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The Jost solutions to the Schrödinger equation with an additional complex potential

Khatira E. Abbasova, Agil Kh. Khanmamedov⋆, Sevindj M. Bagirova

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Abstract. *We consider the differential equation $-y'' + xe^{ix}y + q(x)\,y = k^2y$. Using transformation operators, we obtain representations of solutions of this equation with conditions at infinity. Estimates for the kernels of the transformation operators are obtained.*

Keywords. Schrödinger equation · non-self-adjoint differential operator · the space $L_2(-\infty, +\infty)$ · transformation operator · the Jost solution.

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1 Introduction and main results

In many aspects of the theory of inverse problems of spectral analysis, an important role is played by so-called transformation operators. The latter first appeared in the theory of generalized translation operators of J. Delsarte [1] and B.M. Levitan [5]. For arbitrary Sturm-Liouville equations, transformation operators were constructed by A.Ya. Povzner [9]. V.A. Marchenko [6] used transformation operators for studying inverse spectral problems and the asymptotic behavior of the spectral function of the singular Sturm-Liouville operator. It should be remarked that in the effective solution of various inverse problems of scattering theory, an important role is played by the transformation operators with a conditional which were discovered by B.Ya. Levin [4] Similar problems for the Schrödinger equation with unbounded potentials were considered in [3, 8, 10].

We consider the differential equation

$$-y'' + xe^{ix}y + q(x)\,y = k^2y, \tag{1.1}$$

⋆ Corresponding author

Kh.E. Abbasova,
Azerbaijan State University of Economics (UNEC), AZ 1001, Baku, Azerbaijan
E-mail: abbasova_xatira@unec.edu.az

A.Kh. Khanmamedov
Baku State University, AZ 1148, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan Baku, Azerbaijan
Azerbaycan University, Baku, AZ-1073 Azerbaijan
E-mail: agil_khanmamedov@yahoo.com

S.M. Bagirova
Azerbaijan State Agricultural University, AZ 2000, Ganja, Azerbaijan
E-mail: bagirovasevindj@rambler.ru

where $q(x)$ is a continuously differentiable function with bounded support and is a complex parameter. If $q(x) = 0$, then from [2], the equation (1.1) has unique solution $f_0(x,k)$, which can be given as a series

$$f_0\left(x,k\right) = e^{ikx} + \sum_{n=1}^{\infty}\sum_{s=0}^{n} p_{ns}\left(k\right) x^s e^{i(n+k)x}. \tag{1.2}$$

Here $p_{ns}(k)$ is a regular rational function with poles at the points $k = -\frac{j}{2}, j = 1, 2, ..., n$ and multiplicities at most $j+1$, while the series (1.2) admits term-by-term differentiation with respect to x any number of times for $k \neq -\frac{n}{2}, n = 1, 2, ...$ It was proved in [2] (see also [7]), for any k with $Imk > 0$, the function $f_0(x,k)$ belongs to $L_2\left(0, +\infty\right)$ and the function belongs to $L_2\left(-\infty, 0\right)$. Moreover, the functions $f_0(x,k)$ and $f_0(x,-k)$ form the fundamental system of solutions of equation (1.1) for $k \neq 0$ when $q(x) = 0$.

This paper is devoted to the study of the solutions of (1.1) with asymptotic conditions

$$f_{\pm}\left(x,k\right) = f_0\left(x, \pm k\right) + o\left(1\right), x \to \pm\infty.$$

We shall derive the integral representation, which is usually called the Jost translation representation between $f_{\pm}\left(x,k\right)$ and $f_0\left(x,\pm k\right)$. The obtained results can be used to study the spectral properties of the non-self-adjoint differential operator L, generated by the differential expression $l(y) = -y'' + xe^{ix}y + q(x)y$ in the space $L_2\left(-\infty, +\infty\right)$.

The main result of the present paper is as follows.

