Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **41** (4), 153-165 (2021).

Horizontal lift in the semi-tangent bundle and its applications

Furkan Yildirim

Received: 26.02.2021 / Revised: 23.04.2021 / Accepted: 15.05.2021

Abstract. *The main aim of the present paper is to study, using the pullback bundle, the complete and horizontal lifts of vector and affinor (tensor of type (1,1)) fields and to investigate their applications.*

Keywords. Vector field, complete lift, horizontal lift, pullback bundle, cotangent bundle, semi-tangent bundle

Mathematics Subject Classification (2010): 53A45, 53C55, 55R10.

1 Introduction

Let M_n be an $n-$dimensional differentiable manifold of class C^∞ and finite dimension n, and $T^*(M_n)$ the cotangent bundle determined by a natural projection (submersion) $\pi_1 : T^*(M_n) \to M_n$. We use the notation $(x^i) = (x^{\overline{\alpha}}, x^\alpha)$, where the indices $i, j, ...$ have range in $\{1, 2, ..., 2n\}$, the indices $\alpha, \beta, ...$ have range in $\{1, 2, ..., n\}$ and the indices $\overline{\alpha}, \overline{\beta}, ...$have range in $\{n+1, n+2, ..., 2n\}$, x^α are coordinates in M_n, $x^{\overline{\alpha}} = p_\alpha$ are fiber coordinates of the cotangent bundle $T^*(M_n)$. If $(x^{i'}) = (x^{\overline{\alpha}'}, x^{\alpha'})$ is another system of local adapted coordinates in the cotangent bundle $T^*(M_n)$, then we have

$$\begin{cases} x^{\overline{\alpha}'} = \frac{\partial x^\beta}{\partial x^{\alpha'}} p_\beta, \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right). \end{cases} \tag{1.1}$$

The Jacobian of (1.1) has components

$$(A^{i'}_j) = \left(\frac{\partial x^{i'}}{\partial x^j}\right) = \begin{pmatrix} A^\beta_{\alpha'} & p_\sigma A^{\beta'}_\beta A^\sigma_{\beta'\alpha'} \\ 0 & A^{\alpha'}_\beta \end{pmatrix},$$

where $A^\beta_{\alpha'} = \frac{\partial x^\beta}{\partial x^{\alpha'}}$, $A^\sigma_{\beta'\alpha'} = \frac{\partial^2 x^\sigma}{\partial x^{\beta'}\partial x^{\alpha'}}$. Let $T_{\mathrm{p}}(M_n)$ be the tangent space at a point p of M_n (p= $\pi_1(\widetilde{\mathrm{p}})$, $\widetilde{\mathrm{p}} = (x^{\overline{\alpha}}, x^\alpha) \in T^*(M_n)$). If $x^\alpha = dx^\alpha(x^\beta)$ are components of x in tangent space $T_{\mathrm{p}}(M_n)$ with respect to the natural base $\{\partial_\alpha\}$ $(\partial_\alpha = \frac{\partial}{\partial x^\alpha})$, then by definition the set $t(M_n)$ of all points $(x^I) = (x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$, $x^{\overline{\overline{\alpha}}} = y^\alpha$; $I, J, ... = 1, ..., 3n$ with projection

The paper was supported by TUBITAK project MFAG-118F176.

F. Yildirim
Department of Mathematics, Faculty of Sci. Atatürk University, Narman Vocational Training School, 25530, Erzurum Turkey
E-mail: furkan.yildirim@atauni.edu.tr

$\pi_2 : t(M_n) \to T^*(M_n)$ (i.e. $\pi_2 : (x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}}) \to (x^{\overline{\alpha}}, x^{\alpha})$) is a semi-tangent [1], [7], [9] (pullback) bundle of the tangent bundle by submersion $\pi_1 : T^*(M_n) \to M_n$ (for the pullback bundle, see [2], [3], [5], [8]). It is clear that the pullback bundle $t(M_n)$ of the tangent bundle $T(M_n)$ also has the natural bundle structure over M_n, its bundle projection $\pi : t(M_n) \to M_n$ being defined by $\pi : (x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}}) \to (x^{\alpha})$, and hence $\pi = \pi_1 \circ \pi_2$. Thus $(t(M_n), \pi_1 \circ \pi_2)$ is the step-like bundle [4] or composite bundle [[6], p.9]. The main aim of this paper is to study complete and horizontal lifts of vector fields and tensor fields of type (1,1) from cotangent bundle $T^*(M_n)$ to semi-tangent (pullback) bundle $(t(M_n), \pi_2)$.

We denote by $\Im_q^p(T^*(M_n))$ and $\Im_q^p(M_n)$ the modules over $F(T^*(M_n))$ and $F(M_n)$ of all tensor fields of type (p,q) on $T^*(M_n)$ and M_n respectively, where $F(T^*(M_n))$ and $F(M_n)$ denote the rings of real-valued C^{∞} −functions on $T^*(M_n)$ and M_n, respectively

To a transformation (1.1) of local coordinates of $T^*(M_n)$, there corresponds on $t(M_n)$ the coordinate transformation

$$\begin{cases} x^{\overline{\alpha}'} = \frac{\partial x^{\beta}}{\partial x^{\alpha'}} p_{\beta}, \\ x^{\alpha'} = x^{\alpha'}\left(x^{\beta}\right), \\ x^{\overline{\overline{\alpha}}'} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} y^{\beta}. \end{cases} \tag{1.2}$$

The Jacobian of (1.2) is given by

$$\overline{A} = (A_J^{I'}) = \begin{pmatrix} A_{\alpha'}^{\beta} & p_{\sigma} A_{\beta}^{\beta'} A_{\beta'\alpha'}^{\sigma} & 0 \\ 0 & A_{\beta}^{\alpha'} & 0 \\ 0 & A_{\beta\varepsilon}^{\alpha'} y^{\varepsilon} & A_{\beta}^{\alpha'} \end{pmatrix}, \tag{1.3}$$

where

$$A_{\beta}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}}, \quad A_{\alpha'}^{\beta} = \frac{\partial x^{\beta}}{\partial x^{\alpha'}}, \quad A_{\beta\varepsilon}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta} \partial x^{\varepsilon}}, \quad A_{\beta'\alpha'}^{\alpha} = \frac{\partial^2 x^{\alpha}}{\partial x^{\beta'} \partial x^{\alpha'}}.$$

From $Det(A_{\beta}^{\alpha'}) \neq 0$, we see that

$$Det\, \overline{A} \neq 0.$$

2 Vertical Lifts

Let $X \in \Im_0^1(T^*(M_n))$, i.e. $X = X^{\alpha}\partial_{\alpha}$. On putting

$$^{vv}X = (^{vv}X^{\alpha}) = \begin{pmatrix} 0 \\ 0 \\ X^{\alpha} \end{pmatrix}, \tag{2.1}$$

from (1.3), we easily see that $^{vv}X' = \overline{A}(^{vv}X)$. The vector field ^{vv}X is called the vertical lift of X to $t(M_n)$.

Let ω be an 1-form with local components ω_{α} on M_n, so that ω is a 1-form with local expression $\omega = \omega_{\alpha} dx^{\alpha}$. On putting

$$^{vv}\omega = (0,\ \omega_{\alpha},\ 0), \tag{2.2}$$

we have a vector field $^{vv}\omega$ on $t(M_n)$. In fact, from (1.3) we easily see that $(^{vv}\omega)' = \left(\overline{A}\right)^{-1} (^{vv}\omega)$, where $\overline{(A)}^{-1} = (A_{J'}^{I})$ is the inverse matrix of $\overline{A}$.

The covector field thus introduced is called the vertical lift of the 1-form ω to $t(M_n)$. For the natural coframe $\{dx^\alpha\}$ in each U, from (2.2) we have in $\pi^{-1}(U)$

$$^{vv}(dx^\alpha) = dx^\alpha$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$.

