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# On a boundary value problem with spectral parameter quadratically contained in the boundary condition

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**Abstract.** *A boundary value problem generated on an interval by a diffusion equation with real coefficients and nonseparated boundary conditions is considered. One of these boundary conditions includes the quadratic function of the spectral parameter. Some spectral properties of the boundary value problem are studied. It is proved that the eigenvalues are real and nonzero and that there are no associated functions to the eigenfunctions, and an asymptotic formula for the spectrum of the problem is derived.*

**Keywords.** diffusion equation, nonseparated boundary conditions, eigenvalues, asymptotics.

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## 1 Introduction

Consider the boundary value problem generated on the interval $[0,\pi]$ by the diffusion differential equation

$$y'' + [\lambda^2 - 2\lambda p(x) - q(x)]y = 0 \tag{1.1}$$

and boundary conditions

$$\begin{array}{l}(m\lambda^2 + \alpha\lambda + \beta)y(0) + y'(0) + \omega y(\pi) = 0,\\ -\bar{\omega}y(0) + \gamma y(\pi) + y'(\pi) = 0,\end{array} \tag{1.2}$$

where the functions $p(x) \in W_2^1[0,\pi]$, $q(x) \in L_2[0,\pi]$ are real, $\lambda$ is a spectral parameter, $\omega$ is a complex number, $\bar{\omega}$ is the complex conjugate of $\omega$, $m, \alpha,\ \beta,\ \gamma$ are the real numbers. We denote by $W_2^n[0,\pi]$ the S.L. Sobolev space of functions $f(x),\ x \in [0,\pi]$, where the functions $f^{(m)}(x),\ m = 0, 1, 2, \dots, n-1$, are absolutely continuous and $f^{(n)}(x) \in L_2[0,\pi]$. Problem (1.1) - (1.2) will be denoted by $P$.

For $\omega = 0$, the boundary conditions (1.2) are separated. In this case, the spectral properties of the Sturm-Liouville and diffusion operators were studied in [2–5, 9, 13, 16–18, 20] and other works. In [1, 6–8, 10–12, 15, 16, 19, 21] direct and inverse spectral problems for equation (1.1) (for $p(x) \equiv 0$ and $p(x) \not\equiv 0$) with various types of nonseparated boundary conditions are investigated.

In this paper some spectral properties of the boundary value problem $P$ in case $m\omega \neq 0$, when one of the nonseparated boundary conditions contains a quadratic function of the spectral parameter are studied. It is proved that the eigenvalues are real and nonzero and that there are no associated functions to the eigenfunctions, and an asymptotic formula for the spectrum of the problem $P$ is derived.

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## 2 Some spectral properties of the boundary value problem $P$

In this section we will assume everywhere that $m > 0$ and the following condition is satisfied: for all functions $y(x) \in W_2^2[0, \pi]$, $y(x) \not\equiv 0$ satisfying conditions (1.2), the following inequality holds:

$$\begin{aligned} &Q = \gamma\left|y(\pi)\right|^2 - 2\omega Re\left[\overline{y(0)}\, y(\pi)\right] - \beta\left|y(0)\right|^2 \\ &+ \int_0^\pi \left\{\left|y'(x)\right|^2 + q(x)\left|y(x)\right|^2\right\} dx > 0. \end{aligned} \tag{2.1}$$

Note that inequality (2.1) is certainly satisfied if

$$\beta \le 0,\ \gamma \ge 0,\ |\omega| \le \sqrt{|\beta|\,\gamma},\ q(x) > 0.$$

Indeed, for $q(x) > 0$ the integral in (2.1) is positive. It's clear that

$$Re\left[\omega\overline{y(0)}y(\pi)\right] \le |\omega| \cdot |y(0)| \cdot |y(\pi)|.$$

Then the expression in (2.1) outside the integral is nonnegative, since for $\beta \le 0,\ \gamma \ge 0,\ |\omega| \le \sqrt{|\beta|\,\gamma}$ we have

$$-\beta\left|y(0)\right|^2 - 2Re\left[\omega\overline{y(0)}y(\pi)\right] + \gamma\left|y(\pi)\right|^2$$

$$\ge -\beta\left|y(0)\right|^2 - 2\left|\omega\right| \cdot \left|y(0)\right| \cdot \left|y(\pi)\right| + \gamma\left|y(\pi)\right|^2$$

$$\ge |\beta| \cdot \left|y(0)\right|^2 - 2\sqrt{|\beta|\,\gamma}\left|y(0)\right| \cdot \left|y(\pi)\right| + \gamma\left|y(\pi)\right|^2$$

$$= \left[\sqrt{|\beta|}\left|y(0)\right| - \sqrt{\gamma}\left|y(\pi)\right|\right]^2 \ge 0.$$