Theorem 1.1 *For any $k \neq -\frac{n}{2}, n = 1, 2, ...$ from the complex plane, equation (1.1) has solutions $f_+\left(x,k\right)$ and $f_-\left(x,k\right)$, which can be represented in the form*

$$f_+\left(x,k\right) = f_0\left(x,k\right) + \int_x^{+\infty} K\left(x,t\right) f_0\left(t,k\right) dt \tag{1.3}$$

and

$$f_-\left(x,k\right) = f_0\left(x,-k\right) + \int_{-\infty}^{x} A\left(x,t\right) f_0\left(t,-k\right) dt. \tag{1.4}$$

Moreover,

$$K\left(x,x\right) = \frac{1}{2}\int_x^{+\infty} q\left(t\right) dt, \tag{1.5}$$

$$A\left(x,x\right) = \frac{1}{2}\int_{-\infty}^{x} q\left(t\right) dt. \tag{1.6}$$

2 Proof of the theorem

Without loss of generality, we consider the case " + " and assume that $x \geq 0$. We shall use the following notation

$$p\left(x\right) = xe^{ix}, \sigma\left(x\right) = \frac{1}{2}\int_x^{+\infty} \left|q\left(t\right)\right| dt.$$

We first consider the following lemmas before turning to the proof of the theorem.

Lemma 2.1 *If $q(x)$ is a continuously differentiable function with bounded support, then the integral equation*

$$U(\xi_0,\eta_0)=\frac{1}{2}\int_{\xi_0}^{+\infty}q(\xi)\,d\xi$$

$$+\int_0^{\eta_0}\int_{\xi_0}^{+\infty}\left[p(\xi-\eta)-p(\xi+\eta)+q(\xi-\eta)\right]U(\xi,\eta)\,d\xi d\eta \qquad (2.1)$$

has one and only one solution $U(\xi_0,\eta_0)$. Furthermore, if $q(x)=0$ when $x>a$, then

$$U(\xi_0,\eta_0)=0 \text{ when } \xi_0\geq a. \qquad (2.2)$$

Proof. Using the method of successive approximation, let

$$U_0(\xi_0,\eta_0)=\frac{1}{2}\int_{\xi_0}^{+\infty}q(\xi)\,d\xi, \qquad (2.3)$$

$$U_n(\xi_0,\eta_0)=\int_0^{\eta_0}\int_{\xi_0}^{+\infty}\left[p(\xi-\eta)-p(\xi+\eta)+q(\xi-\eta)\right]U_{n-1}(\xi,\eta)\,d\xi d\eta. \qquad (2.4)$$

Because the function $q(x)$ with bounded support, there exists an $a>0$ such that $q(x)=0$ for $x>a$. By induction with respect to n , we have

$$U_n(\xi_0,\eta_0) \text{ for } \xi_0>2a, n=0,1,2,.... \qquad (2.5)$$

For any $R>0$, suppose that $0<\eta_0<R$, $0<\xi_0<+\infty$. By (2.3), we have

$$|U(\xi_0,\eta_0)|\leq\sigma(\xi_0).$$

Taking the notation

$$M=\max_{\substack{0\leq\xi\leq 2a\\ 0\leq\eta\leq R}}|p(\xi-\eta)-p(\xi+\eta)+q(\xi-\eta)|$$

into account, we obtain

$$|U_1(\xi_0,\eta_0)|\leq\sigma(\xi_0)(M\eta_0).$$

Using induction, by (2.4) we next prove that

$$|U_n(\xi_0,\eta_0)|\leq\sigma(\xi_0)\frac{1}{n!}(M\eta_0)^n. \qquad (2.6)$$

Hence the series

$$U(\xi_0,\eta_0)=\sum_{n=0}^{\infty}U_n(\xi_0,\eta_0) \qquad (2.7)$$

is uniformly and absolutely convergent, so $U(\xi_0,\eta_0)$ is the solution of the integral equation (2.1). From (2.6) and (2.7), it follows that

$$|U(\xi_0,\eta_0)|\leq\sigma(\xi_0)\exp(M\eta_0). \qquad (2.8)$$

This implies obviously the uniqueness of the solution to the equation (2.1). The assertion (2.2) is justified by (2.5) and (2.7).