3 $\gamma-$ Operator

For any $F \in \Im_1^1(T^*(M_n))$, if we take account of (1.3), we can prove that $(\gamma F)' = \overline{A}(\gamma F)$, where γF is a vector field defined by

$$\gamma F = (\gamma F^A) = \begin{pmatrix} -p_\sigma F_\alpha^\sigma \\ 0 \\ y^\varepsilon F_\varepsilon^\alpha \end{pmatrix} \tag{3.1}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$.
If $\omega \in \Im_1^0(M_n)$ and $F \in \Im_1^1(T^*(M_n))$ then

$$^{vv}\omega(\gamma F) = 0.$$

Let $T \in \Im_2^1(M_n)$. On putting

$$\gamma T = (\gamma T_B^A) = \begin{pmatrix} 0 & -p_\sigma T_{\beta\ \alpha}^{\sigma} & 0 \\ 0 & 0 & 0 \\ 0 & y^\varepsilon T_{\varepsilon\ \beta}^{\alpha} & 0 \end{pmatrix},$$

from (1.3), we easily see that $\gamma T_{B'}^{A'} = A_A^{A'} A_{B'}^B \gamma T_B^A$, where $\overline{(A)}^{-1} = (A_{B'}^B)$ is the inverse matrix of $\overline{A}$.
If $X \in \Im_0^1(T^*(M_n))$ and $T \in \Im_2^1(M_n)$, then

$$(\gamma T)^{vv} X = 0.$$

4 Complete Lift of Vector Fields

Let $X \in \Im_0^1(T^*(M_n))$, i.e. $X = X^\alpha \partial_\alpha$. The complete lift ^cX of X to cotangent bundle is defined by $^cX = X^\alpha \partial_\alpha - p_\beta(\partial_\alpha X^\beta)\partial_{\overline{\alpha}}$ [[10], p.236]. On putting

$$^{cc}X = (^{cc}X^\alpha) = \begin{pmatrix} -p_\varepsilon(\partial_\alpha X^\varepsilon) \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix}, \tag{4.1}$$

from (1.3), we easily see that $^{cc}X' = \overline{A}\,(^{cc}X)$. The vector field ^{cc}X is called the complete lift of $^cX \in \Im_0^1(T^*(M_n))$ to $t(M_n)$.

Theorem 4.1 *Let $X, Y \in \Im_0^1(T^*(M_n))$. For the Lie product, we have*

(i) $[^{cc}X, ^{cc}Y] = ^{cc}[X, Y]$ *(i.e.* $L_{^{cc}X}\, ^{cc}Y = ^{cc}(L_X Y)$*),*
(ii) $[^{cc}X, ^{vv}Y] = ^{vv}[X, Y]$,
(iii) $[^{vv}X, ^{vv}Y] = 0$,
(iv) $[^{cc}X, \gamma F] = \gamma(L_X F)$

for any $F \in \Im_1^1(T^*(M_n))$, where L_X the operator of Lie derivation with respect to X.

Proof. (i) If $X,Y\in\Im_0^1(T^*(M_n))$ and $\begin{pmatrix}[^{cc}X,^{cc}Y]^{\overline{\beta}}\\ [^{cc}X,^{cc}Y]^{\beta}\\ [^{cc}X,^{cc}Y]^{\overline{\overline{\beta}}}\end{pmatrix}$ are components of $[^{cc}X,^{cc}Y]$ with respect to the coordinates $(x^{\overline{\beta}},x^{\beta},x^{\overline{\overline{\beta}}})$ on $t(M_n)$, then we have

$$[^{cc}X,^{cc}Y]^J=(^{cc}X)^I\partial_I(^{cc}Y)^J-(^{cc}Y)^I\partial_I(^{cc}X)^J.$$

As the first coordinate, if $J=\overline{\beta}$, we obtain

$$\begin{aligned}[^{cc}X,^{cc}Y]^{\overline{\beta}}&=(^{cc}X)^I\partial_I(^{cc}Y)^{\overline{\beta}}-(^{cc}Y)^I\partial_I(^{cc}X)^{\overline{\beta}}\\&=(^{cc}X)^{\overline{\alpha}}\partial_{\overline{\alpha}}(^{cc}Y)^{\overline{\beta}}+(^{cc}X)^{\alpha}\partial_{\alpha}(^{cc}Y)^{\overline{\beta}}+(^{cc}X)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(^{cc}Y)^{\overline{\beta}}\\&\quad-(^{cc}Y)^{\overline{\alpha}}\partial_{\overline{\alpha}}(^{cc}X)^{\overline{\beta}}-(^{cc}Y)^{\alpha}\partial_{\alpha}(^{cc}X)^{\overline{\beta}}-(^{cc}Y)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(^{cc}X)^{\overline{\beta}}\\&=p_\varepsilon\partial_\alpha X^\varepsilon(\partial_\beta Y^\alpha)-X^\alpha\partial_\alpha p_\varepsilon(\partial_\beta Y^\varepsilon)-p_\varepsilon\partial_\alpha Y^\varepsilon(\partial_\beta X^\alpha)+Y^\alpha\partial_\alpha p_\varepsilon(\partial_\beta X^\varepsilon)\\&=p_\varepsilon(\partial_\beta Y^\alpha\partial_\alpha X^\varepsilon-X^\alpha\partial_\alpha\partial_\beta Y^\varepsilon-\partial_\beta X^\alpha\partial_\alpha Y^\varepsilon+Y^\alpha\partial_\alpha\partial_\beta X^\varepsilon)\\&=-p_\varepsilon(\partial_\beta(X^\alpha\partial_\alpha Y^\varepsilon-Y^\alpha\partial_\alpha X^\varepsilon))\\&=-p_\varepsilon(\partial_\beta[X,Y]^\varepsilon)\end{aligned}$$

by virtue of (4.1). As the second coordinate, if $J=\beta$, we obtain

$$\begin{aligned}[^{cc}X,^{cc}Y]^{\beta}&=(^{cc}X)^I\partial_I(^{cc}Y)^{\beta}-(^{cc}Y)^I\partial_I(^{cc}X)^{\beta}\\&=(^{cc}X)^{\overline{\alpha}}\partial_{\overline{\alpha}}(^{cc}Y)^{\beta}+(^{cc}X)^{\alpha}\partial_{\alpha}(^{cc}Y)^{\beta}+(^{cc}X)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(^{cc}Y)^{\beta}\\&\quad-(^{cc}Y)^{\overline{\alpha}}\partial_{\overline{\alpha}}(^{cc}X)^{\beta}-(^{cc}Y)^{\alpha}\partial_{\alpha}(^{cc}X)^{\beta}-(^{cc}Y)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(^{cc}X)^{\beta}\\&=(^{cc}X)^{\alpha}\partial_{\alpha}(^{cc}Y)^{\beta}-(^{cc}Y)^{\alpha}\partial_{\alpha}(^{cc}X)^{\beta}\\&=X^\alpha\partial_\alpha Y^\beta-Y^\alpha\partial_\alpha X^\beta\\&=[X,Y]^\beta\end{aligned}$$

by virtue of (4.1). As the third coordinate, if $J=\overline{\overline{\beta}}$, then we obtain

$$\begin{aligned}[^{cc}X,^{cc}Y]^{\overline{\overline{\beta}}}&=(^{cc}X)^I\partial_I(^{cc}Y)^{\overline{\overline{\beta}}}-(^{cc}Y)^I\partial_I(^{cc}X)^{\overline{\overline{\beta}}}\\&=(^{cc}X)^{\overline{\alpha}}\partial_{\overline{\alpha}}(^{cc}Y)^{\overline{\overline{\beta}}}+(^{cc}X)^{\alpha}\partial_{\alpha}(^{cc}Y)^{\overline{\overline{\beta}}}+(^{cc}X)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(^{cc}Y)^{\overline{\overline{\beta}}}\\&\quad-(^{cc}Y)^{\overline{\alpha}}\partial_{\overline{\alpha}}(^{cc}X)^{\overline{\overline{\beta}}}-(^{cc}Y)^{\alpha}\partial_{\alpha}(^{cc}X)^{\overline{\overline{\beta}}}-(^{cc}Y)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(^{cc}X)^{\overline{\overline{\beta}}}\\&=X^\alpha\partial_\alpha(y^\varepsilon\partial_\varepsilon Y^\beta)+y^\varepsilon\partial_\varepsilon X^\alpha\partial_{\overline{\overline{\alpha}}}y^\sigma\partial_\sigma Y^\beta\\&\quad-Y^\alpha\partial_\alpha(y^\varepsilon\partial_\varepsilon X^\beta)-y^\varepsilon\partial_\varepsilon Y^\alpha\partial_{\overline{\overline{\alpha}}}y^\sigma\partial_\sigma X^\beta\\&=y^\varepsilon X^\alpha\partial_\alpha\partial_\varepsilon Y^\beta+y^\varepsilon\left(\partial_\varepsilon X^\sigma\right)\left(\partial_\sigma Y^\beta\right)-y^\varepsilon Y^\alpha\partial_\alpha\partial_\varepsilon X^\beta-y^\varepsilon\left(\partial_\varepsilon Y^\sigma\right)\left(\partial_\sigma X^\beta\right)\\&=y^\varepsilon\partial_\varepsilon[X,Y]^\beta\end{aligned}$$

by virtue of (4.1). On the other hand, we know that $^{cc}[X,Y]$ have components

$$^{cc}[X,Y]=\begin{pmatrix}-p_\varepsilon(\partial_\beta[X,Y]^\varepsilon)\\ [X,Y]^\beta\\ y^\varepsilon\partial_\varepsilon[X,Y]^\beta\end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\beta}},x^{\beta},x^{\overline{\overline{\beta}}})$ on $t(M_n)$. Thus, we have (i) of Theorem 4.1.

