1 = [h_1 x], \quad a_{n+1} = \left[h_1 h_2 \cdots h_n h_{n+1} \left(x - \sum_{j=1}^{n} \frac{a_j}{h_1 h_2 \cdots h_j} \right) \right].$$

Then $0 \leq a_n \leq h_n - 1$ for all $n \in \mathbb{N}$ and $x = \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{h_1 h_2 \cdots h_n}$.

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A direct consequence of Theorem 1.1 is that any $x \in [0,1)$ has an expansion

$$x = \sum_{n=1}^{\infty} \frac{a_n}{h_1 h_2 \cdots h_n}$$

for some sequence $\{a_n\}_{n=1}^{\infty}$ of integers satisfying $0 \le a_n \le h_n - 1$ for all $n \in \mathbb{N}$. As an application, we generalize the ternary Cantor function f.

Theorem 1.2 *Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $h_n \ge 2$. There exists uniquely a non-decreasing continuous function $f : [0,1] \to [0,1]$ such that*

$$f\left(\sum_{n=1}^{N} \frac{a_n}{h_1 h_2 h_3 \cdots h_n}\right) = \sum_{n=1}^{N} \frac{1}{2^n}\left[\frac{a_n}{h_n - 1}\right],$$

if $N \in \mathbb{N}$ and $a_n \in \{0, h_n - 1\}$ for all $n = 1, 2, \ldots, N$.

Ikeda considered the case of $h_n = n, 2n, 2n-1$ in [2].

Note that the Cantor function can be obtained as a special case of $h_n = 3$ for each n. We refer to [1] for a detailed account of the Cantor function. The Cantor function, named after Georg Cantor, is an example of a function which has a remarkable property: Its derivative vanishes almost everywhere. This applies to the function f above. To this end, we specify the points where f is not differentiable.

Theorem 1.3 *Assume that*

$$\lim_{N\to\infty} \frac{2^N}{h_1 h_2 \cdots h_N} = 0 \tag{1.1}$$

Then the set C of all points where f is not differentiable takes the form:

$$C = \left\{\sum_{n=1}^{\infty} \frac{a_n}{h_1 h_2 \cdots h_n} \; : \; a_n \in \{0, h_n - 1\} \quad (n = 1, 2, \ldots)\right\}.$$

We recall the notion of the Hausdorff dimension according to [1]. First for an interval I, we write $\ell(I)$ to denote its length. Let $A \subset \mathbb{R}$ be an arbitrary set. Its δ-covering is the collection of open intervals having radius less than δ whose union covers A. Let $s \ge 0$. Then define

$$\mathcal{H}^s_\delta(A) \equiv \inf\left\{\sum_{j=1}^{\infty} \omega_s \ell(I_j)^s \; : \; \{I_j\}_{j=1}^{\infty} \text{ is a } \delta\text{-covering of } A\right\}.$$

Here ω_s is a constant given by $\omega_s \equiv \pi^{\frac{s}{2}} \Gamma\left(\frac{s+2}{2}\right)^{-1}$, where Γ denotes the Gamma function. Let $s \ge 0$ again. Then define $\mathcal{H}^s(A) \equiv \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A)$. Finally its Hausdorff dimension is given by $\dim_{\mathcal{H}}(A) \equiv \inf\{s \ge 0 \, : \, \mathcal{H}^s(A) = 0\}$. Thanks to the result in the book [1, p. 72], we see that the Hausdorff dimension is

$$\dim_{\mathcal{H}}(C) = \liminf_{N\to\infty} \frac{N}{\log_2(h_1 h_2 \cdots h_N)},$$

as long as $\{h_N\}_{N=1}^{\infty}$ is a sequence in $\mathbb{N} \cap [3, \infty)$ satisfying (1.1).

Let $\alpha \in [0, n]$ be arbitrary. In view of this formula, we see that if we choose h_j's suitably, then we can arrange $\dim_{\mathcal{H}}(C) = \alpha$. The rest of the paper is devoted to the proof of these theorems.