Lemma 2.2 *Suppose $q(x)$ is a continuously differentiable function with bounded support. Then the solution $U(\xi_0, \eta_0)$ of the integral equation (2.1) satisfies the following differential equation*

$$\frac{\partial^2 U(\xi_0, \eta_0)}{\partial \xi_0 \partial \eta_0} + [p(\xi - \eta) - p(\xi + \eta) + q(\xi - \eta)] U(\xi_0, \eta_0) = 0 \tag{2.9}$$

and

$$U(\xi_0, 0) = \frac{1}{2} \int_{\xi_0}^{+\infty} q(\xi)\, d\xi. \tag{2.10}$$

Proof. From (2.1) the differentiability of $U(\xi_0, \eta_0)$ is evident. Differentiating equation (2.1) directly, we get the equation (2.9). Putting $\xi_0 = 0$ in (2.1), we get the result (2.10). We now let $\xi_0 = \frac{t+x}{2}, \eta_0 = \frac{t-x}{2}$ and express the function $K(x,t) = U(\xi_0, \eta_0)$ as a function of x, t . Then the function $K(x,t)$ is twice continuously differentiable. Moreover, from the two preceding lemmas we get the following lemma.

Lemma 2.3 *Suppose $q(x)$ is a continuously differentiable function with bounded support. Then the function $K(x,t) = U\left(\frac{t+x}{2}, \frac{t-x}{2}\right)$ satisfies both the differential equation*

$$\frac{\partial^2 K(x,t)}{\partial x^2} - [p(x) + q(x)] K(x,t) = \frac{\partial^2 K(x,t)}{\partial t^2} - p(t) K(x,t) \tag{2.11}$$

and the condition

$$K(x,x) = \frac{1}{2} \int_x^{+\infty} q(t)\, dt.$$

Furthermore, if $q(x) = 0$ when $x > a$, then $K(x,t) = 0$ when $x + t > 2a$.

Now the theorem can be proved. By differentiation from (1.3), we have

$$f'_+(x,k) = f'_0(x,k) - K'(x,x) f_0(x,k) + \int_x^{+\infty} K'_x(x,t) f_0(t,k)\, dt \tag{2.12}$$

$$f''_+(x,k) = f''_0(x,k) - \frac{dK(x,x)}{dx} f_0(x,k) - K(x,x) f'_0(x,k)$$
$$-K'_x(x,t) f_0(x,k) + \int_x^{+\infty} K''_{xx}(x,t) f_0(t,k)\, dt. \tag{2.13}$$

From Lemma 2.3, it is easily seen that when t sufficiently large, $K(x,t) = 0$, so the last terms of (1.3), (2.12), (2.13) are integrable. From

$$-f''_0(x,k) + p(x) f_0(x,k) = k^2 f_0(x,k) \tag{2.14}$$

and (1.3), we have

$$k^2 f_+(x,k) = k^2 f_0(x,k)$$
$$+ \int_x^{+\infty} K(x,t) p(t) f_0(t,k)\, dt - \int_x^{+\infty} K(x,t) f''_0(t,k)\, dt. \tag{2.15}$$

Hence, integrating by parts, we obtain

$$\int_x^{+\infty} K(x,t) f''_0(t,k)\, dt = -K(x,x) f'_0(x,k) - \int_x^{+\infty} K'_t(x,t) f'_0(t,k)\, dt$$

$$= -K\left(x,x\right)f_0'\left(x,k\right) + K_t'\left(x,x\right)f_0\left(x,k\right) + \int_x^{+\infty} K_{tt}''\left(x,t\right)f_0\left(t,k\right)dt. \qquad (2.16)$$

By virtue of (1.3) and (2.13)-(2.16), we have

$$-f_+''\left(x,k\right) + p\left(x\right)f_+\left(x,k\right) - k^2 f_+\left(x,k\right)$$

$$= \int_x^{+\infty} \left[K_{tt}''\left(x,t\right) - K_{xx}''\left(x,t\right) + K\left(x,t\right)\left(p\left(x\right) + q\left(x\right) - p\left(t\right)\right)\right] f_0\left(t,k\right)dt$$

$$+ \left[2\frac{dK\left(x,x\right)}{dx} + q\left(x\right)\right] f_0\left(x,k\right).$$

From the lemma 2.3 and the last relation, $f_+(x,k)$ satisfies equation (1.1). Furthermore, by virtue of (2.8)-(2.14), it follows that $f_+(x,k) = f_0(x,k)$ when x sufficiently large. Hence, the $f_+(x,k)$ is a Jost solution. Thus, the proof of the theorem is complete.

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