(ii) If $X, Y \in \Im_0^1(T^*(M_n))$ and $\begin{pmatrix} [^{cc}X,^{vv}Y]^{\overline{\beta}} \\ [^{cc}X,^{vv}Y]^{\beta} \\ [^{cc}X,^{vv}Y]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[^{cc}X,^{vv}Y]$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t(M_n)$, then we have

$$[^{cc}X,^{vv}Y]^J = (^{cc}X)^I \partial_I (^{vv}Y)^J - (^{vv}Y)^I \partial_I (^{cc}X)^J.$$

As the first coordinate, if $J = \overline{\beta}$, we obtain

$$\begin{aligned}
[^{cc}X,^{vv}Y]^{\overline{\beta}} &= (^{cc}X)^I \partial_I (^{vv}Y)^{\overline{\beta}} - (^{vv}Y)^I \partial_I (^{cc}X)^{\overline{\beta}} \\
&= -(^{vv}Y)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}X)^{\overline{\beta}} - (^{vv}Y)^{\alpha} \partial_{\alpha} (^{cc}X)^{\overline{\beta}} - (^{vv}Y)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}X)^{\overline{\beta}} \\
&= (^{vv}Y)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} p_{\varepsilon} (\partial_{\alpha} X^{\varepsilon}) \\
&= 0
\end{aligned}$$

by virtue of (2.1) and (4.1). As the second coordinate, if $J = \beta$, we obtain

$$\begin{aligned}
[^{cc}X,^{vv}Y]^{\beta} &= (^{cc}X)^I \partial_I (^{vv}Y)^{\beta} - (^{vv}Y)^I \partial_I (^{cc}X)^{\beta} \\
&= -(^{vv}Y)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}X)^{\beta} - (^{vv}Y)^{\alpha} \partial_{\alpha} (^{cc}X)^{\beta} - (^{vv}Y)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}X)^{\beta} \\
&= -(^{vv}Y)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} X^{\beta} \\
&= 0
\end{aligned}$$

by virtue of (2.1) and (4.1). As the third coordinate, if $J = \overline{\overline{\beta}}$, then we obtain

$$\begin{aligned}
[^{cc}X,^{vv}Y]^{\overline{\overline{\beta}}} &= (^{cc}X)^I \partial_I (^{vv}Y)^{\overline{\overline{\beta}}} - (^{vv}Y)^I \partial_I (^{cc}X)^{\overline{\overline{\beta}}} \\
&= (^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{vv}Y)^{\overline{\overline{\beta}}} + (^{cc}X)^{\alpha} \partial_{\alpha} (^{vv}Y)^{\overline{\overline{\beta}}} + (^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{vv}Y)^{\overline{\overline{\beta}}} \\
&\quad -(^{vv}Y)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}X)^{\overline{\overline{\beta}}} - (^{vv}Y)^{\alpha} \partial_{\alpha} (^{cc}X)^{\overline{\overline{\beta}}} - (^{vv}Y)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}X)^{\overline{\overline{\beta}}} \\
&= (^{cc}X)^{\alpha} \partial_{\alpha} (^{vv}Y)^{\overline{\overline{\beta}}} - (^{vv}Y)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}X)^{\overline{\overline{\beta}}} \\
&= X^{\alpha} \partial_{\alpha} Y^{\beta} + Y^{\alpha} \partial_{\overline{\overline{\alpha}}} y^{\varepsilon} \partial_{\varepsilon} X^{\beta} \\
&= X^{\alpha} \partial_{\alpha} Y^{\beta} + Y^{\alpha} \partial_{\alpha} X^{\beta} \\
&= [X, Y]^{\beta}
\end{aligned}$$

by virtue of (2.1) and (4.1). On the other hand, we know that the vertical lift $^{vv}[X, Y]$ of $[X, Y]$ has components of the form

$$^{vv}[X, Y] = \begin{pmatrix} 0 \\ 0 \\ [X, Y]^{\beta} \end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t(M_n)$. Thus, we have (ii) of Theorem 4.1.

(iii) If $X, Y \in \Im_0^1(T^*(M_n))$ and $\begin{pmatrix} [^{vv}X,^{vv}Y]^{\overline{\beta}} \\ [^{vv}X,^{vv}Y]^{\beta} \\ [^{vv}X,^{vv}Y]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[^{vv}X,^{vv}Y]$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t(M_n)$, then we have

$$[^{vv}X,^{vv}Y]^J = (^{vv}X)^I \partial_I (^{vv}Y)^J - (^{vv}Y)^I \partial_I (^{vv}X)^J.$$

As the first coordinate, if $J = \overline{\beta}$, we obtain

$$\begin{aligned}[][^{vv}X,^{vv}Y]^{\overline{\beta}} &= (^{vv}X)^I \partial_I (^{vv}Y)^{\overline{\beta}} - (^{vv}Y)^I \partial_I (^{vv}X)^{\overline{\beta}} \\ &= 0\end{aligned}$$

by virtue of (2.1). As the second coordinate, if $J = \beta$, we obtain

$$\begin{aligned}[][^{vv}X,^{vv}Y]^{\beta} &= (^{vv}X)^I \partial_I (^{vv}Y)^{\beta} - (^{vv}Y)^I \partial_I (^{vv}X)^{\beta} \\ &= 0\end{aligned}$$

by virtue of (2.1). As the third coordinate, if $J = \overline{\overline{\beta}}$, then we obtain

$$\begin{aligned}[][^{vv}X,^{vv}Y]^{\overline{\overline{\beta}}} &= (^{vv}X)^I \partial_I (^{vv}Y)^{\overline{\overline{\beta}}} - (^{vv}Y)^I \partial_I (^{vv}X)^{\overline{\overline{\beta}}} \\ &= (^{vv}X)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{vv}Y)^{\overline{\overline{\beta}}} + (^{vv}X)^{\alpha} \partial_{\alpha} (^{vv}Y)^{\overline{\overline{\beta}}} + (^{vv}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{vv}Y)^{\overline{\overline{\beta}}} \\ &\quad - (^{vv}Y)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{vv}X)^{\overline{\overline{\beta}}} - (^{vv}Y)^{\alpha} \partial_{\alpha} (^{vv}X)^{\overline{\overline{\beta}}} - (^{vv}Y)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{vv}X)^{\overline{\overline{\beta}}} \\ &= 0\end{aligned}$$

by virtue of (2.1). Thus, we have (iii) of Theorem 4.1.

