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On a linear inverse problem for the three-dimensional Tricomi equation with nonlocal boundary conditions of periodic type in a prismatic unbounded domain

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Abstract. *In this article the correctness of a linear inverse problem for the three-dimensional Tricomi equation with nonlocal boundary conditions of periodic type is considered in a prismatic unbounded domain. The existence and uniqueness theorems for a generalized solution to a linear inverse problem with nonlocal boundary conditions of periodic type are proved in a certain class of integrable functions. The "ε-regularization", a priori estimates, approximation sequences and Fourier transform methods are applied.*

Keywords. three-dimensional Tricomi equations, linear inverse problem, nonlocal boundary conditions of periodic type, problem correctness, "ε-regularization" method, a priori estimation method, approximation sequences method, Fourier transforms.

Mathematics Subject Classification (2010): 35M10.

1 Formulation of the problem

Different kind of inverse problems for the classical equations (parabolic, elliptic and hyperbolic types) are studied by many mathematicians (see, for examples, [1], [3], [16], [17], [19], [33], [34], [35]). Mixed differential equations of first and second type are considered, in particular, in [4], [5], [6], [7], [26], [30], [31] in bounded domains. Inverse problems for equations of mixed type (especially, for the Tricomi equation) in unbounded domains are much less studied.

In this paper, we study the unique solubility of inverse problems for the three-dimensional Tricomi equation with nonlocal boundary conditions of periodic type in an unbounded prismatic domain. We note, the method of reduction of the inverse problem to direct with nonlo-

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cal boundary conditions of periodic type for a family of loaded Tricomi integro-differential equations is suggested in [8] in a bounded rectangular domain.

We remind that as a loaded equation is typically called a partial differential equation containing the values of certain functionals of the solution of the equation in the coefficients or in the right-hand side [4], [5], [6], [7], [8], [9], [10], [17], [22], [26], [30], [31], [32]. Some inverse boundary problems for partial differential and integro-differential equations are considered in [2], [13], [15], [24], [25], [36], [37], [38].
In the domain $Q = Q_1 \times \mathbb{R} = (-1,1) \times (0,T) \times \mathbb{R}$ we consider the three-dimensional Tricomi equation:

$$Lu = x\, u_{tt} - \Delta\, u + a(x,t)\, u_t + c(x,t)\, u = \psi(x,t,z), \tag{1.1}$$

where $\Delta u = u_{xx} + u_{zz}$ is the Laplace operator, $\psi(x,t,y) = g(x,t,y) + h(x,t)\, f(x,t,y)$, the functions $g(x,t,y)$ and $f(x,t,y)$ are given and the function $h(x,t)$ is unknown.

We need to introduce definitions of several function spaces and designations. The Fourier transform of function $u(x,t,z)$ we denote by

$$\hat{u}(x,t,\lambda) = (2\pi)^{-1/2} \int\limits_{-\infty}^{+\infty} u(x,t,z)\, e^{-i\lambda z} dz$$

and the inverse Fourier transform – by

$$u(x,t,z) = (2\pi)^{-1/2} \int\limits_{-\infty}^{+\infty} \hat{u}(x,t,\lambda)\, e^{i\lambda z} d\lambda.$$

Now, by the aid of the Fourier transform, we determine the space $W_2^{l,s}(Q)$ with the norm

$$\|u\|^2_{W_2^{l,s}(Q)} = (2\pi)^{-1/2} \int\limits_{-\infty}^{+\infty} \left(1 + |\lambda|^2\right)^s \|\hat{u}\,(x,t,\lambda)\|^2_{W_2^l(Q_1)}\, d\lambda, \tag{A}$$

where s, l are any finite positive integers. The Sobolev space is defined by $W_2^l(Q_1)$ (for $l = 0,\ W_2^0(Q_1) = L_2(Q_1)$) with the scalar product $(u, \vartheta)_l$ and with the norm

$$\|\vartheta\|_l^2 = \|\vartheta\|^2_{W_2^l(Q_1)} = \sum_{|\alpha| \le l} \int\limits_{Q_1} |D^\alpha \vartheta|^2\, dxdt,$$

where α is multi-index, D^α is generalized derivative on variables x and t. It is obvious that the space $W_2^{l,s}(Q)$ with the norm (A) is a Hilbert space [14], [18], [20], [21], [23], [27].

Linear inverse problem. Find the pair of functions $\{u(x,t,z),\ h(x,t)\}$ satisfying the equation (1.1) in the domain Q, the following with nonlocal boundary conditions of periodic type

$$\gamma D_t^p\, u|_{t=0} = D_t^p\, u|_{t=T}\,, \tag{1.2}$$

$$D_x^p\, u|_{x=-1} = D_x^p\, u|_{x=1} \tag{1.3}$$

and the additional condition

$$u(x,t,\ell_0) = \varphi_0\,(x,t), \tag{1.4}$$

where $p \in \{0,1\},\ D_t^p u = \frac{\partial^p u}{\partial\, t^p},\quad D_t^0\, u = u,\ \gamma$ is some nonzero constant, $\ell_0 \in \mathbb{R}$ with the function $h(x,t)$ belongs to the class

$$U = \Big\{ (u,h) \,|\, u \in W_2^{2,s}(Q);\ h \in W_2^2(Q_1),\ s \ge 3 \Big\}.$$

Definition 1. As a generalized solution to the problem (1.1)–(1.4) we will call the function $u(x,t,z) \in U$, which satisfies the equation (1.1) with the conditions (1.2)–(1.4) almost everywhere.

Let all the coefficients of the equation (1.1) be sufficiently smooth functions in the domain Q and let the following conditions be satisfied.