**Definition 2.1** *A complex number $\lambda_0$ is called an eigenvalue of a boundary value problem $P$, if the equation* (1.1) *has a nontrivial solution $y_0(x)$ for $\lambda = \lambda_0$ that satisfies boundary conditions* (1.2)*; in this case $y_0(x)$ is called the eigenfunction of the problem $P$ which corresponds to the eigenvalue $\lambda_0$. The set of eigenvalues is called the spectrum of the problem $P$. Functions*

$$y_1(x), y_2(x), \ldots, y_r(x)$$

*are called associated functions of the eigenfunction $y_0(x)$ if these functions have an absolutely continuous derivative and satisfy the differential equations*

$$y_j''(x) + \left[\lambda_0^2 - 2\lambda p(x) - q(x)\right] y_j(x) + [2\lambda_0 - 2p(x)]y_{j-1}(x) + y_{j-2}(x) = 0$$

*and boundary conditions*

$$\begin{aligned} &(m\lambda_0^2 + \alpha\lambda_0 + \beta)y_j(0) + y_j'(0) + \omega\, y_j(\pi) + (2m\lambda_0 + \alpha)y_{j-1}(0) + my_{j-2}(0) = 0, \\ &-\bar{\omega}y_j'(0) + \gamma y_j(\pi) + y_j'(\pi) = 0, \\ &j = 1, 2, 3, ..., r \quad (y_{-1}(x) \equiv 0). \end{aligned} \tag{2.2}$$

**Theorem 2.1** *The eigenvalues of the boundary value problem $P$ are real and nonzero.*

**Proof.** Let $\lambda_0$ be the eigenvalue of the problem $P$ and $y_0\left(x\right)$ be the corresponding eigenfunction. We put

$$ly_0 = -y_0''(x) + q(x)y_0(x).$$

We denote by $(f,g)$ the usual scalar product of functions $f\left(x\right)$ and $g\left(x\right)$ in space $L_2[0,\pi]$:

$$(f,g) = \int_0^\pi f(x)\overline{g(x)}dx.$$

Scalarly multiplying both sides of the equality

$$y_0''(x) + ({\lambda_0}^2 - 2\lambda_0 p(x) - q(x))y_0(x) = 0$$

by $y_0(x)$, we get

$$(y_0'', y_0) + {\lambda_0}^2(y_0, y_0) - 2\lambda_0(py_0, y_0) - (qy_0, y_0) = 0$$

The last equality can be rewritten as

$${\lambda_0}^2(y_0, y_0) - 2\lambda_0(py_0, y_0) - (ly_0, y_0) = 0, \tag{2.3}$$

It is obvious that

$$(ly_0, y_0) = \int_0^\pi \left(-y_0''\left(x\right) + q(x)y_0(x)\right)\overline{y_0\left(x\right)}dx$$

$$= -\int_0^\pi y_0''\left(x\right)\overline{y_0\left(x\right)}dx + \int_0^\pi q(x)\left|y_0(x)\right|^2 dx. \tag{2.4}$$

Applying the formula of integration by parts to the integral $\int_0^\pi y_0''\left(x\right)\overline{y_0\left(x\right)}dx$, we have

$$\int_0^\pi \overline{y_0\left(x\right)}d(y_0'(x)) = \overline{y_0(x)}y_0'(x)\Big|_0^\pi - \int_0^\pi y_0'(x)d(\overline{y_0(x)})$$

$$= \overline{y_0(\pi)}y'(\pi) - \overline{y_0(0)}y'(0) - \int_0^\pi \left|y_0'(x)\right|^2 dx.$$

Therefore, relation (2.4) can be written as follows:

$$(ly_0, y_0) = y_0'\left(0\right)\overline{y_0\left(0\right)} - y_0'\left(\pi\right)\overline{y_0\left(\pi\right)} + \int_0^\pi \left(\left|y_0'\left(x\right)\right|^2 + q\left(x\right)\left|y_0\left(x\right)\right|^2\right)dx. \tag{2.5}$$

According to the boundary conditions (1.2)

$$\begin{array}{l} y_0'(0) = -\omega\, y_0(\pi) - (m{\lambda_0}^2 + \alpha\lambda_0 + \beta)y_0(0), \\ y_0'(\pi) = \bar{\omega}y_0(0) - \gamma y_0(\pi). \end{array}$$