(iv) If $F \in \Im_1^1(T^*(M_n))$, $X \in \Im_0^1(T^*(M_n))$ and $\begin{pmatrix} [^{cc}X, \gamma F]^{\overline{\beta}} \\ [^{cc}X, \gamma F]^{\beta} \\ [^{cc}X, \gamma F]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[^{cc}X, \gamma F]$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t(M_n)$, then we have

$$[^{cc}X, \gamma F]^J = (^{cc}X)^I \partial_I (\gamma F)^J - (\gamma F)^I \partial_I (^{cc}X)^J.$$

As the first coordinate, if $J = \overline{\beta}$, we obtain

$$\begin{aligned}[][^{cc}X, \gamma F]^{\overline{\beta}} &= (^{cc}X)^I \partial_I (\gamma F)^{\overline{\beta}} - (\gamma F)^I \partial_I (^{cc}X)^{\overline{\beta}} \\ &= (^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\gamma F)^{\overline{\beta}} + (^{cc}X)^{\alpha} \partial_{\alpha} (\gamma F)^{\overline{\beta}} + (^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\gamma F)^{\overline{\beta}} \\ &\quad - (\gamma F)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}X)^{\overline{\beta}} - (\gamma F)^{\alpha} \partial_{\alpha} (^{cc}X)^{\overline{\beta}} - (\gamma F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}X)^{\overline{\beta}} \\ &= (^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\gamma F)^{\overline{\beta}} + (^{cc}X)^{\alpha} \partial_{\alpha} (\gamma F)^{\overline{\beta}} - (\gamma F)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}X)^{\overline{\beta}} \\ &= p_\sigma (\partial_\alpha X^\sigma) \partial_{\overline{\alpha}} p_\sigma F_\beta^\sigma - X^\alpha \partial_\alpha p_\sigma F_\beta^\sigma - p_\sigma F_\alpha^\sigma \partial_{\overline{\alpha}} p_\sigma (\partial_\beta X^\sigma) \\ &= p_\sigma (\partial_\alpha X^\sigma) F_\beta^\alpha - X^\alpha \partial_\alpha p_\sigma F_\beta^\sigma - p_\sigma F_\alpha^\sigma (\partial_\beta X^\alpha) \\ &= -p_\sigma (X^\alpha \partial_\alpha F_\beta^\sigma - \partial_\alpha X^\sigma F_\beta^\alpha + \partial_\beta X^\alpha F_\alpha^\sigma) \\ &= -p_\sigma (L_X F)_\beta^\sigma\end{aligned}$$

by virtue of (3.1) and (4.1). As the second coordinate, if $J = \beta$, we obtain

$$\begin{aligned}[][^{cc}X, \gamma F]^{\beta} &= (^{cc}X)^I \partial_I (\gamma F)^{\beta} - (\gamma F)^I \partial_I (^{cc}X)^{\beta} \\ &= (^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\gamma F)^{\beta} + (^{cc}X)^{\alpha} \partial_{\alpha} (\gamma F)^{\beta} + (^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\gamma F)^{\beta} \\ &\quad - (\gamma F)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}X)^{\beta} - (\gamma F)^{\alpha} \partial_{\alpha} (^{cc}X)^{\beta} - (\gamma F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}X)^{\beta} \\ &= 0\end{aligned}$$

by virtue of (3.1) and (4.1). As the third coordinate, if $J = \overline{\overline{\beta}}$, then we obtain

$$\begin{aligned}
[{}^{cc}X, \gamma F]^{\overline{\overline{\beta}}} &= ({}^{cc}X)^I \partial_I (\gamma F)^{\overline{\overline{\beta}}} - (\gamma F)^I \partial_I ({}^{cc}X)^{\overline{\overline{\beta}}} \\
&= ({}^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\gamma F)^{\overline{\overline{\beta}}} + ({}^{cc}X)^{\alpha} \partial_{\alpha} (\gamma F)^{\overline{\overline{\beta}}} + ({}^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\gamma F)^{\overline{\overline{\beta}}} \\
&\quad - (\gamma F)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}X)^{\overline{\overline{\beta}}} - (\gamma F)^{\alpha} \partial_{\alpha} ({}^{cc}X)^{\overline{\overline{\beta}}} - (\gamma F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} ({}^{cc}X)^{\overline{\overline{\beta}}} \\
&= X^{\alpha} \partial_{\alpha} y^{\varepsilon} F_{\varepsilon}^{\beta} + y^{\varepsilon} \partial_{\varepsilon} X^{\alpha} \partial_{\overline{\overline{\alpha}}} y^{\varepsilon} F_{\varepsilon}^{\beta} - y^{\varepsilon} F_{\varepsilon}^{\alpha} \partial_{\overline{\overline{\alpha}}} y^{\varepsilon} \partial_{\varepsilon} X^{\beta} \\
&= y^{\varepsilon} X^{\alpha} \partial_{\alpha} F_{\varepsilon}^{\beta} + y^{\varepsilon} \partial_{\varepsilon} X^{\alpha} F_{\alpha}^{\beta} - y^{\varepsilon} F_{\varepsilon}^{\alpha} \partial_{\alpha} X^{\beta} \\
&= y^{\varepsilon} \left(\partial_{\varepsilon} X^{\alpha} F_{\alpha}^{\beta} + X^{\alpha} \partial_{\alpha} F_{\varepsilon}^{\beta} - F_{\varepsilon}^{\alpha} \partial_{\alpha} X^{\beta} \right) \\
&= y^{\varepsilon} \left(L_X F \right)_{\varepsilon}^{\beta}
\end{aligned}$$

by virtue of (3.1) and (4.1). We know that $\gamma(L_X F)$ have components

$$\gamma(L_X F) = \begin{pmatrix} -p_{\sigma} (L_X F)_{\beta}^{\sigma} \\ 0 \\ y^{\varepsilon} \left(L_X F \right)_{\varepsilon}^{\beta} \end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t(M_n)$. Thus, we have (iv) of Theorem 4.1.

5 Complete Lift of Tensor Fields of Type (1,1)

Suppose now that $F \in \Im_1^1(T^*(M_n))$ and F has local components F_{β}^{α} in a neighborhood U of M_n, $F = F_{\beta}^{\alpha} \partial_{\alpha} \otimes dx^{\beta}$. If we take account of (1.3), we can prove that ${}^{cc}F_{J'}^{I'} = A_I^{I'} A_{J'}^{J} {}^{cc}F_J^I$, where ${}^{cc}F$ is an affinor field defined by

$${}^{cc}F = \left({}^{cc}F_J^I \right) = \begin{pmatrix} F_{\alpha}^{\beta} & p_{\sigma} (\partial_{\beta} F_{\alpha}^{\sigma} - \partial_{\alpha} F_{\beta}^{\sigma}) & 0 \\ 0 & F_{\beta}^{\alpha} & 0 \\ 0 & y^{\varepsilon} \partial_{\varepsilon} F_{\beta}^{\alpha} & F_{\beta}^{\alpha} \end{pmatrix}, \tag{5.1}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}})$ on $t(M_n)$. We call ${}^{cc}F$ the complete lift of the tensor field F of type (1,1) to $t(M_n)$.

Proof. For simplicity we take only ${}^{cc}F_{\overline{\beta}'}^{\overline{\alpha}'}$. In fact,

$$\begin{aligned}
{}^{cc}F_{\overline{\beta}'}^{\overline{\alpha}'} &= A_{\overline{\alpha}}^{\overline{\alpha}'} A_{\overline{\beta}'}^{\overline{\beta}} \, {}^{cc}F_{\overline{\beta}}^{\overline{\alpha}} + A_{\overline{\alpha}}^{\overline{\alpha}'} A_{\overline{\beta}'}^{\beta} \, {}^{cc}F_{\beta}^{\overline{\alpha}} + A_{\overline{\alpha}}^{\overline{\alpha}'} A_{\overline{\beta}'}^{\overline{\overline{\beta}}} \, {}^{cc}F_{\overline{\overline{\beta}}}^{\overline{\alpha}} \\
&\quad + A_{\alpha}^{\overline{\alpha}'} A_{\overline{\beta}'}^{\overline{\beta}} \, {}^{cc}F_{\overline{\beta}}^{\alpha} + A_{\alpha}^{\overline{\alpha}'} A_{\overline{\beta}'}^{\beta} \, {}^{cc}F_{\beta}^{\alpha} + A_{\alpha}^{\overline{\alpha}'} A_{\overline{\beta}'}^{\overline{\overline{\beta}}} \, {}^{cc}F_{\overline{\overline{\beta}}}^{\alpha} \\
&\quad + A_{\overline{\overline{\alpha}}}^{\overline{\alpha}'} A_{\overline{\beta}'}^{\overline{\beta}} \, {}^{cc}F_{\overline{\beta}}^{\overline{\overline{\alpha}}} + A_{\overline{\overline{\alpha}}}^{\overline{\alpha}'} A_{\overline{\beta}'}^{\beta} \, {}^{cc}F_{\beta}^{\overline{\overline{\alpha}}} + A_{\overline{\overline{\alpha}}}^{\overline{\alpha}'} A_{\overline{\beta}'}^{\overline{\overline{\beta}}} \, {}^{cc}F_{\overline{\overline{\beta}}}^{\overline{\overline{\alpha}}} \\
&= A_{\overline{\alpha}}^{\overline{\alpha}'} A_{\overline{\beta}'}^{\overline{\beta}} \, {}^{cc}F_{\overline{\beta}}^{\overline{\alpha}} \\
&= A_{\alpha'}^{\alpha} A_{\beta}^{\beta'} F_{\alpha}^{\beta} \\
&= F_{\alpha'}^{\beta'} .
\end{aligned}$$

Thus we have ${}^{cc}F_{\overline{\beta}'}^{\overline{\alpha}'} = F_{\alpha'}^{\beta'}$. Similarly, we can easily find another components of ${}^{cc}F_{J'}^{I'}$.

