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Global bifurcation from infinity in the linearizable Sturm-Liouville problem with indefinite weight and spectral parameter in the boundary condition

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Abstract. *This paper is devoted to the study of the global bifurcation from infinity of the nonlinear Sturm-Liouville problem with an indefinite weight function and a spectral parameter in the boundary condition. We prove that there are two pairs of two classes of global continua of nontrivial solutions bifurcating from points of $\mathbb{R} \times \{\infty\}$ corresponding to positive and negative eigenvalues of the linear problem obtained by setting the nonlinear term equal to zero. These continua possess the usual nodal properties in some neighborhoods of the asymptotic bifurcation points of this problem. Moreover, each of these continua either intersects $\mathbb{R} \times \{0\}$), or intersects some asymptotic bifurcation point in a certain class with fixed oscillation count, or the projection of this continuum onto $\mathbb{R} \times \{0\}$ is unbounded.*

Keywords. Sturm-Liouville problem with indefinite weight, spectral parameter in the boundary condition, eigenfunction, asymptotic bifurcation point, global continua

Mathematics Subject Classification (2010): 34B09, 34B15, 34C10, 34C23, 34K11, 34L20, 34K18, 47J10, 47J15.

1 Introduction

In this paper, we consider the nonlinear Sturm-Liouville problem

$$\ell y \equiv -(p\,(x)\,y')' + q(x)y = \lambda r(x)\,y + g(x, y, y', \lambda),\ x \in (0, 1), \tag{1.1}$$

$$b_0 y(0) = d_0 p(0) y'(0), \tag{1.2}$$

$$(a_1\lambda + b_1)y(1) = p(1)y'(1), \tag{1.3}$$

which was started in [17, 18]. Here $\lambda \in \mathbb{R}$ is a parameter, $p \in C^1\left([0, 1]; (0, +\infty)\right)$, $q \in C\left([0, 1]; [0, +\infty)\right)$, $r \in C\left([0, 1]; \mathbb{R}\right)$ and there exist $x_0,\ x_1 \in [0, 1]$ such that $r(x_0)r(x_1) < 0$, $b_0,\ d_0,\ a_1,\ b_1$ are real constants such that

$$\begin{gathered} |b_0| + |d_0| > 0,\ b_0\,d_0 \geq 0 \text{ and if } b_0 = 0, \text{ then } q \not\equiv 0, \text{ and} \\ a_1 > 0,\ b_1 \leq 0. \end{gathered} \tag{1.4}$$

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The nonlinear term $g \in C\left([0,1] \times \mathbb{R}^3; \mathbb{R}\right)$ and satisfy the following conditions:

$$ug(x, u, s, 0\lambda) \leq 0,\ (x, u, s, \lambda) \in [0,1] \times \mathbb{R}^3, \tag{1.5}$$

$$g(x, u, s, \lambda) = o\left(|u| + |s|\right) \text{ as } |u| + |s| \to +\infty \tag{1.6}$$

uniformly in $(x, \lambda) \in [0,1] \times \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$.

Since the second half of the last century, bifurcation of solutions of nonlinear eigenvalue problems have been intensively studied. Note that bifurcation of nonlinear eigenvalue problems (for ordinary and partial differential equations) arise in the study of various problems of mechanics, physics, mathematical physics, biology, and also other areas of natural science (see [3, 5, 13, 15, 20] and bibliography therein); for example, the considered problem (1.1)-(1.3) arises when describing the selection-migration process in population genetics (see [12, 14]).

In the case when $r > 0$ and $a_1 = 0$ or $a_1 \neq 0$ the global bifurcation from zero or infinity of nontrivial solutions of problem (1.1)-(1.3) was considered before in [9, 12, 22-26]. These papers was shown the existence of two families of unbounded continua of nontrivial solutions in $\mathbb{R} \times C^1[0,1]$ emanating from bifurcation points and intervals of the line of trivial solutions or the line $\mathbb{R} \times \{\infty\}$ corresponding to the eigenvalues of the linear problem obtained from (1.1)-(1.3) by putting $g \equiv 0$. Moreover, these continua possess the usual nodal properties in some neighborhood of these bifurcation points. Similar global bifurcation results in nonlinear eigenvalue problems for ordinary differential equations of fourth order and one-dimensional Dirac system (both with a spectral parameter in the boundary condition, also without a spectral parameter in the boundary conditions) are obtained in [1-4, 6, 8].