Condition 1: The periodicity conditions: $a(x,0) = a(x,T),\ c(x,0) = c(x,T)$; the nonlocal conditions: $\gamma\, g(x,0,z) = g(x,T,z),\ \ \gamma\, f(x,0,z) = f(x,T,z)$ the smoothness conditions: $f(x,t,l_0) = f_0(x,t) \in C^{0,1}_{x,t}(Q_1),\ |f_0(x,t)| \geq \eta > 0,\ f \in W^{3,s}_2(Q),\ g \in W^{1,s}_2(Q)$;

Condition 2: $\varphi_0(x,t) \in W^3_2(Q_1)$,

$\gamma D^q_t \varphi_0\big|_{t=0} = D^q_t \varphi_0\big|_{t=T},\ \ q = 0,1,2. \quad D^p_x \varphi_0\big|_{x=-1} = D^p_x \varphi_0\big|_{x=},\ p = 0,1.$

The unique solubility of the problem (1.1)–(1.4) will be proved by the help of the Fourier transform. We consider the traces of the equation (1.1) at $z = \ell_0$:

$$Lu\big(x,t,\ell_0\big) = xu_{tt}\big(x,t,\ell_0\big) - u_{xx}\big(x,t,\ell_0\big) - u_{zz}\big(x,t,\ell_0\big) +$$

$$+ a(x,t)u_t\big(x,t,\ell_0\big) + c(x,t)u\big(x,t,\ell_0\big) = \psi\big(x,t,\ell_0\big).$$

Taking into account the condition (1.4) and that $f_0 \neq 0$, we determine a formally unknown function $h(x,t)$ in the form of the integral

$$h(x,t) = \frac{1}{f_0(x,t)}\left[\varPhi_0 + \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \lambda^2 e^{i\lambda\ell_0} \hat{u}(x,t,\lambda)d\lambda\right],$$

where $\varPhi_0 = L_0\varphi_0 - g_0,\ L_0\varphi_0 = x\varphi_{0_{tt}} - \varphi_{0_{xx}} + a(x,t)\,\varphi_{0_t} + c\,(x,t)\,\varphi_0$.

For the function $\hat{u}(x,t,\lambda)$ in the domain $Q_1 = (-1,1) \times (0,T)$ we obtain the loaded Tricomi integro-differential equation:

$$L\hat{u} = x\hat{u}_{tt} - \hat{u}_{xx} + a(x,t)\,\hat{u}_t + \big(c(x,t) + \lambda^2\big)\hat{u} = \hat{g}(x,t,\lambda)$$

$$+ \frac{\hat{f}(x,t,\lambda)}{f_0(x,t)}\left[\varPhi_0 + \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \lambda^2 e^{i\lambda\ell_0} \hat{u}(x,t,\lambda)d\lambda\right] \equiv \hat{F}(\hat{u}) \tag{1.5}$$

with nonlocal boundary conditions of periodic type

$$\gamma D^p_t\ \hat{u}|_{t=0} = D^p_t\ \hat{u}|_{t=T}\,, \tag{1.6}$$

$$D^p_x\ \hat{u}|_{x=-1} = D^p_x\ \hat{u}|_{x=1}\,; p = 0,1, \tag{1.7}$$

where

$$\hat{f}(x,t,\lambda) = (2\pi)^{-1/2} \int\limits_{-\infty}^{+\infty} f(x,t,z)\, e^{-i\lambda z} dz,\ \ \lambda \in \mathbb{R}$$

is the Fourier transform of function $f(x,t,z)$ in variable z.

The main result is given by the following theorem.

Theorem 1.1 *Let conditions 1 and 2 be satisfied,* $2a(x,t) + \mu\, x > B_1 > 0,\ \mu c(x,t) - c_t(x,t) > b_2 > 0$ *for all* $x \in \overline{Q_1}$, *where* $\mu = \frac{2}{T}\ln|\,\gamma\,| > 0,\ |\,\gamma\,| > 1$. *Suppose that there exist some positive numbers* σ, $c(\sigma^{-1})$ *such that for* $b_0 = \min\big\{B_1,\, \mu,\, b_2\big\}$ *we have* $b_0 - c\big(\sigma^{-1}\big) = \delta > 0$, $M\,\|\,f\,\|^2_{W^{3,s}_2(Q)} < \frac{1}{2}$, $c(\sigma^{-1}) = 14\mu^2\sigma^{-1} > 0$, *where*

$M = \mathrm{const}\Big(\sigma\,\mu^2 m\,\delta^{-1}\eta^{-2}\,\|f_0\|_{C^{0,1}_{x,t}(Q_1)}\Big)$, $m = 10c_1c_2c_3$, $c_1 = \int\limits_{-\infty}^{+\infty}\frac{\lambda^4 d\lambda}{\left(1+|\lambda|^2\right)^s} < \infty$, $s \geq 3$, $c_i (i = 2, 3)$ *are the coefficients of the Sobolev embedding theorems.*

Then the functions

$$u(x,t,z) = (2\pi)^{-1/2}\int\limits_{-\infty}^{+\infty}\hat{u}(x,t,\lambda)\,e^{i\lambda z}d\lambda, \tag{1.8}$$

$$h(x,t) = \frac{1}{f_0(x,t)}\left[\Phi_0 + \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\lambda^2 e^{i\lambda\ell_0}\hat{u}(x,t,\lambda)d\lambda\right] \tag{1.9}$$

are the unique pair of solutions to the linear inverse problem (1.1)–(1.4) from the class U.

Proof. We prove the theorem according to the following scheme:

1. We show that the function $u(x,t,y)$, defined by the formula (1.8), satisfies the additional condition (1.4).

2. We show the unique solubility to the problem (1.5)–(1.7). In this order, we will study the unique solubility of the problem for the family of loaded Tricomi integro-differential equations of the third order with a small parameter (auxiliary problem).

3. With the help of this auxiliary problem, we study the unique solubility of the problem (1.5)–(1.7).

4. Using the unique solubility of the problem (1.5)–(1.7), we prove the unique solubility of linear inverse problem (1.1)–(1.4).