Theorem 5.1 *If F and G are affinor fields on $T^*(M_n)$, and $X \in \Im_0^1(T^*(M_n))$, then*

(i) ${}^{cc}F({}^{cc}X) = {}^{cc}(FX) - \gamma(L_X F) + {}^{vv}(\gamma(L_X F))$,
(ii) ${}^{cc}F({}^{vv}X) = {}^{vv}(F \circ X)$,
(iii) ${}^{cc}F(\gamma G) = \gamma(F \circ G)$.

Proof. (i) If $X \in \Im_0^1(T^*(M_n))$ and $F \in \Im_1^1(T^*(M_n))$, from (2.1), (4.1) and (5.1), we have

$$\begin{aligned}
{}^{cc}F{}^{cc}X &= \begin{pmatrix} F_\alpha^\beta & p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & F_\beta^\alpha \end{pmatrix} \begin{pmatrix} -p_\varepsilon(\partial_\beta X^\varepsilon) \\ X^\beta \\ y^\varepsilon \partial_\varepsilon X^\beta \end{pmatrix} \\
&= \begin{pmatrix} -p_\varepsilon(\partial_\beta X^\varepsilon) F_\alpha^\beta + p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) X^\beta \\ F_\beta^\alpha X^\beta \\ y^\varepsilon \partial_\varepsilon F_\beta^\alpha X^\beta + F_\beta^\alpha y^\varepsilon \partial_\varepsilon X^\beta \end{pmatrix} \\
&= \begin{pmatrix} -p_\varepsilon(\partial_\beta X^\varepsilon) F_\alpha^\beta + p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) X^\beta \\ (FX)^\alpha \\ F_\beta^\alpha y^\varepsilon \partial_\varepsilon X^\beta + y^\varepsilon \partial_\varepsilon F_\beta^\alpha X^\beta \end{pmatrix} \\
&= \begin{pmatrix} -p_\sigma \partial_\alpha (FX)^\sigma \\ (FX)^\alpha \\ y^\varepsilon \partial_\varepsilon (FX)^\alpha \end{pmatrix} + \begin{pmatrix} p_\sigma(X^\beta \partial_\beta F_\alpha^\sigma - (\partial_\alpha X^\beta) F_\beta^\sigma - (\partial_\beta X^\sigma) F_\alpha^\beta) \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -p_\sigma \partial_\alpha (FX)^\sigma \\ (FX)^\alpha \\ y^\varepsilon \partial_\varepsilon (FX)^\alpha \end{pmatrix} - \begin{pmatrix} -p_\sigma (L_X F)_\alpha^\sigma \\ 0 \\ y^\varepsilon \, (L_X F)_\varepsilon^\alpha \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon \, (L_X F)_\varepsilon^\alpha \end{pmatrix} \\
&= {}^{cc}(FX) - \gamma(L_X F) + {}^{vv}(\gamma(L_X F)),
\end{aligned}$$

which prove (i) of Theorem 5.1.

(ii) If $X \in \Im_0^1(T^*(M_n))$ and $F \in \Im_1^1(T^*(M_n))$, from (2.1) and (5.1), we have

$$\begin{aligned}
{}^{cc}F\,{}^{vv}X &= \begin{pmatrix} F_\alpha^\beta & p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & F_\beta^\alpha \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ X^\beta \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ F_\beta^\alpha X^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (F \circ X)^\alpha \end{pmatrix} = {}^{vv}(F \circ X),
\end{aligned}$$

which gives equation (ii) of Theorem 5.1.

(iii) If $F, G \in \Im_1^1(T^*(M_n))$, then, by (3.1) and (5.1), we find

$$\begin{aligned}
{}^{cc}F(\gamma G) &= \begin{pmatrix} F_\alpha^\beta & p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & F_\beta^\alpha \end{pmatrix} \begin{pmatrix} -p_\sigma G_\beta^\sigma \\ 0 \\ y^\varepsilon G_\varepsilon^\beta \end{pmatrix} \\
&= \begin{pmatrix} -p_\sigma F_\alpha^\beta G_\beta^\sigma \\ 0 \\ y^\varepsilon F_\beta^\alpha G_\varepsilon^\beta \end{pmatrix} = \begin{pmatrix} -p_\sigma \, (F \circ G)_\alpha^\sigma \\ 0 \\ y^\varepsilon \, (F \circ G)_\varepsilon^\alpha \end{pmatrix} = \gamma(F \circ G).
\end{aligned}$$

6 Horizontal Lifts of Vector Fields

Let $X \in \Im_0^1(T^*(M_n))$, i.e. $X = X^\alpha \partial_\alpha$. Then we define the horizontal lift ${}^{HH}X$ of X by

$$ {}^{HH}X = {}^{cc}X - \gamma(\nabla X) $$

on $t(M_n)$. Where ∇ is a symmetric affine connection in a differentiable manifold M_n. Then, remembering that ${}^{cc}X$ and $\gamma(\nabla X)$ have, respectively, local componenets

$$ {}^{cc}X = \left({}^{cc}X^A\right) = \begin{pmatrix} -p_\varepsilon(\partial_\alpha X^\varepsilon) \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix}, \quad \gamma(\nabla X) = \left(\gamma(\nabla X)^A\right) = \begin{pmatrix} -p_\varepsilon(\nabla_\alpha X^\varepsilon) \\ 0 \\ y^\varepsilon \nabla_\varepsilon X^\alpha \end{pmatrix} $$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ on $t(M_n)$. $\nabla_\alpha X^\varepsilon$ being the covariant derivative of X^ε, i.e.,

$$ (\nabla_\alpha X^\varepsilon) = \partial_\alpha X^\varepsilon + X^\beta \Gamma_{\beta\alpha}^\varepsilon. $$

We find that the horizontal lift ${}^{HH}X$ of X has the components

$$ {}^{HH}X = \left({}^{HH}X^A\right) = \begin{pmatrix} X^\beta \Gamma_{\beta\alpha} \\ X^\alpha \\ -\Gamma_\beta^\alpha X^\beta \end{pmatrix} \tag{6.1} $$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ on $t(M_n)$. Where

$$ \Gamma_\beta^\alpha = y^\varepsilon \Gamma_{\varepsilon\ \beta}^{\ \alpha}, \quad \Gamma_{\beta\alpha} = p_\varepsilon \Gamma_{\beta\alpha}^\varepsilon. \tag{6.2} $$

Theorem 6.1 *If $X, Y \in \Im_0^1(T^*(M_n))$ then*

(i) $[{}^{HH}X, {}^{HH}Y] = {}^{HH}[X, Y] - \gamma R(X, Y)$,
(ii) $[{}^{HH}X, {}^{vv}Y] = {}^{vv}(\nabla_X Y)$,

where R is the curvature tensor of the affine connection ∇ is given by $(L_X\nabla)_Y = \nabla_Y \nabla X + R(X, Y)$.

Proof. (i) If X and Y are vector fields on $T^*(M_n)$, and $\begin{pmatrix} [{}^{HH}X, {}^{HH}Y]^{\overline{\beta}} \\ [{}^{HH}X, {}^{HH}Y]^\beta \\ [{}^{HH}X, {}^{HH}Y]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[{}^{HH}X, {}^{HH}Y]$ with respect to the coordinates $(x^{\overline{\beta}}, x^\beta, x^{\overline{\overline{\beta}}})$ on $t(M_n)$, then by (6.1), we have

$$ \begin{aligned} [{}^{HH}X, {}^{HH}Y]^J &= {}^{HH}X^I \partial_I ({}^{HH}Y)^J - {}^{HH}Y^I \partial_I ({}^{HH}X)^J \\ &= {}^{HH}X^{\overline{\alpha}} \partial_{\overline{\alpha}} {}^{HH}Y^J + {}^{HH}X^\alpha \partial_\alpha {}^{HH}Y^J + {}^{HH}X^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} {}^{HH}Y^J \\ &\quad - {}^{HH}Y^{\overline{\alpha}} \partial_{\overline{\alpha}} {}^{HH}X^J - {}^{HH}Y^\alpha \partial_\alpha {}^{HH}X^J - {}^{HH}Y^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} {}^{HH}X^J \\ &= p_\varepsilon X^\beta \Gamma_{\beta\alpha}^\varepsilon \partial_{\overline{\alpha}} {}^{HH}Y^J + X^\alpha \partial_\alpha {}^{HH}Y^J - y^\varepsilon \Gamma_{\varepsilon\ \beta}^{\ \alpha} X^\beta \partial_{\overline{\overline{\alpha}}} {}^{HH}Y^J \\ &\quad - p_\varepsilon Y^\beta \Gamma_{\beta\alpha}^\varepsilon \partial_{\overline{\alpha}} {}^{HH}X^J - Y^\alpha \partial_\alpha {}^{HH}X^J + y^\varepsilon \Gamma_{\varepsilon\ \beta}^{\ \alpha} Y^\beta \partial_{\overline{\overline{\alpha}}} {}^{HH}X^J \end{aligned} $$