2 Proofs

2.1 Proof of Theorem 1.1

If $x = 1$, then the conclusion is trivial; assume otherwise. We induct on N to show that a_N, whose definition is as in Theorem 1.1, fulfills $a_N \in \{0, 1, \ldots, h_N - 1\}$. In the base case of $N = 1$, the conclusion follows from the fact that $0 \le h_1 x < h_1$. Suppose that the conclusion holds for some $N = N_0 \ge 1$. We observe that

$$0 \le h_1 h_2 \cdots h_{N_0} \left(x - \sum_{n=1}^{N_0 - 1} \frac{a_n}{h_1 h_2 \cdots h_n} \right) - a_{N_0} < 1$$

from the definition of a_{N_0}. Hence,

$$0 \le h_1 h_2 \cdots h_{N_0} h_{N_0+1} \left(x - \sum_{n=1}^{N_0} \frac{a_n}{h_1 h_2 \cdots h_n} \right) < h_{N_0+1},$$

implying that $a_{N_0+1} \in \{0, 1, 2, \ldots, h_{N_0+1} - 1\}$.

It remains to establish that x can be expanded as above. But this is a direct consequence of the inequality:

$$0 \le x - \sum_{n=1}^{N_0} \frac{a_n}{h_1 h_2 \cdots h_n} < \frac{1}{h_1 h_2 \cdots h_{N_0}}.$$

2.2 Proof of Theorem 1.2

Fix N first. Let f_N be a non-decreasing piecewise linear function interpolating linearly between the points

$$\left(\sum_{n=1}^{N} \frac{a_n}{h_1 h_2 \cdots h_n}, \sum_{n=1}^{N} \frac{1}{2^n} \left[\frac{a_n}{h_n - 1} \right] \right), (1, 1),$$

where each a_n moves over the set $\{0, h_n - 1\}$. For the proof of Theorem 1.2, we need the following observation:

Lemma 2.1 *For all $N \in \mathbb{N}$ and $x \in [0, 1]$, $|f_N(x) - f_{N+1}(x)| \le 2^{-N}$.*

Proof. We may assume $0 < x < 1$; otherwise the conclusion is trivial. Let $[2^N f(x)] = k$. Then $k < 2^N$, since $f(x) < 1$. With this in mind, let $x_N^- = f_N^{-1}(k \cdot 2^{-N})$ and $x_N^+ = f_N^{-1}((k+1)2^{-N})$. We claim that $x_N^\pm$ takes the form

$$x_N^\pm = \sum_{n=1}^{N} \frac{a_n^\pm}{h_1 h_2 \cdots h_n}, \quad a_n^\pm \in \{0, h_n - 1\} \quad (n = 1, 2, \ldots, N).$$

In fact, there exists a subset A of $\{1, 2, \ldots, n\}$ such that

$$k \cdot 2^{-N} = \sum_{n \in A} \frac{1}{2^n}.$$

If we set

$$a_n^- = \chi_A(n)(h_n - 1) \quad (n = 1, 2, \ldots, N),$$

then we have the desired sequence. We can go through the same argument as x_N^- for x_N^+. Since

$$f_N(x_N^-) = f_{N+1}(x_N^-) = \frac{k}{2^N} \le f_N(x), f_{N+1}(x) \le f_{N+1}(x_N^-) = f_{N+1}(x_N^+) = \frac{k+1}{2^N},$$

we obtain the desired result.

We prove Theorem 1.2.

We start with the existence. Thanks to Lemma 2.1, the limit $f = \lim_{N\to\infty} f_N$ exists in $L^\infty([0,1])$. Since each f_N is continuous, so is f. Since

$$f_M\left(\sum_{n=1}^N \frac{a_n}{h_1h_2h_3\cdots h_n}\right) = \sum_{n=1}^N \frac{1}{2^n}\left[\frac{a_n}{h_n-1}\right],$$

if $M, N \in \mathbb{N}$ satisfies $M \geq N$ and $a_n \in \{0, h_{n-1}\}$ for all $n = 1, 2, \ldots, N$, letting $M \to \infty$, we obtain the desired function f.