In the case when the weight function r changes sign and $a_1 = 0$ in [7] and [21] was proved the existence of four families of unbounded continua of nontrivial solutions of problem (1.1)-(1.3) emanating from bifurcation points and intervals of the line of trivial solutions or the line $\mathbb{R} \times \{\infty\}$ corresponding to the eigenvalues of the linear problem (1.1)-(1.3) with $g \equiv 0$ and possessing the some nodal properties in some neighborhood of these bifurcation points. Similar results in nonlinear eigenvalue problems for elliptic partial differential equations of second order with indefinite weight were obtained in [5, 13, 19].

The papers [18] and [17] the author studied the global bifurcation from zero of nontrivial solutions to problem (1.1)-(1.3) in the cases: (a) the function g satisfies condition (1.6) as $|u|+|s| \to 0$, and (b) the function g can be represented in the form $g_1 + g_2$, where g_1 satisfies condition (a) and g_2 has sublinear growth with respect to the variable u. In these works it was proved that there exists four families of unbounded continua of nontrivial solutions in $\mathbb{R} \times C^1[0,1]$ emanating from bifurcation points and intervals of the line of trivial solutions corresponding to the eigenvalues of the linear problem (1.1)-(1.3) with $g \equiv 0$ and contained in the classes of functions with usual nodal properties.

In this paper, we study the global bifurcation from infinity of nontrivial solutions to problem (1.1)-(1.3).

The structure of this paper is as follows. In Section 2 we reduces problem (1.1)-(1.3) to the nonlinear operator equation, and prove that the corresponding nonlinear operator is asymptotically linear. In Section 3 using the approach used in [23] and [24], we transform this bifurcation from infinity problem to the bifurcation from zero problem with completely continuous operators. Next combining global bifurcation results in [14], [18] and [22] we prove the existence of global continua of solutions bifurcating from infinity which are similar to those obtained in [6] and [7].

2 Reduction of problem (1.1)-(1.3) to an operator equation and the necessary auxiliary results

Let $H = L_2(0,1) \oplus \mathbb{C}$ be a Hilbert space with the inner product

$$(\hat{y}, \hat{v})_H = (\{y, \alpha\}, \{v, \beta\}) = (y, v)_{L_2} + a_1^{-1} \alpha \, \bar{\beta},$$

where $(y, v)_{L_2}$ is a scalar product in $L_2(0,1)$. In H we define a linear operator

$$A\hat{y} = A\{y, \alpha\} = \{\ell(y), p(1)y'(1) - b_1 y(1)\},$$

on the domain

$$D(A) = \{\hat{y} = \{y, \alpha\} \in H : y,\, py' \in AC[0,1],\, \ell(y) \in L_2(0,1), \\ b_0 y(0) = d_0 p(0)\, y'(0), \alpha = a_1\, y(1)\}.$$

In view of condition (1.4) it follows from [16, §§ 2-3; 18, Lemma 2.1] (see also [10, Lemma 2.1]) that the operator A is self-adjoint and positive definite on $D(A)$.

We also define the operators $R : H \to H$ and $G : \mathbb{R} \times D(A) \to H$ as follows:

$$R\hat{y} = R\{y, \alpha\} = \{ry, \alpha\},\; G(\lambda, \hat{y}) = G(\lambda, \{y, \alpha\}) = \{g(\,\cdot\,, y,\, y', \lambda), 0\},$$

where $\alpha = a_1\, y'(1)$. Thus the nonlinear eigenvalue problem (1.1)-(1.3) is reduced to the following equivalent nonlinear eigenvalue problem

$$A\hat{y} = \lambda R\hat{y} + G(\lambda, \hat{y}),\; \hat{y} \in D(A), \tag{2.1}$$

i.e., between the solutions of problems (1.1)-(1.3) and (2.1) we can establish a one-to-one correspondence

$$(\lambda, y) \leftrightarrow (\lambda, \hat{y}),\; \hat{y} = \{y, \alpha\},\, \alpha = a_1 y(1). \tag{2.2}$$

We consider the linear eigenvalue problem

$$\begin{gathered} (\ell y)(x) = \lambda r(x) y(x),\; x \in (0,1), \\ b_0 y(0) = d_0 p(0) y'(0), \\ (a_1 \lambda + b_1) y(1) = p(1) y'(1). \end{gathered} \tag{2.3}$$

which obtained from (1.1)-(1.3) by setting $g \equiv 0$. It follows from [10, Theorem 3.2] that the eigenvalues of the linear problem (2.3) are all real, simple and form two unbounded sequences

$$0 < \lambda_1^+ < \lambda_2^+ < \,\ldots\, < \lambda_n^+ < \,\ldots$$

and

$$0 > \lambda_1^- > \lambda_2^- > \,\ldots\, > \lambda_n^- > \,\ldots\,.$$

For each $n \in \mathbb{N}$ the eigenfunction $y_n^+(x)$ ($y_n^-(x)$), corresponding to the eigenvalue λ_n^+ (λ_n^-), has exactly $n-1$ simple nodal zeros in $(0,1)$ (recall that the zero of a function is called a nodal zero if the function changes sign at this zero; the zero of a function is called a simple nodal zero if the derivative of the function at this zero is nonzero).