As the first coordinate, if $J=\overline{\beta}$, we obtain

$$\begin{aligned}
[{}^{HH}X,\ {}^{HH}Y]^{\overline{\beta}} &= p_\varepsilon X^\beta \Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}}\, {}^{HH}Y^{\overline{\overline{\beta}}} + X^\alpha\partial_\alpha\, {}^{HH}Y^{\overline{\beta}} - y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} X^\beta \partial_{\overline{\overline{\alpha}}}\, {}^{HH}Y^{\overline{\overline{\beta}}} \\
&\quad - p_\varepsilon Y^\beta \Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}}\, {}^{HH}X^{\overline{\overline{\beta}}} - Y^\alpha\partial_\alpha\, {}^{HH}X^{\overline{\beta}} + y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} Y^\beta \partial_{\overline{\overline{\alpha}}}\, {}^{HH}X^{\overline{\overline{\beta}}} \\
&= p_\varepsilon X^\beta \Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}} p_\varepsilon Y^\alpha \Gamma^\varepsilon_{\alpha\beta} + X^\alpha\partial_\alpha\left(p_\varepsilon Y^\alpha \Gamma^\varepsilon_{\alpha\beta}\right) - y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} X^\beta \partial_{\overline{\overline{\alpha}}} p_\varepsilon Y^\alpha \Gamma^\varepsilon_{\alpha\beta} \\
&\quad - p_\varepsilon Y^\beta \Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}} p_\varepsilon X^\alpha \Gamma^\varepsilon_{\alpha\beta} - Y^\alpha\partial_\alpha\left(p_\varepsilon X^\alpha \Gamma^\varepsilon_{\alpha\beta}\right) + y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} Y^\beta \partial_{\overline{\overline{\alpha}}} p_\varepsilon X^\alpha \Gamma^\varepsilon_{\alpha\beta} \\
&= p_\varepsilon X^\beta \Gamma^\varepsilon_{\beta\alpha} Y^\theta \Gamma^\alpha_{\theta\ \beta} + X^\alpha\partial_\alpha(p_\varepsilon Y^\alpha \Gamma^\varepsilon_{\alpha\beta}) - p_\varepsilon Y^\beta \Gamma^\varepsilon_{\beta\alpha} X^\theta \Gamma^\alpha_{\theta\ \beta} - Y^\alpha\partial_\alpha(p_\varepsilon X^\alpha \Gamma^\varepsilon_{\alpha\beta}) \\
&= p_\varepsilon X^\alpha Y^\theta \Gamma^\varepsilon_{\alpha\sigma}\Gamma^\sigma_{\theta\ \beta} + p_\varepsilon X^\alpha Y^\theta \partial_\alpha \Gamma^\varepsilon_{\theta\ \beta} + p_\varepsilon X^\alpha(\partial_\alpha Y^\alpha)\Gamma^\varepsilon_{\alpha\beta} \\
&\quad - p_\varepsilon X^\alpha Y^\theta \Gamma^\varepsilon_{\theta\ \sigma}\Gamma^\sigma_{\alpha\ \beta} - p_\varepsilon Y^\alpha X^\theta \partial_\alpha \Gamma^\varepsilon_{\theta\ \beta} - p_\varepsilon Y^\alpha(\partial_\alpha X^\alpha)\Gamma^\varepsilon_{\alpha\beta} \\
&= [p_\varepsilon(X^\alpha(\partial_\alpha Y^\alpha) - Y^\alpha(\partial_\alpha X^\alpha))\Gamma^\varepsilon_{\alpha\beta}] \\
&\quad + p_\varepsilon[X^\alpha Y^\theta(\partial_\alpha \Gamma^\varepsilon_{\theta\ \beta} - \partial_\theta \Gamma^\varepsilon_{\alpha\beta} + \Gamma^\varepsilon_{\alpha\sigma}\Gamma^\sigma_{\theta\ \beta} - \Gamma^\varepsilon_{\theta\ \sigma}\Gamma^\sigma_{\alpha\ \beta})] \\
&= p_\varepsilon[X,Y]^\alpha \Gamma^\varepsilon_{\alpha\beta} + p_\varepsilon(R(X,Y))^\varepsilon_\beta
\end{aligned}$$

by virtue of (6.1). As the second coordinate, if $J=\beta$, we obtain

$$\begin{aligned}
[{}^{HH}X,\ {}^{HH}Y]^\beta &= p_\varepsilon X^\beta \Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}}\, {}^{HH}Y^\beta + X^\alpha\partial_\alpha\, {}^{HH}Y^\beta - y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} X^\beta \partial_{\overline{\overline{\alpha}}}\, {}^{HH}Y^\beta \\
&\quad - p_\varepsilon Y^\beta \Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}}\, {}^{HH}X^\beta - Y^\alpha\partial_\alpha\, {}^{HH}X^\beta + y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} Y^\beta \partial_{\overline{\overline{\alpha}}}\, {}^{HH}X^\beta \\
&= p_\varepsilon X^\beta \Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}} Y^\beta + X^\alpha\partial_\alpha Y^\beta - y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} X^\beta \partial_{\overline{\overline{\alpha}}} Y^\beta \\
&\quad - p_\varepsilon Y^\beta \Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}} X^\beta - Y^\alpha\partial_\alpha X^\beta + y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} Y^\beta \partial_{\overline{\overline{\alpha}}} X^\beta \\
&= X^\alpha\partial_\alpha Y^\beta - Y^\alpha\partial_\alpha X^\beta \\
&= [X,Y]^\beta
\end{aligned}$$

by virtue of (6.1). As the third coordinate, if $J=\overline{\overline{\beta}}$, then we obtain

$$\begin{aligned}
[{}^{HH}X,\ {}^{HH}Y]^{\overline{\overline{\beta}}} &= y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} X^\beta \partial_{\overline{\alpha}} y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} Y^\alpha - X^\alpha\partial_\alpha\left(y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} Y^\alpha\right) - p_\varepsilon X^\beta \Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\overline{\alpha}}} y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} Y^\alpha \\
&\quad - y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} Y^\beta \partial_{\overline{\alpha}} y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} X^\alpha + Y^\alpha\partial_\alpha\left(y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} X^\alpha\right) + p_\varepsilon Y^\beta \Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\overline{\alpha}}} y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} X^\alpha \\
&= y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} X^\beta \partial_{\overline{\alpha}} y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} Y^\alpha - X^\alpha\partial_\alpha\left(y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} Y^\alpha\right) \\
&\quad - y^\varepsilon \Gamma^\alpha_{\varepsilon\ \beta} Y^\beta \partial_{\overline{\alpha}} y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} X^\alpha + Y^\alpha\partial_\alpha\left(y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} X^\alpha\right) \\
&= -X^\alpha\partial_\alpha\left(y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} Y^\alpha\right) + y^\varepsilon X^\beta \Gamma^\alpha_{\varepsilon\ \beta} Y^\theta \Gamma^\beta_{\alpha\theta} + Y^\alpha\partial_\alpha\left(y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} X^\alpha\right) - y^\varepsilon Y^\beta \Gamma^\alpha_{\varepsilon\ \beta} X^\theta \Gamma^\beta_{\alpha\theta} \\
&= -y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} X^\alpha\left(\partial_\alpha Y^\alpha\right) - y^\varepsilon X^\alpha Y^\theta \partial_\alpha \Gamma^\beta_{\theta\ \varepsilon} - y^\varepsilon X^\alpha Y^\theta \Gamma^\beta_{\alpha\ \gamma}\Gamma^\gamma_{\theta\ \varepsilon} \\
&\quad + y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha} Y^\alpha\left(\partial_\alpha X^\alpha\right) + y^\varepsilon X^\alpha Y^\theta \partial_\theta \Gamma^\beta_{\alpha\ \varepsilon} - y^\varepsilon X^\alpha Y^\theta \Gamma^\beta_{\theta\ \gamma}\Gamma^\gamma_{\alpha\ \varepsilon} \\
&= -\left[y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha}\left(X^\alpha\left(\partial_\alpha Y^\alpha\right) - Y^\alpha\left(\partial_\alpha X^\alpha\right)\right)\right] \\
&\quad - y^\varepsilon\left[X^\alpha Y^\theta\left(\partial_\alpha \Gamma^\beta_{\theta\ \varepsilon} - \partial_\theta \Gamma^\beta_{\alpha\ \varepsilon} - \Gamma^\beta_{\alpha\ \gamma}\Gamma^\gamma_{\theta\ \varepsilon} + \Gamma^\beta_{\theta\ \gamma}\Gamma^\gamma_{\alpha\ \varepsilon}\right)\right] \\
&= -\left[y^\varepsilon \Gamma^\beta_{\varepsilon\ \alpha}[X,Y]^\alpha\right] - y^\varepsilon(R(X,Y))^\beta_\varepsilon
\end{aligned}$$

