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Algorithm for determining of sweep coefficients for solving one system of hyperbolic equations of the second order

Nihan A.Aliyev * · **Rashad M.Tagiyev** · **Orhan Z.Namazov**

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Abstract. *The dependence of the volume of gas in a gas and liquid-gas mixture on the corresponding gas pressure is sought as a linear function. For the first-order two-dimensional hyperbolic equations system describing the motion of a gas and liquid-gas mixture, in accordance with the gaslift process of oil production in the annular space and in the lift, to the problem with initial condition is applying the sweep method. As a result, two equations were obtained, one quasilinear, and the other an ordinary differential equation. For a quasi-linear equation an analytical solution has been obtained in the special case.*

Keywords. Gas lift, Hyperbolic type equations, Differential equations, Sweep method, Quasilinear equations

Mathematics Subject Classification (2010): 37N40, 46N10, 47N10, 91B55

1 Introduction

In this problem, the sweep method is applied to the partial derivative hyperbolic type differential equations describing the movement during the oil production by gaslift method [1, 2, 7, 11]. In contrast to [3, 4], to find a solution, the substitution is made here

$$Q(x,t) = L(x,t) \cdot P(x,t) + \alpha(t) \cdot K(x).$$

It is shown that when applying the sweep method to the equations of motion, the coefficients of the functions $L(x,t)$and $K(x,t)$ are found using two differential equations. The first of them is the solution of the classical quasilinear partial differential equation, and the second is the solution of the ordinary differential equation. The solution of the quasilinear equation is sought in the implicit form. For the obtained quasilinear equation, the characteristic method was used [3, 6].

* Corresponding author

N.A. Aliyev
Baku State University, AZ 1148, Baku, Azerbaijan
E-mail: nihan@aliev.info

R.M. Tagiyev
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
Baku State University. Institute of Aplied Mathematics, Baku, Azerbaijan

O.Z. Namazov
Baku State University, AZ 1148, Baku, Azerbaijan
E-mail: orxannamazov886@gmail.com

2 Problem statement

It is known [5, 8, 10] that the system of differential equations of hyperbolic type with partial derivatives, which characterizes the motion of gas in the annular space and gas-liquid mixture in the lift in the process of gas lift, is in the following form:

$$\begin{cases} \frac{\partial P_i(x,t)}{\partial t} = -\frac{c_i^2}{F_i} \cdot \frac{\partial Q_i(x,t)}{\partial x}, & i = 1, 2 \\ \frac{\partial Q_i(x,t)}{\partial t} = -F_i \frac{\partial P_i(x,t)}{\partial x} - 2a_i Q_i(x,t), & t > 0, \;\; x \in (0,\, 2l)\,, \end{cases} \tag{2.1}$$

$$\begin{cases} P\,(0,t) = P_0\,(t)\,, \\ Q\,(0,t) = Q_0\,(t)\,. \end{cases} \tag{2.2}$$

Here $P_i(x,t)$ is gas pressure injected into well (gas-liquid mixture in the lift), $Q_i(x,t)$ is the volume of gas, c_i is the the speed of sound in the relevant environment, l is the depth of the well, the parameter a_i is found through the expression $2a_i = \frac{g}{\omega_c} + \frac{\lambda\omega}{2D}$. In this expression λ is the hydraulic resistance coefficient, g is gravity acceleration, D is effective diameter of annular space and lift. Indices 1 and 2 are the parameters that describe the movement in the annular space and in the lifting pipe, respectively.
By changing the role of the arguments x and t in this system, we can write equation (2.1) in the form of the following equivalent equations [4, 6, 7],

$$\begin{cases} \frac{\partial P_i(x,t)}{\partial x} = -\frac{c^2}{F} \cdot \frac{\partial Q_i(x,t)}{\partial t}, \\ \frac{\partial Q_i(x,t)}{\partial x} = -F \frac{\partial P_i(x,t)}{\partial t} - 2a Q_i(x,t), \end{cases} \quad x \in (0, +\infty), \;\; t > 0. \tag{2.3}$$

And the initial conditions will be in the following form

$$\begin{cases} P_i\,(x,0) = P_0\,(x)\,, \\ Q_i\,(x,0) = Q_0\,(x)\,, & x \in (0, +\infty). \end{cases} \tag{2.4}$$