Note that the linear problem is reduces to the following operator equation

$$A\hat{y} = \lambda R\hat{y},\; \hat{y} \in D(A). \tag{2.4}$$

We introduce the notation:

$$bc_0 = \{y \in C^1[0,1] : b_0 y(0) = d_0 y'(0)\}.$$

By E we denote the Banach space $C^1[0,1] \cap bc_0$ with the norm

$$\|y\|_1 = \|y\|_\infty + \|y'\|_\infty, \;\; \|y\|_\infty = \max_{x\in[0,1]} |y(x)|.$$

Let $\hat{E} = E \oplus \mathbb{C}$ be the Banach space with the following norm

$$\|\hat{y}\|_1 = \|\{y, \alpha\}\|_1 = \|y\|_1 + |\alpha| \;.$$

Note that if $\{y, \alpha\} \in D(A)$, then by $p \in C^1[0,1]$ we get $y' \in AC[0,1]$. Consequently, we have $y \in C^1[0,1]$, which implies that

$$D(A) \subseteq \hat{E}.$$

Now by $\hat{E}^0$ we denote the Banach space $C^0[0,1] \oplus \mathbb{R}$ with the norm

$$\|\hat{y}\|_0 = \|\{y, \alpha\}\|_0 = \|y\|_0 + |\alpha|, \;\; \|y\|_0 = \max_{x\in[0,1]} |y(x)|.$$

By the definition of operators A, R, and G it follows that

$$A : \hat{E} \to \hat{E}^0, \;\; R : \hat{E}^0 \to \hat{E}^0, \;\; G : \mathbb{R} \times \hat{E} \to \hat{E}^0.$$

Since the operator A is positive definite on $D(A)$, it follows from Lemma 3.3 of [9] that there exists

$$\mathcal{A} = A^{-1} : \hat{E}^0 \to D(A)$$

which is a compact and continuous mapping. Then problem (2.1) we can rewrite in the following equivalent form

$$\hat{y} = \lambda \mathcal{A} R \hat{y} + \mathcal{A} G(\lambda, \hat{y}), \;\; \hat{y} \in D(A). \tag{2.5}$$

Denote:

$$\mathcal{R} = \mathcal{A} R \;\text{ and }\; \mathcal{G} = \mathcal{A} G.$$

Then, in turn, problem (2.5) can be rewritten as follows:

$$\hat{y} = \lambda \mathcal{R} \hat{y} + \mathcal{G}(\lambda, \hat{y}). \tag{2.6}$$

Note that

$$\mathcal{R} : \hat{E} \to \hat{E} \;\text{ and } \mathcal{G} : \mathbb{R} \times \hat{E} \to \hat{E}.$$

It follows from definition of operator R that

$$\|R\hat{y}\|_0 \le r_1 \|\hat{y}\|_1, \;\; r_1 = \max_{x\in[0,1]} |r(x)|,$$

and, therefore, the operator $\mathcal{R}$ is completely continuous due to the complete continuity of $\mathcal{A}$. Moreover, $\mathcal{G}$ is a continuous operator.

Lemma 2.1 *For any bounded interval $\Lambda \in \mathbb{R}$ the relations*

$$\mathcal{G}(\lambda, \hat{y}) = o\left(\|\hat{y}\|_1\right) \text{ as } \|\hat{y}\|_1 \to +\infty \tag{2.7}$$

holds uniformly in $\lambda \in \Lambda$.