Now let us go to the realization of this scheme. So, we prove that $u(x,t,\ell_0) = \varphi_0\,(x,t)$. Suppose that

$$u\,(x,t,\ell_0) = (2\pi)^{-1/2}\int\limits_{-\infty}^{+\infty}\hat{u}(x,t,\lambda)e^{i\lambda\ell_0}d\lambda = \omega\,(x,t) \neq \varphi_0(x,t).$$

We consider the function $\vartheta(x,t) = \omega(x,t) - \varphi_0(x,t)$ in the domain Q_1. Multiplying the problem (1.5)–(1.7) by $\frac{e^{i\lambda\ell_0}}{\sqrt{2\pi}}$ and integrating by the parameter λ from $-\infty$ to ∞ and taking into account the conditions of the theorem, we obtain the following differential equation

$$L_0\vartheta = x\vartheta_{tt} - \vartheta_{xx} + a\,(x,t)\,\vartheta_t + c\,(x,t)\vartheta = 0 \tag{1.10}$$

with nonlocal boundary conditions of periodic type

$$\gamma D_t^p\,\vartheta|_{t=0} = D_t^p\,\vartheta|_{t=T}\,; \tag{1.11}$$

$$D_x^p\,\vartheta|_{x=-1} = D_x^p\,\vartheta|_{x=1}\,; p = 0,1, \tag{1.12}$$

The uniqueness of the solution to problems (1.10)–(1.12) is proven in [8], [11],[12]. Therefore, it follows that $\vartheta(x,t) = 0$, i.e. $\omega(x,t) = \varphi_0(x,t)$.

2 A family of loaded integro-differential equations of the third order with a small parameter

In the domain $Q_1 = (-1,1) \times (0,T)$ we consider the following family of loaded integro-differential equations of the third order with a small parameter:

$$L_\varepsilon \hat{u}_\varepsilon = -\varepsilon \hat{u}_{\varepsilon ttt} + L_0 \hat{u}_\varepsilon + \lambda^2 \, \hat{u}_\varepsilon = \hat{g}(x,t,\lambda)$$

$$+\frac{\hat{f}\,(x,t,\lambda)}{f_0(x,t)} \left[\Phi_0 + \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \lambda^2 e^{i\lambda \ell_0} \hat{u}_\varepsilon(x,t,\lambda) d\lambda \right] \equiv \hat{F}(\hat{u}_\varepsilon) \tag{2.1}$$

with nonlocal boundary conditions of periodic type

$$\gamma D_t^q \, \hat{u}_\varepsilon|_{t=0} = D_t^q \, \hat{u}_\varepsilon|_{t=T} \, , \; q = 0,1,2, \tag{2.2}$$

$$D_x^p \, \hat{u}_\varepsilon|_{x=-1} = D_x^p \, \hat{u}_\varepsilon|_{x=1} \, , \; q = 0,1, \tag{2.3}$$

where ε is a small positive number. Further, for the correctness of the problem (2.1)–(2.3) we introduce the following notation of a space of generalized functions:

$$W_i(Q_1,\mathbb{R}) = \left\{ \hat{\vartheta} \,\middle|\, \hat{\vartheta} \in W_2^i(Q_1), \; i = 0,1,2; \; \left(1 + |\lambda|^2\right)^{s/2} \left\| \hat{\vartheta} \right\|_{W_2^i(Q_1)} \in L_2(\mathbb{R}); \right.$$

$$\left. W_2^0(Q_1) = L_2(Q_1) \right\}$$

with the norm

$$\left\langle \hat{\vartheta} \right\rangle_i^2 = \int\limits_{-\infty}^{\infty} \left(1 + |\lambda|^2\right)^s \left\| \hat{\vartheta} \right\|_{W_2^i(Q_1)}^2 d\lambda. \tag{B}$$

It is obvious that the space $W_i(Q_1,\mathbb{R})$ with the given norm is Hilbert space [14], [18], [20], [21], [23], [27]. From the definition of the space $W_2^i(Q_1)$, $i = 0,1,2$, it follows the following embedding $W_2(Q_1,\mathbb{R}) \subset W_1(Q_1,\mathbb{R}) \subset W_0(Q_1,\mathbb{R})$. By the symbol

$$W(Q_1,\mathbb{R}) = \left\{ \hat{\vartheta} \,\middle|\, \hat{\vartheta} \in W_2(Q_1,\mathbb{R}), \; \frac{\partial^3 \hat{\vartheta}}{\partial t^3} = \hat{\vartheta}_{ttt} \in W_0(Q_1,\mathbb{R}) \right\}$$

we denote the class of functions satisfying the corresponding conditions (2.2)–(2.3).

Definition 2. As a generalized solution to the problem (2.1)–(2.3) we will call the function $\hat{\vartheta}(x,t,\lambda) \in W(Q_1, \mathbb{R})$, satisfying the equation (2.1) almost everywhere.

In order to proof the solubility of the problems (2.1)–(2.3) we consider nonlocal boundary conditions of periodic type for a family of loaded integro-differential equations of the third order with a small parameter:

$$L_\varepsilon \hat{u}_\varepsilon^{(\theta)} = -\varepsilon \hat{u}_{\varepsilon ttt}^{(\theta)} + L_0 \hat{u}_\varepsilon^{(\theta)} + \lambda^2 \hat{u}_\varepsilon^{(\theta)} = \hat{g}(x,t,\lambda)$$

$$+\frac{\hat{f}(x,t,\lambda)}{f_0(x,t)} \left[\Phi_0 + \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \lambda^2 e^{i\,\lambda\,\ell_0} \hat{u}_\varepsilon^{(\theta-1)}(x,t,\lambda)\, d\lambda \right] \equiv \hat{F}\left(u_\varepsilon^{(\hat{\theta-1})} \right) \tag{2.4}$$

with nonlocal boundary conditions of periodic type

$$\gamma D_t^q \, \hat{u}_\varepsilon^{(\theta)} \Big|_{t=0} = D_t^q \, \hat{u}_\varepsilon^{(\theta)} \Big|_{t=T} , \; q = 0,1,2, \tag{2.5}$$

$$D_x^p\,\hat{u}_\varepsilon^{(\theta)}\Big|_{x=-1}=D_x^p\,\hat{u}_\varepsilon^{(\theta)}\Big|_{x=1},\ p=0,1, \tag{2.6}$$

where $\varepsilon>0,\quad \theta=0,1,2,...$

Lemma 1. *Suppose that all the conditions of the theorem* 1.1 *be satisfied. Then the solution to problem* (2.4)–(2.6) *satisfies the following estimates*