We prove the uniqueness. Choose any continuous function $g : [0,1] \to [0,1]$ with the same property as g. Let $x \in [0,1)$ and define $x_N^+ = f_N^{-1}(2^{-N}[2^N f(x)+1])$ and $x_N^- = f_N^{-1}(2^{-N}[2^N f(x)])$ as in the proof of Lemma 2.1. Then since we know the value of $f(x_N^\pm)$ and $g(x_N^\pm)$, we have $f(x_N^-) = g(x_N^-) \leq f(x), g(x) \leq f(x_N^+) = g(x_N^+) = f(x_N^-) + 2^{-N}$. Letting $N \to \infty$, we obtain $g(x) = \lim_{N\to\infty} g(x_N^-) = \lim_{N\to\infty} f(x_N^-) = f(x)$. Thus $f \equiv g$.

3 Non-differentiablilty points of f in Theorem 1.2

Here and below it is understood that $\sum_{n=1}^{0} x_n = 0$ for any sequnce $\{x_n\}_{n=1}^\infty$. We start with an additional property of f.

Proposition 3.1 *The function f in Theorem 1.2 satisfies*

$$f\left(\sum_{n=1}^N \frac{a_n}{h_1h_2h_3\cdots h_n} + \frac{1}{h_1h_2\cdots h_{N+1}}\right) = \sum_{n=1}^N \frac{1}{2^n}\left[\frac{a_n}{h_n-1}\right] + \frac{1}{2^{N+1}},$$

if $N \in \mathbb{N}$ and $a_n \in \{0, h_n - 1\}$ for all $n = 1, 2, \ldots, N$.

Proof. Let $M \geq N$. Simply observe that

$$\sum_{n=1}^N \frac{a_n}{h_1h_2h_3\cdots h_n} + \frac{1}{h_1h_2\cdots h_{n+1}} = \sum_{n=1}^N \frac{a_n}{h_1h_2h_3\cdots h_n} + \sum_{n=N+2}^{\infty} \frac{h_n-1}{h_1h_2\cdots h_n}$$

and that

$$f\left(\sum_{n=1}^N \frac{a_n}{h_1h_2h_3\cdots h_n} + \sum_{n=N+2}^{M} \frac{h_n-1}{h_1h_2\cdots h_n}\right) = \sum_{n=1}^N \frac{1}{2^n}\left[\frac{a_n}{h_n-1}\right] + \sum_{n=N+2}^{M} \frac{1}{2^n}.$$

If we let $M \to \infty$, we obtain the desired result.

Corollary 3.1

1) If $N \in \mathbb{N} \cup \{0\}$ and $a_n \in \{0, h_{n-1}\}$, $t \in [1, h_{N+1} - 1]$ for all $n = 1, 2, \ldots, N$, then

$$f\left(\sum_{n=1}^N \frac{a_n}{h_1h_2h_3\cdots h_n} + \frac{t}{h_1h_2\cdots h_{N+1}}\right) = \sum_{n=1}^N \frac{1}{2^n}\left[\frac{a_n}{h_n-1}\right] + \frac{1}{2^{N+1}}$$

for all.

2) If $N \in \mathbb{N} \cup \{0\}$ and $a_n \in \{0, h_{n-1}\}$, $t \in (1, h_{N+1} - 1)$ for all $n = 1, 2, \ldots, N$, then

$$f'\left(\sum_{n=1}^{N} \frac{a_n}{h_1 h_2 h_3 \cdots h_n} + \frac{t}{h_1 h_2 \cdots h_{N+1}}\right) = 0.$$

3) The function f is almost everywhere differentiable if (1.1) holds.

Since $h_N \geq 2$ for each N, (1.1) fails if and only if $h_N = 2$ with a finite number of exceptions. So, (1.1) is a natural assumption.

Proof.

1 Simply observe that

$$f\left(\sum_{n=1}^{N} \frac{a_n}{h_1 h_2 h_3 \cdots h_n} + \frac{1}{h_1 h_2 \cdots h_{N+1}}\right) = f\left(\sum_{n=1}^{N} \frac{a_n}{h_1 h_2 h_3 \cdots h_n} + \frac{h_{N+1} - 1}{h_1 h_2 \cdots h_{N+1}}\right)$$
$$= \sum_{n=1}^{N} \frac{1}{2^n}\left[\frac{a_n}{h_n - 1}\right] + \frac{1}{2^{N+1}}$$

and that f is non-decreasing.