Proof. By virtue of (1.6) for any sufficiently small $\varepsilon > 0$ there exists sufficiently large $\Delta_\varepsilon > 0$ such that for any $(x, u, s, \lambda) \in [0,1] \times \mathbb{R}^3$, $|u| + |s| > \Delta_\varepsilon$, $\lambda \in \Lambda$, the relation

$$|g(x, u, s, \lambda)| < \frac{\varepsilon\,(|u| + |s|)}{2\,\|\mathcal{A}\|} \tag{2.8}$$

holds. In other hand since $g \in C\left([0,1] \times \mathbb{R}^3; \mathbb{R}\right)$ it follows that there exists positive constant M_ε depending on ε such that for any $(x, u, s, \lambda) \in [0,1] \times \mathbb{R}^3$, $|u| + |s| \leq \Delta_\varepsilon$, $\lambda \in \Lambda$, the following inequality holds:

$$|g(x, u, s, \lambda)| \leq M_\varepsilon. \tag{2.9}$$

Let us choose a sufficiently large number $\Delta_{\varepsilon,1} > \Delta_\varepsilon$ so that

$$\frac{M_\varepsilon}{\Delta_{\varepsilon,1}} < \frac{\varepsilon}{2\,\|\mathcal{A}\|}\,. \tag{2.10}$$

Then, for any $\lambda \in \Lambda$ and $\hat{y} \in \hat{E}$ with $\|y\|_1 > \Delta_{\varepsilon,1}$, we have

$$\|\mathcal{G}(\lambda, y)\|_1 \leq \|\mathcal{A}\|\,\|G(\lambda, y)\|_0 = \|\mathcal{A}\| \max_{x\in[0,1]} |g(x, y(x), y'(x), \lambda)| \leq \|\mathcal{A}\|$$

$$\times \max\left\{ \max_{\substack{x\in[0,1],\\ |y(x)|+|y'(x)|\leq \Delta_\varepsilon,\\ \lambda\in\Lambda}} |g(x, y(x), y'(x), \lambda)|,\ \max_{\substack{x\in[0,1],\\ |y(x)|+|y'(x)|>\Delta_\varepsilon,\\ \lambda\in\Lambda}} |g(x, y(x), y'(x), \lambda)| \right\}$$

$$\leq \|\mathcal{A}\| \max\left\{M_\varepsilon, \frac{\varepsilon\,\|y\|_1}{2\,\|\mathcal{A}\|}\right\} \leq \|\mathcal{A}\| \max\left\{\frac{\varepsilon\,\Delta_{\varepsilon,1}}{2\,\|\mathcal{A}\|}, \frac{\varepsilon\,\|y\|_1}{2\,\|\mathcal{A}\|}\right\} \leq \frac{\varepsilon\,\|y\|_1}{2} < \varepsilon\,\|y\|_1. \tag{2.11}$$

The proof of this lemma is complete.

3 Global bifurcation from infinity of solutions to problem (1.1)-(1.3)

To study the global bifurcation of solutions to problem (1.1)-(1.3) we will use the subsets $S_n^{\sigma,\nu}$ and $\hat{S}_n^{\sigma,\nu}$, $n \in \mathbb{N}$, $\sigma \in \{+, -\}$, $\nu \in \{+, -\}$, of E and $\hat{E}$, respectively, with fixed oscillation count, which defined in [18, § 3]. Moreover, adding the points (λ, ∞), $\lambda \in \mathbb{R}$, to $\mathbb{R} \times \hat{E}$ and defining an appropriate topology on the resulting set, we obtain that (λ, ∞) is an element of $\mathbb{R} \times \hat{E}$.

Recall that problem (1.1)-(1.3) is equivalent to problem (2.6). For problem (2.6) we have the following global bifurcation result.

Theorem 3.1 *For each $n \in \mathbb{N}$, each $\sigma \in \{+, -\}$ and each $\nu \in \{+, -\}$ there exists an unbounded component $\hat{\mathcal{C}}_n^{\sigma,\nu}$ of the set of nontrivial solutions of* (2.6) *and the neighborhood $\hat{\mathcal{Q}}_n^{\sigma,\nu}$ of the point $(\lambda_n^\sigma, \infty)$ such that*

(i) $\hat{\mathcal{C}}_n^{\sigma,\nu} \subset \mathbb{R}^\sigma \times \hat{E}$, *where* $\mathbb{R}^+ = (0, +\infty)$ *and* $\mathbb{R}^- = (-\infty, 0)$;

(i) $\hat{\mathcal{C}}_n^{\sigma,\nu} \cap \hat{\mathcal{Q}}_n^{\sigma,\nu} \subset \mathbb{R}^\sigma \times \hat{S}_n^{\sigma,\nu}$;