$(I).\ \frac{\varepsilon}{\delta}\left\langle\hat{u}_{\varepsilon tt}^{(\theta)}\right\rangle_0^2+\left\langle u_\varepsilon^{(\theta)}\right\rangle_1^2\le \mathrm{const}(\widetilde{\theta},\,\widetilde{\varepsilon},\,\widetilde{\lambda}),$

$(II).\ \frac{\varepsilon}{\delta}\left\langle\hat{u}_{\varepsilon ttt}^{(\theta)}\right\rangle_0^2+\left\langle \hat{u}_\varepsilon^{(\theta)}\right\rangle_2^2\le \mathrm{const}(\widetilde{\theta},\,\widetilde{\varepsilon},\,\widetilde{\lambda}).$

The $\mathrm{const}(\widetilde{\theta},\widetilde{\varepsilon},\widetilde{\lambda})$ *is independent from the parameters* $\theta,\varepsilon,\lambda$.

Proof. Consider the identity

$$2\left(L\hat{u}_\varepsilon^{(\theta)},\,e^{-\mu t}\hat{u}_{\varepsilon t}^{(\theta)}\right)_0=2\left(\hat{F}\big(\hat{u}_\varepsilon^{(\theta-1)}\big),\,e^{-\mu t}\hat{u}_{\varepsilon t}^{(\theta)}\right)_0, \tag{2.7}$$

where the constant $\mu>0$ we will choose later.

Taking into account the conditions of the Theorem 1.1, integrating identical equation (2.7) by parts and applying the Cauchy inequality with σ ([18], [27]), it is easy to obtain the following inequality

$$2\int\limits_{Q_1}L\hat{u}_\varepsilon^{(\theta)}\,e^{-\mu\,t}\,\hat{u}_{\varepsilon t}^{(\theta)}dxdt\ge\varepsilon\left\|\hat{u}_{\varepsilon tt}^{(\theta)}\right\|_0^2+\int\limits_{Q_1}e^{-\mu t}\Big[(2a+\mu\,x)\,\hat{u}_{\varepsilon t}^{2(\theta)}$$

$$+\mu\,\hat{u}_{\varepsilon x}^{2(\theta)}+\mu\,\lambda^2\hat{u}_\varepsilon^{2(\theta)}+(\mu\,c-c_t)\,\hat{u_\varepsilon^{2(\theta)}}\Big]\,dxdt$$

$$-\int\limits_{\partial Q_1}e^{-\mu t}\Big\{x\hat{u}_{\varepsilon t}^{2(\theta)}e_t-\hat{u}_{\varepsilon x}^{2(\theta)}e_t-2\hat{u}_{\varepsilon t}^{(\theta)}\hat{u}_{\varepsilon x}^{(\theta)}e_x+(c+\lambda^2)\hat{u}_\varepsilon^{(\theta)}e_t\Big\}ds, \tag{2.8}$$

where $\mu-\mathrm{const}>0,\ \overrightarrow{e}=\big(e_x=(\overrightarrow{e},x);\,e_t=(\overrightarrow{e},t)\big)$ is the unit vector of the inward normal values to the boundary $\partial\,Q_1$. The conditions of the Theorem 1.1 provides non-negativity of the integral in the domain Q. By virtue of the nonlocal boundary conditions of periodic type (2.5), (2.6) and the conditions of Theorem 1.1, by the choice of $\gamma^2=e^{\mu T}$ we obtain that the boundary integrals will be equal to zero. Thus, from the inequality

$$2\int\limits_{Q_1}L\hat{u}_\varepsilon^{(\theta)}\,e^{-\mu t}\,\hat{u}_{\varepsilon t}^{(\theta)}dxdt\ge\varepsilon\left\|\hat{u}_{\varepsilon tt}^{(\theta)}\right\|_0^2+\int\limits_{Q_1}e^{-\mu t}\Big(B_1\,\hat{u}_{\varepsilon t}^{2(\theta)}+\mu\,\hat{u}_{\varepsilon x}^{2(\theta)}$$

$$+\mu\,\lambda^2\hat{u}_\varepsilon^{2(\theta)}+b_2\,\hat{u}_\varepsilon^{2(\theta)}\Big)dxdt\ge\varepsilon\left\|\hat{u}_{\varepsilon tt}^{(\theta)}\right\|_0^2+b_0\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_1^2, \tag{2.9}$$

applying the Cauchy inequality with σ to the identity (2.7), we obtain

$$\left|2\left(\hat{F}\big(\hat{u}_\varepsilon^{(\theta-1)}\big),\,e^{-\mu t}\hat{u}_{\varepsilon t}^{(\theta)}\right)_0\right|$$

$$\le\left|2\bigg(\hat{g}+\frac{\hat{f}(x,t,\lambda)}{f_0(x,t)}\bigg[\varPhi_0+\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\lambda^2e^{i\,\lambda\,\ell_0}\,\hat{u}_\varepsilon^{(\theta-1)}(x,t,\lambda)d\lambda\bigg],e^{-\mu\,t}\hat{u}_{\varepsilon t}^{(\theta)}\bigg)_0\right|$$

$$\le 3\sigma\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_1^2+\sigma^{-1}\left[\,\|\hat{g}\|_0^2+\eta^{-2}\left\|\hat{f}\right\|_{C(Q_1)}^2\Big(T_0\,\|\varphi\|_{W_2^2(Q_1)}^2+\|g_0\|_0^2\Big)\right]$$

$$+c_1\sigma\eta^{-2}\left\|\hat{f}\right\|_{C(Q_1)}\int\limits_{-\infty}^{\infty}\left(1+|\lambda|^2\right)^s\left\|\hat{u}_\varepsilon^{(\theta-1)}\right\|_1^2 d\lambda, \tag{2.10}$$

where $T_0=2\max\left\{1,\|a\|_{C(Q_1)};\ \|c\|_{C(Q_1)}\right\}$.