by virtue of (6.1). We know that ${}^{HH}[X,Y]-\gamma R(X,Y)$ have components

$$
{}^{HH}[X,Y]-\gamma R(X,Y)=\begin{pmatrix} p_\varepsilon[X,Y]^\alpha\Gamma^\varepsilon_{\alpha\beta} \\ [X,Y]^\beta \\ -y^\varepsilon\Gamma^\beta_{\varepsilon\ \alpha}[X,Y]^\alpha \end{pmatrix}-\begin{pmatrix} -p_\varepsilon(R(X,Y))^\varepsilon_\beta \\ 0 \\ y^\varepsilon(R(X,Y))^\beta_\varepsilon \end{pmatrix}
=\begin{pmatrix} p_\varepsilon[X,Y]^\alpha\Gamma^\varepsilon_{\alpha\beta}+p_\varepsilon(R(X,Y))^\varepsilon_\beta \\ [X,Y]^\beta \\ -y^\varepsilon\Gamma^\beta_{\varepsilon\ \alpha}[X,Y]^\alpha-y^\varepsilon(R(X,Y))^\beta_\varepsilon \end{pmatrix}
$$

with respect to the coordinates $(x^{\overline{\beta}},x^\beta,x^{\overline{\overline{\beta}}})$ on $t(M_n)$. Thus, we have $[\,{}^{HH}X,\ {}^{HH}Y]={}^{HH}[X,Y]-\gamma R(X,Y)$.

(ii) If $X,Y\in\Im^1_0(T^*(M_n))$, and $\begin{pmatrix}[\,{}^{HH}X,{}^{vv}Y]^{\overline{\beta}} \\ [\,{}^{HH}X,{}^{vv}Y]^\beta \\ [\,{}^{HH}X,{}^{vv}Y]^{\overline{\overline{\beta}}}\end{pmatrix}$ are components of $[\,{}^{HH}X,{}^{vv}Y]$ with respect to the coordinates $(x^{\overline{\beta}},x^\beta,x^{\overline{\overline{\beta}}})$ on $t(M_n)$, then by (2.1) and (6.1), we have

$$
\begin{aligned}
[\,{}^{HH}X,{}^{vv}Y]^J &= {}^{HH}X^I\partial_I({}^{vv}Y^J)-{}^{vv}Y^I\partial_I\,{}^{HH}X^J \\
&= {}^{HH}X^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{vv}Y^J)+{}^{HH}X^\alpha\partial_\alpha({}^{vv}Y^J)+{}^{HH}X^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{vv}Y^J) \\
&\quad -{}^{vv}Y^{\overline{\alpha}}\partial_{\overline{\alpha}}\,{}^{HH}X^J-{}^{vv}Y^\alpha\partial_\alpha\,{}^{HH}X^J-{}^{vv}Y^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}\,{}^{HH}X^J \\
&= p_\varepsilon X^\beta\Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}}({}^{vv}Y^J)+X^\alpha\partial_\alpha({}^{vv}Y^J)-y^\varepsilon\Gamma^\alpha_{\varepsilon\ \beta}X^\beta\partial_{\overline{\overline{\alpha}}}({}^{vv}Y^J)-Y^\alpha\partial_{\overline{\overline{\alpha}}}\,{}^{HH}X^J.
\end{aligned}
$$

As the first coordinate, if $J=\overline{\beta}$, we obtain

$$
\begin{aligned}
[\,{}^{HH}X,{}^{vv}Y]^{\overline{\beta}} &= p_\varepsilon X^\beta\Gamma^\varepsilon_{\beta\alpha}\partial^{vv}_{\overline{\alpha}}Y^{\overline{\beta}}+X^\alpha\partial^{vv}_\alpha Y^{\overline{\beta}}-y^\varepsilon\Gamma^\alpha_{\varepsilon\ \beta}X^\beta\partial^{vv}_{\overline{\overline{\alpha}}}Y^{\overline{\beta}}-Y^\alpha\partial_{\overline{\overline{\alpha}}}\,{}^{HH}X^{\overline{\beta}} \\
&= -Y^\alpha\partial_{\overline{\overline{\alpha}}}p_\varepsilon X^\beta\Gamma^\varepsilon_{\beta\alpha} \\
&= 0
\end{aligned}
$$

by virtue of (2.1) and (6.1). As the second coordinate, if $J=\beta$ we obtain

$$
\begin{aligned}
[\,{}^{HH}X,{}^{vv}Y]^\beta &= p_\varepsilon X^\beta\Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}}({}^{vv}Y^\beta)+X^\alpha\partial_\alpha({}^{vv}Y^\beta)-y^\varepsilon\Gamma^\alpha_{\varepsilon\ \beta}X^\beta\partial_{\overline{\overline{\alpha}}}({}^{vv}Y^\beta)-Y^\alpha\partial_{\overline{\overline{\alpha}}}\,{}^{HH}X^\beta \\
&= -Y^\alpha\partial_{\overline{\overline{\alpha}}}X^\beta \\
&= 0
\end{aligned}
$$

by virtue of (2.1) and (6.1). As the third coordinate, if $J=\overline{\overline{\beta}}$, then we obtain

$$
\begin{aligned}
[\,{}^{HH}X,{}^{vv}Y]^{\overline{\overline{\beta}}} &= p_\varepsilon X^\beta\Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}}({}^{vv}Y^{\overline{\overline{\beta}}})+X^\alpha\partial_\alpha({}^{vv}Y^{\overline{\overline{\beta}}})-y^\varepsilon\Gamma^\alpha_{\varepsilon\ \beta}X^\beta\partial_{\overline{\overline{\alpha}}}({}^{vv}Y^{\overline{\overline{\beta}}})-Y^\alpha\partial_{\overline{\overline{\alpha}}}\,{}^{HH}X^{\overline{\overline{\beta}}} \\
&= p_\varepsilon X^\beta\Gamma^\varepsilon_{\beta\alpha}\partial_{\overline{\alpha}}({}^{vv}Y^{\overline{\overline{\beta}}})+X^\alpha\partial_\alpha({}^{vv}Y^{\overline{\overline{\beta}}})-y^\varepsilon\Gamma^\alpha_{\varepsilon\ \beta}X^\beta\partial_{\overline{\overline{\alpha}}}({}^{vv}Y^{\overline{\overline{\beta}}})-Y^\alpha\partial_{\overline{\overline{\alpha}}}\,{}^{HH}X^{\overline{\overline{\beta}}} \\
&= X^\alpha\partial_\alpha Y^\beta+Y^\alpha\partial_{\overline{\overline{\alpha}}}y^\varepsilon\Gamma^\beta_{\varepsilon\ \theta}X^\theta \\
&= X^\theta\partial_\theta Y^\beta+Y^\alpha X^\theta\Gamma^\beta_{\alpha\ \theta} \\
&= X^\theta(\partial_\theta Y^\beta+\Gamma^\beta_{\theta\ \alpha}Y^\alpha)=(\nabla_X Y)^\beta
\end{aligned}
$$

by virtue of (2.1) and (6.1). On the other hand the vertical lift ${}^{vv}(\nabla_X Y)$ of $(\nabla_X Y)$ has components of the form

$$
{}^{vv}(\nabla_X Y)=\begin{pmatrix} 0 \\ 0 \\ (\nabla_X Y)^\beta \end{pmatrix}
$$

with respect to the coordinates $(x^{\overline{\beta}},x^\beta,x^{\overline{\overline{\beta}}})$ on $t(M_n)$. Thus we have (ii) of Theorem 6.1.