(iii) *either* $\hat{\mathcal{C}}_n^{\sigma,\nu}$ *meets* $(\lambda_{n'}^\sigma, \infty)$ *with respect to the set* $\mathbb{R} \times \hat{S}_{n'}^{\sigma,\nu'}$ *for some* $(n', \nu') \neq (n, \nu)$*, or* $\hat{\mathcal{C}}_n^{\sigma,\nu}$ *meets* $(\lambda, \hat{0})$ *for some* $\lambda \in \mathbb{R}^\sigma$*, or projection* $P_{\mathbb{R}}^\sigma(\hat{\mathcal{C}}_n^{\sigma,\nu})$ *of* $\hat{\mathcal{C}}_n^{\sigma,\nu}$ *onto* $\mathbb{R}^\sigma \times \{\hat{0}\}$ *is unbounded, where* $\hat{0} = \{0, 0\}$.

Proof. To prove the theorem using the approach used in [23] and [24], we transform the bifurcation from infinity problem (2.6) to the bifurcation from zero problem. For this purpose, by following [24], we consider the following map

$$T : (\lambda, \hat{y}) \to (\lambda, \hat{v}) = \left(\lambda, \frac{\hat{y}}{\|\hat{y}\|_1^2}\right), \, (\lambda, \hat{y}) \in \mathbb{R} \times \hat{E}, \, \hat{y} \neq \hat{0}. \tag{3.1}$$

Let $\hat{\mathcal{C}}$ be the set of nontrivial solutions of problem (2.6). If $(\lambda, \hat{y}) \in \hat{\mathcal{C}}$, then we have the following relations

$$\|\hat{v}\|_1 = \frac{1}{\|\hat{y}\|_1}, \ \|\hat{y}\|_1 = \frac{1}{\|\hat{v}\|_1} \ \text{and} \ \hat{y} = \frac{\hat{v}}{\|\hat{v}\|_1^2}. \tag{3.2}$$

Hence under the inverse transformation T^{-1}, we get

$$T^{-1} : (\lambda, \hat{v}) \to (\lambda, \hat{y}) = \left(\lambda, \frac{\hat{v}}{\|\hat{v}\|_1^2}\right), \tag{3.3}$$

Dividing both parts (2.6) by $\|\hat{y}\|_1^2$ and using (3.2), we get

$$\hat{v} = \lambda \mathcal{R} \hat{v} + \|\hat{v}\|_1^2 \, \mathcal{G}\left(\lambda, \frac{\hat{v}}{\|\hat{v}\|_1^2}\right). \tag{3.4}$$

Let $\hat{\mathcal{G}} : \mathbb{R} \times \hat{E} \to \hat{E}$ be the continuous operator defined by

$$\hat{\mathcal{G}}(\lambda, \hat{v}) = \begin{cases} \|\hat{v}\|_1^2 \, \mathcal{G}\left(\lambda, \frac{v}{\|\hat{v}\|_1^2}\right) & \text{if } \hat{v} \neq \hat{0}, \\ 0 & \text{if } \hat{v} = \hat{0}. \end{cases}$$

(the continuity of this operator for $v = 0$ follows from Lemma 2.1). Then problem (3.4) takes the following form

$$\hat{v} = \lambda \mathcal{R} \hat{v} + \hat{\mathcal{G}}\left(\lambda, \hat{v}\right). \tag{3.5}$$

We now prove that the operator $\hat{\mathcal{G}}$ is completely continuous and satisfies the condition:

$$\hat{\mathcal{G}}(\lambda, \hat{v}) = o\left(\|\hat{v}\|_1\right) \text{ as } \|\hat{v}\|_1 \to 0, \tag{3.6}$$

uniformly in $\lambda \in \Lambda$ for any bounded interval $\Lambda \subset \mathbb{R}$. Indeed, by (2.7) (see Lemma 2.1) for any small $\varepsilon > 0$ there exists a sufficiently large $\hat{\Delta}_{\varepsilon,1} > 0$ such that

$$\frac{\|\mathcal{G}(\lambda, \hat{y})\|_1}{\|y\|_1} < \varepsilon \ \text{ for any } \hat{y} \in \hat{E}, \ \|\hat{y}\|_1 > \hat{\Delta}_{\varepsilon,1} \text{ and } \lambda \in \Lambda. \tag{3.7}$$