Combining the inequalities (2.9) and (2.10), we derive

$$\varepsilon\left\|\hat{u}_{\varepsilon tt}^{(\theta)}\right\|_0^2+\left(b_0-3\sigma^{-1}\right)\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_1^2$$

$$\leq\sigma^{-1}\left[\|\hat{g}\|_0^2+\eta^{-2}\left\|\hat{f}\right\|_{C(Q_1)}^2\left(T_0\|\varphi\|_{W_2^2(Q_1)}^2+\|g_0\|_0^2\right)\right]$$

$$+c_1\sigma\eta^{-2}\left\|\hat{f}\right\|_{C(Q_1)}\int\limits_{-\infty}^{\infty}\left(1+|\lambda|^2\right)^s\left\|\hat{u}_\varepsilon^{(\theta-1)}\right\|_1^2 d\lambda. \tag{2.11}$$

Applying embedding theorem of Sobolev: $\left\|\hat{f}\right\|_{C(Q_1)}^2\leq c_2\left\|\hat{f}\right\|_{W_2^2(Q_1)}^2$ to the inequality (2.11), we obtain

$$\varepsilon\left\|\hat{u}_{\varepsilon tt}^{(\theta)}\right\|_0^2+\left(b_0-3\sigma^{-1}\right)\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_1^2$$

$$\leq\sigma^{-1}c_2\left[\|\hat{g}\|_0^2+\eta^{-2}\left\|\hat{f}\right\|_{W_2^2(Q_1)}^2\left(T_0\|\varphi\|_{W_2^2(Q_1)}^2+\|g_0\|_0^2\right)\right]$$

$$+c_1\sigma c_2\eta^{-2}\left\|\hat{f}\right\|_{W_2^2(Q_1)}^2\int\limits_{-\infty}^{\infty}\left(1+|\lambda|^2\right)^s\left\|\hat{u}_\varepsilon^{(\theta-1)}\right\|_1^2 d\lambda. \tag{2.12}$$

By virtue of the conditions of the Theorem 1.1, we have $b_0-3\sigma\geq b_0-c(\sigma^{-1})>\delta>0$. So, dividing the inequality (2.12) into $b_0-c(\sigma^{-1})\geq\delta>0$, multiplying $\left(1+|\lambda|^2\right)^s$ and integrating by λ from $-\infty$ to ∞, we obtain the first recurrent formula

$$\frac{\varepsilon}{\delta}\left\langle\hat{u}_{\varepsilon tt}^{(\theta)}\right\rangle_0^2+\left\langle u_\varepsilon^{(\theta)}\right\rangle_1^2\leq A+c_1c_2\sigma\delta^{-1}\eta^{-2}\left\langle\hat{f}\right\rangle_2^2\left\langle u_\varepsilon^{(\theta-1)}\right\rangle_1^2, \tag{2.13}$$

where

$$A\equiv(\sigma\delta)^{-1}c_2\left[\langle\hat{g}\rangle_0^2+\eta^{-2}\left\langle\hat{f}\right\rangle_2^2\left(T_0\|\varphi\|_{W_2^2(Q_1)}^2+\|g_0\|_0^2\right)\right].$$

By the conditions of the Theorem 1.1, we have

$$c_1c_2\sigma\delta^{-1}\eta^{-2}\left\langle\hat{f}\right\rangle_2^2<M\left\langle\hat{f}\right\rangle_3^2<\frac{1}{2}.$$

So, from the recurrent formulas (2.13), we obtain the validity of estimates (I). Since as the "zero approximation" we take $\left\{\hat{u}_\varepsilon^{(0)}\right\}\equiv\{0\}$, then we have

$$\frac{\varepsilon}{\delta}\left\langle\hat{u}_{\varepsilon tt}^{(0)}\right\rangle_0^2+\left\langle u_\varepsilon^{(0)}\right\rangle_1^2\leq A.$$

Continuing this process, we obtain the first a priori estimate for any function $\left\{\hat{u}_\varepsilon^{(\theta)}\right\},\forall\theta\geq 1$

$$\frac{\varepsilon}{\delta}\left\langle\hat{u}_{\varepsilon tt}^{(\theta)}\right\rangle_0^2+\left\langle u_\varepsilon^{(\theta)}\right\rangle_1^2\leq A\left(1+\sum_{\theta=0}^{\infty}\frac{1}{2^{\theta+1}}\right)\leq 2A.$$

Let us prove the estimate (II). To this end, consider the identity:

$$-2\int\limits_{Q_1} e^{-\mu t}\, L_\varepsilon \hat{u}_\varepsilon^{(\theta)}\, P\hat{u}_\varepsilon^{(\theta)} dxdt = -2\int\limits_{Q_1} e^{-\mu t}\, \hat{F}\big(\hat{u}_\varepsilon^{(\theta-1)}\big)\, P\hat{u}_\varepsilon^{(\theta)} dxdt, \tag{2.14}$$

where $P\hat{u}_\varepsilon^{(\theta)} = \hat{u}_{\varepsilon ttt}^{(\theta)} - \mu\, \hat{u}_{\varepsilon tt}^{(\theta)} + \frac{\mu}{2}\hat{u}_{\varepsilon xx}^{(\theta)} - \mu \hat{u}_{\varepsilon t}^{(\theta)}$.