Here we get problem (2.3) and (2.4), which is equivalent to problem (2.1) and (2.2). Due to the linearity of the equation (2.3), we can sought the dependence of the volume of gas-liquid mixture $Q(x,t)$ on the pressure $P(x,t)$ in the following form [4, 7, 12]:

$$Q_i(x,t) = L_i(x,t) \cdot P_i(x,t) + \alpha(t) \cdot K(x). \tag{2.5}$$

Here $L_i(x,t)$ and $K(x)$ must be defined, and the scalar function $\alpha(t)$ is an arbitrary function satisfying the following condition

$$\alpha(0) = 0, \int_0^\infty \alpha(t)dt = 1, K\,(0) = 0.$$

In the special case $\alpha(t)$ can be chosen as $\alpha(t) = e^{-t}$.
To apply the sweep method, we take the derivative of (2.5) in x and t

$$\begin{cases} \frac{\partial Q_i(x,t)}{\partial x} = \frac{\partial L_i(x,t)}{\partial x} P_i(x,t) + L_i(x,t) \frac{\partial P_i(x,t)}{\partial x} + \alpha(t) K'(x), \\ \frac{\partial Q_i(x,t)}{\partial t} = \frac{\partial L_i(x,t)}{\partial t} P_i(x,t) + L_i(x,t) \frac{\partial P_i(x,t)}{\partial t} + \alpha'(t) K(x). \end{cases}$$

If we write these expressions in the system (2.3), we obtain:

$$\begin{cases} \frac{\partial P_i(x,t)}{\partial x} = -\frac{c^2}{F} \left(\frac{\partial L_i(x,t)}{\partial t} P_i(x,t) + L_i(x,t) \frac{\partial P_i(x,t)}{\partial t} + \alpha'(t) K(x) \right) \\ \frac{\partial L_i(x,t)}{\partial x} P_i(x,t) + L_i(x,t) \frac{\partial P_i(x,t)}{\partial x} + \alpha(t) K'(x) \\ = -F \frac{\partial P_i(x,t)}{\partial t} - 2a \left(L_i(x,t) \cdot P_i(x,t) + \alpha(t) \cdot K(x) \right). \end{cases}$$

Substituting the derivative $\frac{\partial P(x,t)}{\partial x}$ from the first equation of this system into the second equation, we obtain the following expression:

$$\frac{\partial L_i(x,t)}{\partial x}P_i(x,t) - \frac{c^2}{F}\frac{\partial L_i(x,t)}{\partial t}P_i(x,t)L_i(x,t) - \frac{c^2}{F}L_i^2(x,t)\frac{\partial P_i(x,t)}{\partial t} - \frac{c^2}{F}\alpha'(t)K(x)L_i(x,t)$$
$$+\alpha(t)K'(x) = -F\frac{\partial P_i(x,t)}{\partial t} - 2aL_i(x,t)\cdot P_i(x,t) - 2a\alpha(t)\cdot K(x). \tag{2.6}$$

If we simplify equation (2.6):

$$\frac{\partial P_i(x,t)}{\partial t}\left(F - \frac{c^2}{F}L_i^2(x,t)\right) + P_i(x,t)\left(\frac{\partial L_i(x,t)}{\partial x} - \frac{c^2}{F}\frac{\partial L_i(x,t)}{\partial t}L_i(x,t) + 2aL_i(x,t)\right)$$
$$-\frac{c^2}{F}\alpha'(t)K(x)L_i(x,t) + \alpha(t)K'(x) + 2a\alpha(t)\cdot K(x) = 0. \tag{2.7}$$

We integrate the obtained expression (2.7):

$$\int_0^\infty \left[\left(\frac{\partial L_i(x,t)}{\partial x} - \frac{c^2}{F}\frac{\partial L_i(x,t)}{\partial t}L(x,t) + 2aL_i(x,t)\right)P_i(x,t) + \left(F - \frac{c^2}{F}L_i^2(x,t)\right)\frac{\partial P_i(x,t)}{\partial t}\right.$$
$$\left.-\frac{c^2}{F}\alpha'(t)K(x)L_i(x,t) + \alpha(t)K'(x) + 2a\alpha(t)\cdot K(x)\right]dt = 0. \tag{2.8}$$

By integrating by part we get:

$$\int_0^\infty \left(F - \frac{c^2}{F}L_i^2(x,t)\right)\frac{\partial P_i(x,t)}{\partial t}dt = \left(F - \frac{c^2}{F}L_i^2(x,t)\right)P_i(x,t)$$
$$+2\frac{c^2}{F}\int_0^\infty P_i(x,t)L_i(x,t)\frac{\partial L_i(x,t)}{\partial t}dt.$$

Here is accepted that,

$$\left(F - \frac{c^2}{F}L_i^2(x,t)\right)P_i(x,t)\Big|_{t=0}^{\infty} = 0.$$

If we write this obtained expression in (2.8), we get:

$$\int_0^\infty \left[\frac{\partial L_i(x,t)}{\partial x} - \frac{c^2}{F}L_i(x,t)\frac{\partial L_i(x,t)}{\partial t} + 2aL_i(x,t) + 2\frac{c^2}{F}L_i(x,t)\frac{\partial L_i(x,t)}{\partial t}\right]P_i(x,t)dt+$$
$$+K'(x) - \frac{c^2}{F}K(x)\int_0^\infty \alpha'(t)L_i(x,t)dt + 2aK(x) = 0. \tag{2.9}$$