7 Horizontal Lifts of Tensor Fields of Type (1,1)

Suppose now that $F \in \Im_1^1(T^*(M_n))$ and F has local components F_β^α in a neighborhood U of M_n, $F = F_\beta^\alpha \partial_\alpha \otimes dx^\beta$. Then we define the horizontal lift ${}^{HH}F$ of F by

$$ {}^{HH}F = {}^{cc}F - \gamma[\nabla F] \tag{7.1} $$

on $t(M_n)$. Where $[\nabla F]$ is a tensor field of type (1,2) defined by

$$ [\nabla F](X,Y) = -\nabla_X(FY) + \nabla_Y(FX), \tag{7.2} $$

X and Y being arbitrary elements of $\Im_0^1(T^*(M_n))$. From (5.1), (7.1) and (7.2), we see that the horizontal lift ${}^{HH}F$ has components of the form

$$ {}^{HH}F = ({}^{HH}F_J^I) = \begin{pmatrix} F_\alpha^\beta & -\Gamma_{\beta\sigma}F_\alpha^\sigma + \Gamma_{\alpha\sigma}F_\beta^\sigma & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & -\Gamma_\varepsilon^\alpha F_\beta^\varepsilon + \Gamma_\beta^\varepsilon F_\varepsilon^\alpha & F_\beta^\alpha \end{pmatrix} \tag{7.3} $$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ on $t(M_n)$, where F_β^α are local components of F, $\Gamma_{\beta\alpha}^\varepsilon$ componenets of ∇ on $t(M_n)$ and $\Gamma_{\beta\alpha}$, Γ_β^α are defined by (6.2).

Proof. From (5.1), (7.1) and (7.2), we have

$$ \begin{aligned} {}^{HH}F &= \begin{pmatrix} F_\alpha^\beta & -\Gamma_{\beta\sigma}F_\alpha^\sigma + \Gamma_{\alpha\sigma}F_\beta^\sigma & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & -\Gamma_\varepsilon^\alpha F_\beta^\varepsilon + \Gamma_\beta^\varepsilon F_\varepsilon^\alpha & F_\beta^\alpha \end{pmatrix} \\ &= \begin{pmatrix} F_\alpha^\beta & p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & F_\beta^\alpha \end{pmatrix} \\ &\quad - \begin{pmatrix} 0 & -p_\sigma(\partial_\alpha F_\beta^\sigma + \Gamma_\alpha^\sigma{}_\gamma F_\beta^\gamma - \partial_\beta F_\alpha^\sigma - \Gamma_\beta^\sigma{}_\gamma F_\alpha^\gamma) & 0 \\ 0 & 0 & 0 \\ 0 & y^\varepsilon\left(\partial_\varepsilon F_\beta^\alpha + \Gamma_\varepsilon^\alpha{}_\gamma F_\beta^\gamma - \Gamma_\varepsilon^\gamma{}_\beta F_\gamma^\alpha\right) & 0 \end{pmatrix} \\ &= \begin{pmatrix} F_\alpha^\beta & p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & F_\beta^\alpha \end{pmatrix} - \begin{pmatrix} 0 & -p_\sigma([\nabla F](X,Y))_{\beta\alpha}^\sigma & 0 \\ 0 & 0 & 0 \\ 0 & y^\varepsilon\left(\nabla_\varepsilon F_\beta^\alpha\right) & 0 \end{pmatrix} \\ &= {}^{cc}F - \gamma[\nabla F]. \end{aligned} $$

Thus we have (7.3).

Theorem 7.1 *If F and X are affinor and vector fields on $T^*(M_n)$ then*

(i) ${}^{HH}F({}^{vv}X) ={}^{vv}(F \circ X)$,
(ii) ${}^{HH}F({}^{HH}X) = {}^{HH}(FX)$.

Proof. (i) If $X \in \Im_0^1(T^*(M_n))$, $F \in \Im_1^1(T^*(M_n))$, then, by (2.1) and (7.3), we find

$$
\begin{aligned}
{}^{HH}F({}^{vv}X) &= \begin{pmatrix} F_\alpha^\beta & -\Gamma_{\beta\sigma}F_\alpha^\sigma + \Gamma_{\alpha\sigma}F_\beta^\sigma & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & -\Gamma_\varepsilon^\alpha F_\beta^\varepsilon + \Gamma_\beta^\varepsilon F_\varepsilon^\alpha & F_\beta^\alpha \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ X^\beta \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ F_\beta^\alpha X^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (F \circ X)^\alpha \end{pmatrix} =^{vv} (F \circ X),
\end{aligned}
$$

which implies (i) of the Theorem 7.1.

(ii) If F and X are affinor and vector fields on $T^*(M_n)$, then, by (6.1) and (7.3), we have

$$
\begin{aligned}
{}^{HH}F({}^{HH}X) &= \begin{pmatrix} F_\alpha^\beta & -\Gamma_{\beta\sigma}F_\alpha^\sigma + \Gamma_{\alpha\sigma}F_\beta^\sigma & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & -\Gamma_\varepsilon^\alpha F_\beta^\varepsilon + \Gamma_\beta^\varepsilon F_\varepsilon^\alpha & F_\beta^\alpha \end{pmatrix} \begin{pmatrix} p_\sigma X^\alpha \Gamma_{\alpha\,\beta}^\sigma \\ X^\beta \\ -y^\varepsilon \Gamma_{\varepsilon\ \alpha}^{\beta} X^\alpha \end{pmatrix} \\
&= \begin{pmatrix} p_\varepsilon X^\alpha \Gamma_{\alpha\beta}^\varepsilon F_\alpha^\beta + p_\theta \Gamma_{\alpha\varepsilon}^\theta F_\beta^\varepsilon X^\beta - p_\theta \Gamma_{\beta\varepsilon}^\theta F_\alpha^\varepsilon X^\beta \\ F_\beta^\alpha X^\beta \\ -y^\gamma \Gamma_{\gamma\ \varepsilon}^{\alpha} F_\beta^\varepsilon X^\beta + y^\gamma \Gamma_{\gamma\beta}^\varepsilon F_\varepsilon^\alpha X^\beta - y^\varepsilon \Gamma_{\varepsilon\ \theta}^{\beta} X^\theta F_\beta^\alpha \end{pmatrix} \\
&= \begin{pmatrix} p_\sigma (FX)^\beta \Gamma_{\beta\,\alpha}^\sigma \\ (FX)^\alpha \\ -y^\varepsilon \Gamma_{\varepsilon\ \beta}^{\alpha} (FX)^\beta \end{pmatrix} = \begin{pmatrix} (FX)^\beta \Gamma_{\beta\alpha} \\ (FX)^\alpha \\ -\Gamma_\beta^\alpha (FX)^\beta \end{pmatrix} = {}^{HH}(FX).
\end{aligned}
$$

Thus we have (ii) of the Theorem 7.1.

References

1. Duc, T.V.: *Structure presque-transverse*. J. Diff . Geom., **14** (2), 215-219 (1979).
2. Husemoller, D.: Fibre Bundles. Springer, *New York* (1994).
3. Lawson, H.B. and Michelsohn,, M.L.: Spin Geometry. Princeton University Press., *Princeton* (1989).
4. Ostianu, N.M.: Step-fibred spaces, *Tr. Geom. Sem., 5*, VINITI, Moscow, 259-309 (1974).
5. Pontryagin, L.S.: *Characteristic cycles on differentiable manifolds*. Amer. Math. Soc. Translation, **1950** (32), 72p. (1950).
6. Poor, W.A.: Differential Geometric Structures, McGraw-Hil1, *New York* (1981).
7. Salimov, A.A. and Kadiouglu, E.: *Lifts of Derivations to the Semitangent Bundle*, Turk J. Math. **24** (3), 259-266 (2000).
8. Steenrod, N.: The Topology of Fibre Bundles. Princeton University Press., *Princeton* (1951).
9. Vishnevskii, V.V.: *Integrable affinor structures and their plural interpretations*. Geometry, *7.J. Math. Sci.* (New York) **108** (2), 151-187 (2002).
10. Yano, K. and Ishihara, S.: Tangent and Cotangent Bundles. Marcel Dekker, Inc., *New York* (1973).