Reasoning in the same way as in the proof of the estimates (I), integrating (2.14) by parts and taking into account the conditions of Theorem 1.1 and nonlocal boundary conditions of periodic type (2.5) and (2.6), we get

$$\left| -2\int\limits_{Q_1} e^{-\mu t}\, \hat{F}\big(\hat{u}_\varepsilon^{(\theta-1)}\big)\, P\hat{u}_\varepsilon^{(\theta)} dxdt \right| \geq \varepsilon \left\| \hat{u}_{\varepsilon ttt}^{(\theta)} \right\|_0^2 + \int\limits_{Q_1} e^{-\mu t}\Big(B_1 \cdot \hat{u}_{\varepsilon tt}^{2(\theta)}$$

$$+\mu\, \hat{u}_{\varepsilon xx}^{2(\theta)} + \mu^2\, \hat{u}_{\varepsilon tx}^{2(\theta)} \Big) dxdt + b_0 \left\| \hat{u}_\varepsilon^{(\theta)} \right\|_1^2 - 3\mu^2\sigma^{-1} \left\| \hat{u}_\varepsilon^{(\theta)} \right\|_2^2, \tag{2.15}$$

where $c_0\big(\sigma^{-1}\big) = 3\mu^2\sigma^{-1}$. For the coefficients of Cauchy inequality we have $b_0 - c_0(\sigma^{-1}) > b_0 - c(\sigma^{-1}) \geq \delta > 0$, where $b_0 = \min\big\{B_1;\ \mu;\ \lambda^2\mu + b_2\big\}$. Then from the inequality (2.15) we obtain the following estimate

$$\left| -2\int\limits_{Q_1} e^{-\mu t}\, \hat{F}\big(\hat{u}_\varepsilon^{(\theta-1)}\big)\, P\hat{u}_\varepsilon^{(\theta)}\, d\,x\,d\,t \right| \ \geq\ \varepsilon \left\| \hat{u}_{\varepsilon ttt}^{(\theta)} \right\|_0^2 + \delta \left\| \hat{u}_\varepsilon^{(\theta)} \right\|_2^2. \tag{2.16}$$

Applying the Cauchy inequality with σ to the right-hand side of identity (2.14), we get the following inequality

$$\left| -2\int\limits_{Q_1} e^{-\mu t}\, \hat{F}\big(\hat{u}_\varepsilon^{(\theta-1)}\big)\, P\hat{u}_\varepsilon^{(\theta)} dxdt \right|$$

$$\leq \left| 2\bigg(\hat{g} + \frac{\hat{f}(x,t,\lambda)}{f_0(x,t)} \bigg[\varPhi_0 + \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \lambda^2 e^{i\lambda\ell_0}\, \hat{u}_\varepsilon^{(\theta-1)}(x,t,\lambda) d\lambda \bigg];\ e^{-\mu t} P\hat{u}_\varepsilon^{(\theta)} \bigg)_0 \right|$$

$$\leq 11\mu^2\sigma^{-1} \left\| \hat{u}_\varepsilon^{(\theta)} \right\|_2^2 + 8\sigma^{-1}\mu^2 \bigg[\|\hat{g}\|_1^2 + \eta^{-2} \left\| \hat{f} \right\|_{C^1(Q_1)}^2 \Big(T_0 \|\varphi\|_{W_2^3(Q_1)}^2 + \|g_0\|_1^2 \Big) \bigg]$$

$$+10c_1\sigma\mu^2\eta^{-2} \|f_0\|_{C^1(Q_1)} \left\| \hat{f} \right\|_{C^1(Q_1)} \int\limits_{-\infty}^{\infty} \big(1+|\lambda|^2\big)^s \left\| \hat{u}_\varepsilon^{(\theta-1)} \right\|_2^2 d\lambda. \tag{2.17}$$

Combining inequalities (2.16) and (2.17), we get the correlation

$$\varepsilon \left\| \hat{u}_{\varepsilon ttt}^{(\theta)} \right\|_0^2 + \big(b_0 - c(\sigma^{-1})\big) \left\| \hat{u}_\varepsilon^{(\theta)} \right\|_2^2$$

$$\leq 8\sigma^{-1}\mu^2 \bigg[\|\hat{g}\|_1^2 + \eta^{-2} \left\| \hat{f} \right\|_{C^1(Q_1)}^2 \Big(T_0 \|\varphi\|_{W_2^3(Q_1)}^2 + \|g_0\|_1^2 \Big) \bigg]$$

$$+10c_1\,\sigma\,\mu^2\eta^{-2} \|f_0\|_{C^1(Q_1)} \left\| \hat{f} \right\|_{C^1(Q_1)} \int\limits_{-\infty}^{\infty} \big(1+|\lambda|^2\big)^s \left\| \hat{u}_\varepsilon^{(\theta-1)} \right\|_2^2 d\lambda. \tag{2.18}$$

Applying the Sobolev embedding theorem: $\left\|\hat{f}\right\|^2_{C^1(Q_1)} \le c_3 \left\|\hat{f}\right\|^2_{W_2^3(Q_1)}$ to the inequality (2.18), we obtain

$$\varepsilon \left\|\hat{u}^{(\theta)}_{\varepsilon ttt}\right\|^2_0 + \left(b_0 - c(\sigma^{-1})\right)\left\|\hat{u}^{(\theta)}_{\varepsilon}\right\|^2_2$$

$$\le 8\sigma^{-1}c_3\mu^2\left[\|\hat{g}\|_1^2 + \eta^{-2}\left\|\hat{f}\right\|^2_{W_2^3(Q_1)}\left(T_0\|\varphi\|^2_{W_2^3(Q_1)} + \|g_0\|_1^2\right)\right]$$

$$+10c_1c_3\sigma\mu^2\eta^{-2}\|f_0\|_{C^1(Q_1)}\left\|\hat{f}\right\|^2_{W_2^3(Q_1)}\int\limits_{-\infty}^{\infty}\left(1+|\lambda|^2\right)^s\left\|\hat{u}^{(\theta-1)}_{\varepsilon}\right\|^2_2 d\lambda. \qquad (2.19)$$