Assuming that the obtained expression is satisfied regardless of $P(x,t)$, then expression (2.9) can be written as:

$$\frac{\partial L_i(x,t)}{\partial x} + \frac{c^2}{F}L_i(x,t)\frac{\partial L_i(x,t)}{\partial t} + 2aL_i(x,t) = 0, \tag{2.10}$$

$$K'(x) - \frac{c^2}{F}K(x)\int_0^\infty \alpha'(t)L_i(x,t)dt + 2aK(x) = 0. \tag{2.11}$$

The obtained equation (2.10) is a quasilinear differential equation of the first order, and to solve this equation, in the Cauchy problem (2.3) and (2.4) it is proposed to take $\lim_{t\to\infty} Q(x,t) = 0$ and $\lim_{t\to\infty} P(x,t) = 0$.

Since equation (2.11) is a homogeneous equation, we can write $K(x) = 0$. Then the expression (2.5) will be as follows:

$$Q_i(x,t) = L_i(x,t)\cdot P_i(x,t). \tag{2.12}$$

Thus, based on the formulation of problems (2.3) and (2.4), quasilinear equation (2.10) can be solved under the following initial conditions:

$$L_i(x,0) = \varphi(x) = \frac{Q_0(x)}{P_0(x)}. \tag{2.13}$$

Let us use the method of characteristics to solve the quasi-linear equation (2.10) with the initial conditions (2.13):

$$\frac{dx}{1} = \frac{dt}{\frac{c^2}{F}L} = \frac{dL}{-2aL}. \tag{2.14}$$

From here we get the following ordinary differential equations:

$$dL = \frac{-2aF}{c^2}dt, dL = -2aLdx.$$

If we integrate the last two equations, we get the following system of equations:

$$\begin{cases} \ln L_i(x,t) + 2ax = C_1, \\ L_i(x,t) + \frac{2aFt}{c^2} = C_2. \end{cases} \tag{2.15}$$

Here C_1 and C_2 are constants determined from the characteristic equations. Let $t = 0$ in system (2.14):

$$\begin{cases} \ln L_i\,(x,0) + 2ax = C_1, \\ L_i\,(x,0) = C_2 \end{cases}$$

and let take into consideration the initial condition (2.13):

$$\begin{cases} \ln \varphi\,(x) + 2ax = C_1, \\ \varphi\,(x) = C_2. \end{cases} \tag{2.16}$$

It can be seen from the second equation of this system that the function $\varphi\,(x)$ is constant for any value of the variable x. If we write this equation in the form $x = \varphi^{-1}\,(C_2)$ and take it into account in the first equation of the system, we obtain:

$$\ln C_2 - 2a\varphi^{-1}\,(C_2) = C_1,$$

$$\varphi^{-1}\,(C_2) = \frac{1}{2a}\,(\ln C_2 - C_1)\,,$$

$$C_2 = \varphi\left(\frac{1}{2a}\,(\ln C_2 - C_1)\right).$$

Finally, if consider the last equality in (2.14), we get

$$L_i(x,t) + \frac{2aFt}{c^2} = \varphi\left(\frac{-1}{2a}\,(\ln C_2 - C_1)\right),$$

or

$$L_i(x,t) + \frac{2aFt}{c^2} = \varphi\left(\frac{-1}{2a}\ln\left(L_i(x,t) + \frac{2aFt}{c^2}\right) - \frac{1}{2a}\,(\ln L_i(x,t) + 2ax)\right)$$

expressions.

We can write the last expression in the following form:

$$L_i(x,t)+\frac{2aFt}{c^2}=\varphi\left(\frac{-1}{2a}\ln\left(L_i^2(x,t)-\frac{2aFt}{c^2}L_i(x,t)\right)-x\right). \quad (2.17)$$

Taking into account that the function is constant for any value of the argument, we can write (2.15) as follows

$$L_i(x,t)=C-\frac{2aFt}{c^2}. \quad (2.18)$$

It is easy to show that the found expression $L_i(x,t)$ is a solution to the quasilinear equation (2.10) at the initial condition (2.13), see [2, 9].

Now let's define the functions $K(x)$ from equation (2.11). For this, we write the known function $L_i(x,t)$ in (2.12) and solve it taking into account the corresponding initial conditions, and find $K(x)$. Then we can find $Q_i(x,t)$ using $P_i(x,t)$ by writing these obtained functions $L_i(x,t)$ and $K(x)$ in (2.5). Writing the resulting expression for the function $Q_i(x,t)$ into system (1), we obtain the first order linear partial differential equations corresponding to $Q_i(x,t)$ (with the initial condition $Q_i(x,0)=Q_0(x)$). These equations can also be solved by the method of characteristics.

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