Then for any $(\lambda, v) \in \mathbb{R} \times \hat{E}$ such that $\lambda \in \Lambda$ and $\|\hat{v}\|_1 < \hat{\delta}_{\varepsilon,1} = \hat{\Delta}_{\varepsilon,1}^{-1}$ we get

$$\frac{\|\hat{\mathcal{G}}(\lambda, \hat{v})\|_1}{\|\hat{v}\|_1} = \frac{\|\hat{v}\|_1^2 \left\|\mathcal{G}\left(\lambda, \frac{\hat{v}}{\|\hat{v}\|_1^2}\right)\right\|_1}{\|\hat{v}\|_1} = \frac{\left\|\mathcal{G}\left(\lambda, \frac{\hat{v}}{\|\hat{v}\|_1^2}\right)\right\|_1}{\frac{1}{\|\hat{v}\|_1}} = \frac{\|\mathcal{G}(\lambda, \hat{y})\|_1}{\|\hat{y}\|_1} < \varepsilon, \tag{3.8}$$

which implies (3.6).

Let $\hat{B}_\delta$ be an open ball in $\hat{E}$ of radius δ centered at $\hat{0}$, and $\overline{\hat{B}}_\delta$ is the closure of this ball.

Now we will fix the number $\varepsilon > 0$. Then, by (3.8), for any $(\lambda, \hat{v}) \in \Lambda \times \overline{B}_{\delta_{\varepsilon,1}}$ we get the following estimate

$$\|\hat{\mathcal{G}}(\lambda, \hat{v})\|_1 = \frac{\|\hat{\mathcal{G}}(\lambda, \hat{v})\|_1}{\|\hat{v}\|_1} \|\hat{v}\|_1 < \varepsilon \, \|\hat{v}\|_1 < \varepsilon \, \hat{\delta}_{\varepsilon,1}, \tag{3.9}$$

i.e., the set $\hat{\mathcal{G}}(\Lambda \times \overline{B}_{\delta_\varepsilon})$ is bounded in $\hat{E}$. Since for any $\hat{v} \neq \hat{0}$

$$\hat{\mathcal{G}}(\lambda, \hat{v}) = \|\hat{v}\|_1^2 \,\mathcal{G}\left(\lambda, \frac{\hat{v}}{\|\hat{v}\|_1^2}\right) = \|\hat{v}\|_1^2 A^{-1} G\left(\lambda, \frac{\hat{v}}{\|\hat{v}\|_1^2}\right)$$

it follows that $\hat{w} = \hat{\mathcal{G}}(\lambda, \hat{v})$ ($\hat{w} = (w, a_1 w(1))$ satisfies the equation

$$A\hat{w} = \|\hat{v}\|_1^2 G\left(\lambda, \frac{\hat{v}}{\|\hat{v}\|_1^2}\right).$$

Hence the function $w(x)$ satisfies the following differential equation

$$-(p\,(x)\,w(x)')' + q(x)w(x) = \|\hat{v}\|_1^2 \,g\left(x, \frac{v(x)}{\|\hat{v}\|_1^2}, \frac{v'(x)}{\|\hat{v}\|_1^2}, \lambda\right),\ x \in (0,1). \tag{3.10}$$

We rewrite the equation in the following form

$$w''(x) = \frac{1}{p(x)}\left\{p'(x)w(x) - q(x)w(x) + \|\hat{v}\|_1 \,\frac{g(x,y(x),y'(x),\lambda)}{\|\hat{y}\|_1}\right\},\ x \in (0,1). \tag{3.11}$$

Since $\|\hat{v}\|_1 < \hat{\delta}_{\varepsilon,1}$ it follows that $\|\hat{y}\|_1 = \frac{1}{\|\hat{v}\|_1} > \hat{\delta}_{\varepsilon,1}^{-1} = \hat{\Delta}_{\varepsilon,1}$, and consequently, by (2.11) we have

$$\frac{g\,(x, y(x), y'(x), \lambda)}{\|\hat{y}\|_1} < \varepsilon. \tag{3.12}$$

By virtue of (3.9) and (3.12) from (3.11) we get the following inequality

$$|w''(x)| \leq p_0^{-1}\{(\|p\,\|_1 + \|q\|_\infty)\|w\|_1 + \varepsilon\,\delta_{\varepsilon,1}\}$$

$$\leq p_0^{-1}(\|p\,\|_1 + \|q\|_\infty) + 1)\,\varepsilon\,\delta_{\varepsilon,1} \leq p_0^{-1}(\|p\,\|_1 + \|q\|_\infty) + 1),$$

where $p_0 = \min\limits_{x\in[0,1]} p(x)$. Then, based on the Arzela-Ascoli theorem, we can assert that the set

$$\left\{\hat{\mathcal{G}}(\lambda, \hat{v}) : \lambda \in \Lambda,\ v \in \overline{B}_{\delta_{\varepsilon,1}}\right\}$$

is precompact in $\hat{E}$, which means that the operator $\hat{\mathcal{G}} : \mathbb{R} \times E \to \hat{E}$ is completely continuous.