Dividing the inequality (2.19) into $b_0 - c(\sigma^{-1}) \ge \delta > 0$, where $c(\sigma^{-1}) = 14\mu^2\sigma^{-1}$, multiplying by $\left(1+|\lambda|^2\right)^s$ and integrating by λ from $-\infty$ to ∞, we obtain the second recurrent formula

$$\frac{\varepsilon}{\delta}\left\langle\hat{u}^{(\theta)}_{\varepsilon ttt}\right\rangle^2_0 + \left\langle\hat{u}^{(\theta)}_{\varepsilon}\right\rangle^2_2 \le A_1 + 10c_1c_3\sigma\mu^2\eta^{-2}\|f_0\|_{C^1(Q_1)}\left\langle\hat{f}\right\rangle^2_3\left\langle\hat{u}^{(\theta-1)}_{\varepsilon}\right\rangle^2_2, \qquad (2.20)$$

where

$$A_1 \equiv 8c_3\mu^2(\sigma\delta)^{-1}\left[\langle\hat{g}\rangle_1^2 + \eta^{-2}\left\langle\hat{f}\right\rangle^2_3\left(T_0\|\varphi\|^2_{W_2^3(Q_1)} + \|g_0\|_1^2\right)\right].$$

Therefore, by conditions of Theorem 1.1 we have

$$\frac{\varepsilon}{\delta}\left\langle\hat{u}^{(0)}_{\varepsilon ttt}\right\rangle^2_0 + \left\langle u^{(0)}_{\varepsilon}\right\rangle^2_2 \le A_1.$$

Continuing this process, we obtain the second a priori estimate for any function $\left\{\hat{u}^{(\theta)}_{\varepsilon}\right\}$, $\forall\theta \ge 1$

$$\frac{\varepsilon}{\delta}\left\langle\hat{u}^{(\theta)}_{\varepsilon ttt}\right\rangle^2_0 + \left\langle u^{(\theta)}_{\varepsilon}\right\rangle^2_2 \le A_1\left(1 + \sum_{\theta=0}^{\infty}\frac{1}{2^{\theta+1}}\right) \le 2A_1.$$

Hence, we obtain the estimate (II). The lemma 1 is proved.

We introduce a new function from the class $W(Q_1,\mathbb{R})$: $\hat{\vartheta}^{(\theta)}_{\varepsilon} = \hat{u}^{(\theta)}_{\varepsilon} - \hat{u}^{(\theta-1)}_{\varepsilon};\ \varepsilon > 0:$ $\theta = 1,2,3...$ Then the following lemma is valid.

Lemma 2. Suppose that all the conditions of theorem 1.1 be satisfied. Then for the function $\left\{\hat{\vartheta}^{(\theta)}_{\varepsilon}\right\} \in W(Q_1,\mathbb{R})$ allows the following estimates:

(III). $\frac{\varepsilon}{\delta}\left\langle\hat{\vartheta}^{(\theta)}_{\varepsilon tt}\right\rangle^2_0 + \left\langle\hat{\vartheta}^{(\theta)}_{\varepsilon}\right\rangle^2_1 \le A\cdot\left(\frac{1}{2}\right)^{\theta-1};$

(IV). $\frac{\varepsilon}{\delta}\left\langle\hat{\vartheta}^{(\theta)}_{\varepsilon ttt}\right\rangle^2_0 + \left\langle\hat{\vartheta}^{(\theta)}_{\varepsilon}\right\rangle^2_2 \le A_1\cdot\left(\frac{1}{2}\right)^{\theta-1}.$

Proof. From (2.4)–(2.6) for the function $\left\{\hat{\vartheta}^{(\theta)}_{\varepsilon}\right\} \in W(Q_1,\mathbb{R})$ we obtain the following equation

$$L_\varepsilon\hat{\vartheta}^{(\theta)}_{\varepsilon} = -\varepsilon\hat{\vartheta}^{(\theta)}_{\varepsilon ttt} + L_0\hat{\vartheta}^{(\theta)}_{\varepsilon} + \lambda^2\hat{\vartheta}^{(\theta)}_{\varepsilon}$$

$$= \frac{\hat{f}(x,t,\lambda)}{\sqrt{2\pi}f_0(x,t)}\int\limits_{-\infty}^{+\infty}\lambda^2 e^{i\lambda\ell_0}\,\hat{\vartheta}^{(\theta-1)}_{\varepsilon}(x,t,\lambda)\,d\lambda \equiv \hat{F}\left(\hat{\vartheta}^{(\theta-1)}_{\varepsilon}\right) \qquad (2.21)$$

with nonlocal boundary conditions of periodic type

$$\gamma D_t^q \,\hat{\vartheta}_\varepsilon^{(\theta)}\Big|_{t=0} = D_t^q \,\hat{\vartheta}_\varepsilon^{(\theta)}\Big|_{t=T}, \quad q = 0, 1, 2, \tag{2.22}$$

$$D_x^p \,\hat{\vartheta}_\varepsilon^{(\theta)}\Big|_{x=-1} = D_x^p \,\hat{\vartheta}_\varepsilon^{(\theta)}\Big|_{x=1}, \quad p = 0, 1, \tag{2.23}$$

where $\varepsilon > 0, \quad \theta = 1, 2, ...$

Similarly to the proof of Lemma 1 for the function $\left\{\hat{\vartheta}_\varepsilon^{(\theta)}\right\} = \left\{\hat{u}_\varepsilon^{(\theta)}\right\} - \left\{\hat{u}_\varepsilon^{(\theta-1)}\right\} \in W(Q_1, \mathbb{R})$ we obtain the third recurrent formula

$$\frac{\varepsilon}{\delta}\left\langle \hat{\vartheta}_{\varepsilon tt}^{(\theta)}\right\rangle_0^2 + \left\langle \hat{\vartheta}_\varepsilon^{(\theta)}\right\rangle_1^2 \le \frac{1}{2}\left\langle \hat{\vartheta}_\varepsilon^{(\theta-1)}\right\rangle_1^2 \tag{2.24}$$

that, repeating the arguments of Lemma 1 from (2.24) we get a priori estimate (III). Estimate (IV) will be proved similarly. Lemma 2 is proved.