2 Since f is constant on

$$\left(\sum_{n=1}^{N} \frac{a_n}{h_1 h_2 h_3 \cdots h_n} + \frac{1}{h_1 h_2 \cdots h_{N+1}}, \sum_{n=1}^{N} \frac{a_n}{h_1 h_2 h_3 \cdots h_n} + \frac{h_{N+1} - 1}{h_1 h_2 \cdots h_{N+1}}\right),$$

f' vanishes on this interval.

3 Write

$$I_{\emptyset} = \left(\frac{1}{h_1}, \frac{h_1 - 1}{h_1}\right)$$

and

$$I_{(a_1, a_2, \ldots, a_n)} = \left(\sum_{n=1}^{N} \frac{a_n}{h_1 h_2 h_3 \cdots h_n} + \frac{1}{h_1 h_2 \cdots h_{N+1}}, \sum_{n=1}^{N} \frac{a_n}{h_1 h_2 h_3 \cdots h_n} + \frac{h_{N+1} - 1}{h_1 h_2 \cdots h_{N+1}}\right).$$

We remark that

$$\chi_{I_{\emptyset}} + \sum_{N=1}^{\infty} \sum_{a_1 \in \{0, h_1 - 1\}, a_2 \in \{0, h_2 - 1\}, \ldots, a_N \in \{0, h_N - 1\}} \chi_{I_{(a_1, a_2, \ldots, a_n)}} \leq 1,$$

since

$$\frac{1}{h_1 h_2 \cdots h_{N+1}} = \sum_{n=N+2}^{\infty} \frac{h_n - 1}{h_1 h_2 \cdots h_n}$$

for any $n \in \mathbb{N} \cup \{0\}$. Meanwhile,

$$\begin{aligned}
&|I_\emptyset| + \sum_{N=1}^{\infty} \sum_{a_1 \in \{0,h_1-1\}, a_2 \in \{0,h_2-1\}, \dots, a_N \in \{0,h_N-1\}} |I_{(a_1,a_2,\dots,a_n)}| \\
&= \frac{h_1-2}{h_1} + \sum_{N=1}^{\infty} \frac{2^N(h_{N+1}-2)}{h_1 h_2 \cdots h_{N+1}} \\
&= \frac{h_1-2}{h_1} + \sum_{N=1}^{\infty} \left(\frac{2^N}{h_1 h_2 \cdots h_N} - \frac{2^{N+1}}{h_1 h_2 \cdots h_{N+1}} \right) \\
&= 1
\end{aligned}$$

thanks to our assumption (1.1).
Thus, f is almost everywhere differentiable.

As we saw above, f is almost everywhere differentiable under (1.1).

Proposition 3.2 *Assume that* (1.1) *holds. If* $x \in (0,1)$ *has an expansion*

$$x = \sum_{n=1}^{\infty} \frac{a_n}{h_1 h_2 \cdots h_n} \quad (a_n \in \{0, h_n - 1\}),$$

then f *is non-differentiable at* x.

Proof. In fact, we set

$$x_N^+ = \sum_{n=1}^{N} \frac{a_n}{h_1 h_2 \cdots h_n} + \sum_{n=N+1}^{\infty} \frac{h_n - 1}{h_1 h_2 \cdots h_n}, \quad x_N^- = \sum_{n=1}^{N} \frac{a_n}{h_1 h_2 \cdots h_n}.$$

for each N. Thanks to Proposition 3.1, we have

$$x_N^- \le x \le x_N^+ = x_N^{-1} + \frac{1}{h_1 h_2 \cdots h_N}, \quad f(x_N^+) - f(x_N^-) = \frac{1}{2^N}$$

for all $N \in \mathbb{N}$. Consequently,

$$\lim_{N \to \infty} \frac{f(x_N^+) - f(x_N^-)}{x_N^+ - x_N^-} = \infty.$$

Consequently, from Corollary 3.1 and Proposition 3.2, we conclude the proof of Theorem 1.3.

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