In view of relation (3.6), by [20, Ch. 4, § 2, Theorem 2.1] problem (3.5) is linearizable, and the corresponding linearization of this problem at $\hat{v} = \hat{0}$ is given by

$$\hat{v} = \lambda \mathcal{R}\hat{v}. \tag{3.13}$$

It is obvious that the linear problem (3.13) is equivalent to the linear problem (2.4) (or (2.3)), so that all eigenvalues of this problem are real and simple.

Since the operator $\hat{\mathcal{G}}$ is completely continuous which satisfies the condition (3.6) we can apply Theorem 2 of [14] to nonlinear eigenvalue problem (3.5). Then by this theorem, [18, Lemma 4.2] (which holds by condition (1.5)) and [22, Lemma 1.24] for each $n \in \mathbb{N}$ and each $\sigma \in \{+\,,\,-\}$ there exist components $\tilde{\mathcal{C}}_n^{\sigma,+}$ and $\tilde{\mathcal{C}}_n^{\sigma,-}$ of the set $\tilde{\mathcal{C}}$ of nontrivial solutions of problem (3.5), and neighborhoods $\tilde{\mathcal{Q}}_n^{\sigma,+}$ and $\tilde{\mathcal{Q}}_n^{\sigma,-}$ of the point $(\lambda_n^\sigma, 0)$ such that

(1) $\tilde{\mathcal{C}}_n^{\sigma,+} \subset \mathbb{R}^\sigma \times E$ and $\tilde{\mathcal{C}}_n^{\sigma,-} \subset \mathbb{R}^\sigma \times E$;

(2) $\tilde{\mathcal{C}}_n^{\sigma,+} \cap \tilde{\mathcal{Q}}_n^{\sigma,+} \subset \mathbb{R}^\sigma \times \hat{S}_n^{\sigma,+}$ and $\tilde{\mathcal{C}}_n^{\sigma,-} \cap \tilde{\mathcal{Q}}_n^{\sigma,-} \subset \mathbb{R}^\sigma \times \hat{S}_n^{\sigma,-}$;

(3) either $\tilde{\mathcal{C}}_n^{\sigma,+}$ and $\tilde{\mathcal{C}}_n^{\sigma,-}$ are unbounded in $\mathbb{R}^\sigma \times \hat{E}$ or $\tilde{\mathcal{C}}_n^{\sigma,+} \cap \tilde{\mathcal{C}}_n^{\sigma,-} \neq (\lambda_n^\sigma, \hat{0})$.
Statement (3) shows that for the set $\tilde{\mathcal{C}}_n^{\sigma,+}$ (respectively, $\tilde{\mathcal{C}}_n^{\sigma,-}$) at least one of the following holds:

(3. a) $\tilde{\mathcal{C}}_n^{\sigma,+}$ (respectively, $\tilde{\mathcal{C}}_n^{\sigma,-}$) is unbounded in $\mathbb{R}^\sigma \times \hat{E}$ in which case either $\tilde{\mathcal{C}}_n^{\sigma,+}$ (respectively, $\tilde{\mathcal{C}}_n^{\sigma,-}$) meets (λ, ∞) for some $\lambda \in \mathbb{R}^\sigma$, or $\mathrm{pr}_{\mathbb{R}^\sigma \times \{\hat{0}\}}(\tilde{\mathcal{C}}_n^{\sigma,+})$ (respectively, $\mathrm{pr}_{\mathbb{R}^\sigma \times \{\hat{0}\}}(\tilde{\mathcal{C}}_n^{\sigma,-})$) is unbounded;

(3. b) $\tilde{\mathcal{C}}_n^{\sigma,+}$ (respectively, $\tilde{\mathcal{C}}_n^{\sigma,-}$) intersects with the set $\tilde{\mathcal{C}}_n^{\sigma,-}$ (respectively, $\tilde{\mathcal{C}}_n^{\sigma,+}$) for some (λ, u), $u \neq 0$, in which case $\tilde{\mathcal{C}}_n^{\sigma,+}$ (respectively, $\tilde{\mathcal{C}}_n^{\sigma,-}$) meets $(\lambda_n^\sigma, \hat{0})$ with respect to the set $\mathbb{R}^\sigma \times \hat{S}_n^{\sigma,-}$ (respectively, $\mathbb{R}^\sigma \times \hat{S}_n^{\sigma,+}$);