Theorem 2.1 *Suppose that all the conditions of Theorem* 1.1 *be satisfied. Then the problem (2.4)–(2.6) are uniquely solvable in* $W(Q_1, \mathbb{R})$.

Proof. Theorem 2.1 will be proved by the contraction mapping method [11], [12], [29], [28]. Suppose that $\hat{L}_\varepsilon$ is an operator, corresponding to the differential expression (2.4) and conditions (2.5), (2.6). We denote that $\hat{L}_\varepsilon^{-1}$ is a formal inverse operator. We consider the following operator in space $W(Q_1, \mathbb{R})$:

$$\hat{u}_\varepsilon^{(\theta)} = \hat{L}_\varepsilon^{-1} F_s\left(\hat{u}_\varepsilon^{(\theta-1)}\right) \equiv P\hat{u}_\varepsilon^{(\theta-1)}.$$

1. Operator P maps the space $W(Q_1, \mathbb{R})$ in itself. Suppose that $\left\{\hat{u}_\varepsilon^{(\theta-1)}\right\} \in W(Q_1, \mathbb{R})$. Then the Lemma 1 for the problem (2.4)–(2.6) is true. So, the estimate (II) is correct. Hence, follows that for any $\theta = 1, 2, 3...$ we get $\left\{\hat{u}_\varepsilon^{(\theta)}\right\} \in W(Q_1, \mathbb{R})$. Thus $P : W(Q_1, \mathbb{R}) \to W(Q_1, \mathbb{R})$.

2. Let us prove that P is a contractive operator. Suppose that $\left\{\hat{u}_\varepsilon^{(\theta)}\right\}, \left\{\hat{u}_\varepsilon^{(\theta-1)}\right\} \in W(Q_1, \mathbb{R})$. Consider the new function $\left\{\hat{\vartheta}_\varepsilon^{(\theta)}\right\} = \left\{\hat{u}_\varepsilon^{(\theta)}\right\} - \left\{\hat{u}_\varepsilon^{(\theta-1)}\right\}$. For this function the confirmation of Lemma 2 is correct. So, the estimate (IV) is correct:

$$\frac{\varepsilon}{\delta}\left\langle \hat{\vartheta}_{\varepsilon ttt}^{(\theta)}\right\rangle_0^2 + \left\langle \hat{\vartheta}_\varepsilon^{(\theta)}\right\rangle_2^2 \le \left(\frac{1}{2}\right)^{(\theta-1)} \operatorname{const}(\widetilde{\theta}).$$

Therefore P is a contractive operator and, according to a well-known theorem on contracting maps, the problem (2.4)–(2.6) have a unique solution in the space $W(Q_1, \mathbb{R})$, $\varepsilon > 0$. Consequently, we have $\hat{u}_\varepsilon^{(\theta)} \to \hat{u}_\varepsilon$ as $\theta \to \infty$ (see [10], [11], [12], [28], [35]).

3 The family of loaded Tricomi integro-differential equations of the second order

Let us prove the unique solubility of the problem (1.5)–(1.7). The family of loaded integro-differential equation of the third order (2.1) with conditions (2.2), (2.3) we use as "ε-regularizing" equation for (1.5). Suppose that for $\varepsilon > 0$ the function $\{\hat{u}_\varepsilon\} \in W(Q_1, \mathbb{R})$ is a single solution to the problem (2.1)–(2.3). Hence, for $\varepsilon > 0$ the inequality (IV) is valid. According to the theorem on weak compactness ([8], [18]), from the bounded sequence $\{\hat{u}_\varepsilon\}$ can be retrieved the weakly convergent sequence of functions $\left\{\hat{u}_{\varepsilon_j}\right\}$, such that $\hat{u}_{\varepsilon_j} \to \hat{u}$ is weak in $W(Q_1,\mathbb{R})$. Let us show that the limit function $\hat{u}(x,t,\lambda)$ satisfies the equation (1.5) almost everywhere in $W(Q_1,\mathbb{R})$. Indeed, since the sequence $\left\{\hat{u}_{\varepsilon_j}\right\}$ is weakly convergence in $W(Q_1,\mathbb{R})$, then the operator L is linear. So, we have

$$L\hat{u}-F(\hat{u})=L\hat{u}-F(\hat{u}_{\varepsilon_j})-\big[F(\hat{u})-F(\hat{u}_{\varepsilon_j})\big]=\varepsilon_j\hat{u}_{\varepsilon_j ttt}+L_0\big(\hat{u}-\hat{u}_{\varepsilon_j}\big)+\lambda^2\big(\hat{u}-\hat{u}_{\varepsilon_j}\big). \quad (3.1)$$

Passing to the limit in (3.1) as $\varepsilon_j\to 0$, we obtain $L\hat{u}=F(\hat{u})$. It means that the function $\hat{u}(x,t,\lambda)$ is single solution to the problems (1.5)–(1.7) from the class $W(Q_1,\mathbb{R})$(see, [4], [6], [9], [10], [29]). This completes the proof of the Theorem 2.1.

Since all conditions of theorem 1.1 are met, applying the Parseval–Steklov equalities ([20], [27]) to solution of the problem (1.5)–(1.7), we obtain a single solution of the problem (1.1)–(1.4) from the class U.

Remark 3.1 We observe that a linear inverse problems for multi-dimensional equations of Tricomi and Chaplygin types with nonlocal boundary conditions of periodic type in prismatic unbounded domains will be studied in a similar way.

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