(3. c) $\tilde{\mathcal{C}}_n^{\sigma,+} \cap \tilde{\mathcal{C}}_n^{\sigma,-} = (\lambda_{n'}^\sigma, \hat{0})$ for some $n \neq n' \in \mathbb{N}$. Then, by statement (2), in this case $\tilde{\mathcal{C}}_n^{\sigma,+}$ (respectively, $\tilde{\mathcal{C}}_n^{\sigma,-}$) meets $(\lambda_{n'}^\sigma, \hat{0})$ with respect to the set $\mathbb{R}^\sigma \times \hat{S}_{n'}^{\sigma,\nu'}$ for some $\nu' \in \{+, -\}$.

By construction, the inversion (3.3) (i.e. inversion $(\lambda, \hat{v}) \to T^{-1}(\lambda, \hat{y})$) maps $\tilde{\mathcal{C}}$ into $\hat{\mathcal{C}}$. For each $n \in \mathbb{N}$, each $\sigma \in \{+, -\}$, and each $\nu \in \{+, -\}$ we denote by $\hat{\mathcal{C}}_n^{\sigma,\nu}$ and $\hat{\mathcal{Q}}_n^{\sigma,\nu}$ the inverse images $T^{-1}(\tilde{\mathcal{C}}_n^{\sigma,\nu})$ and $T^{-1}(\tilde{\mathcal{Q}}_n^{\sigma,\nu})$ of $\tilde{\mathcal{C}}_n^{\sigma,\nu}$ and $\tilde{\mathcal{Q}}_n^{\sigma,\nu}$, respectively, under the inversion T. Since $\tilde{\mathcal{Q}}_n^{\sigma,\nu}$ is a neighborhood of $(\lambda_n^\sigma, \hat{0})$ it follows that $\hat{\mathcal{Q}}_n^{\sigma,\nu} \subset \mathbb{R}^\sigma \times \hat{E}$ is a neighborhood of $(\lambda_n^\sigma, \infty) \in \mathbb{R}^\sigma \times \hat{E}$. Now the statements (i)-(iii) of the theorem for $\hat{\mathcal{C}}_n^{\sigma,\nu}$ correspond, via T^{-1}, to the statements (1)-(3) for $\tilde{\mathcal{C}}_n^{\sigma,\nu}$ ((here, three alternatives of statement (iii) of the theorem for $\hat{\mathcal{C}}_n^{\sigma,\nu}$ correspond to alternatives (3.a)-(3.c) of statement (3) for $\tilde{\mathcal{C}}_n^{\sigma,\nu}$). The proof of this theorem is complete.

Recall that between the solutions of problems (1.1)-(1.3) and (2.6) (or (2.1)) there is a one-to-one correspondence (2.2). Therefore, Theorem 3.1 immediately implies the following result.

Theorem 3.2 *For each $n \in \mathbb{N}$, each $\sigma \in \{+, -\}$ and each $\nu \in \{+, -\}$ there exists an unbounded component $\mathcal{C}_n^{\sigma,\nu}$ of the set of nontrivial solutions of* (1.1)-(1.3) *and the neighborhood $\mathcal{Q}_n^{\sigma,\nu}$ of the point $(\lambda_n^\sigma, \infty)$ such that*

(i) $\mathcal{C}_n^{\sigma,\nu} \subset \mathbb{R}^\sigma \times E$;

(i) $\mathcal{C}_n^{\sigma,\nu} \cap \mathcal{Q}_n^{\sigma,\nu} \subset \mathbb{R}^\sigma \times S_n^{\sigma,\nu}$;

(iii) *either $\mathcal{C}_n^{\sigma,\nu}$ meets $(\lambda_{n'}^\sigma, \infty)$ with respect to the set $\mathbb{R} \times S_{n'}^{\sigma,\nu'}$ for some $(n', \nu') \neq (n, \nu)$, or $\mathcal{C}_n^{\sigma,\nu}$ meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}^\sigma$, or projection $P_{\mathbb{R}^\sigma \times \{0\}}(\mathcal{C}_n^{\sigma,\nu})$ of $\mathcal{C}_n^{\sigma,\nu}$ onto $\mathbb{R}^\sigma \times \{0\}$ is unbounded.*

Remark 3.1 Following the corresponding reasoning carried out in [6] and [23], we can give examples illustrating all possible cases stated in Theorems 3.1 and 3.2.

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