Then

$$y_0'\left(0\right)\overline{y_0\left(0\right)} - y_0'\left(\pi\right)\overline{y_0\left(\pi\right)} = \overline{y_0\left(0\right)}\left(-\omega\, y_0(\pi) - (m{\lambda_0}^2 + \alpha\lambda_0 + \beta)y_0(0)\right)$$

$$-\overline{y_0\left(\pi\right)}\left(\bar{\omega}y_0(0) - \gamma y_0(\pi)\right) = -\overline{y_0\left(0\right)}\omega y_0(\pi) - \left|y_0(0)\right|^2\left(m{\lambda_0}^2 + \alpha\lambda_0 + \beta\right)$$

$$-\overline{y_0\left(\pi\right)}\overline{\omega}y_0(0) + \gamma\left|y_0(\pi)\right|^2 = -\left|y_0(0)\right|^2\left(m{\lambda_0}^2 + \alpha\lambda_0 + \beta\right)$$

$$-2Re(\omega\overline{y(0)}y(\pi)) + \gamma\left|y_0(\pi)\right|^2.$$

Taking into account the last relation in (2.5), we have

$$(ly_0, y_0) = -\left|y_0(0)\right|^2\left(m{\lambda_0}^2 + \alpha\lambda_0 + \beta\right) - 2Re(\omega\overline{y(0)}y(\pi)) + \gamma\left|y_0(\pi)\right|^2 + A, \tag{2.6}$$

where $A = \int_0^\pi \left[ |y_0' \ (x)|^2 + q(x) \ |y_0(x)|^2 \right] dx$. Substituting (2.6) into (2.3), we obtain

$$\lambda_0{}^2(y_0, y_0) - 2\lambda_0(py_0, y_0) + |y_0(0)|^2 \left(m\lambda_0^2 + \alpha\lambda_0 + \beta\right) + 2Re(\omega\overline{y(0)}y(\pi))$$

$$-\gamma\,|y_0(\pi)|^2 - A = \lambda_0{}^2(y_0, y_0) - 2\lambda_0(py_0, y_0) + m\lambda_0^2\,|y_0(0)|^2 + \alpha\lambda_0\,|y_0(0)|^2$$

$$+\,|\beta y_0(0)|^2 + 2Re(\omega\overline{y(0)}y(\pi)) - \gamma\,|y_0(\pi)|^2 - A = 0$$

or

$$\lambda_0{}^2\left[(y_0, y_0) + m\,|y_0(0)|^2\right] - \lambda_0\left[2(py_0, y_0) - \alpha\,|y_0(0)|^2\right] - [-\beta\,|y_0(0)|$$

$$-2Re(\omega\overline{y(0)}\ y(\pi)) + \gamma\,|y_0(\pi)|^2 + A\Big] = 0. \qquad (2.7)$$

We denote

$$V = (y_0, y_0) + m\,|y_0(0)|^2\,, R = 2(py_0, y_0) - \alpha\,|y_0(0)|^2\,, \qquad (2.8)$$

Taking into account (2.1) and (2.8), from (2.7) we obtain the following quadratic equation for $\lambda_0$:

$$V\lambda_0^2 - R\lambda_0 - Q = 0. \qquad (2.9)$$

It follows from inequalities $m > 0$ and (2.1) that $VQ > 0$, and therefore the discriminant $R^2 + 4VQ$ of the quadratic equation (2.9) is positive. Therefore, the roots of equation (2.9) are real and nonzero. The theorem is proved.

**Corollary 2.1** *If $y_0\,(x)$ is the eigenfunction of the problem $P$ corresponding to the eigenvalue $\lambda_0$, then*

$$2V\lambda_0 - R \neq 0, \qquad (2.10)$$

*where $V$ and $R$ are determined by equalities* (2.8)*. Moreover, the sign of the left side of this inequality coincides with the sign of $\lambda_0$:*

$$\text{sign}\,(2V\lambda_0 - R) = \text{sign}\lambda_0. \qquad (2.11)$$

**Proof.** Solving equation (2.9), we obtain

$$\lambda_0 = \frac{R \pm \sqrt{R^2 + 4VQ}}{2V}. \qquad (2.12)$$

Since $R^2 + 4VQ > 0$, it follows from (2.12) that

$$2V\lambda_0 - R = \pm\sqrt{R^2 + 4VQ} \neq 0.$$

Therefore, (2.10) holds. It is also clear from (2.12), that for $\lambda_0 > 0$ there must be a "+" sign in front of the root, and for $\lambda_0 < 0$ there must be a "-" sign. From here we find that the sign of the expression $2V\lambda_0 - R$ coincides with the sign of $\lambda_0$, i.e. equality (2.11) is true, which should have been proved.

**Theorem 2.2** *The boundary value problem $P$ has no associated functions of the eigenfunctions.*

**Proof.** Let us assume the opposite. Let us suppose there is an associated function $y_1(x)$ of the eigenfunction $y_0(x)$ of problem $P$ , which corresponds to the eigenvalue $\lambda_0$. Then the equalities

$$y_0''(x) + \left[\lambda_0^2 - 2\lambda_0 p(x) - q(x)\right] y_0(x) = 0, \tag{2.13}$$

$$y_1''(x) + \left[\lambda_0^2 - 2\lambda_0 p(x) - q(x)\right] y_1(x) + \left[2\lambda_0 - 2p(x)\right] y_0(x) = 0 \tag{2.14}$$

hold.

Let us pass in equality (2.13) to the complex conjugate and then multiply the resulting equality by $y_1(x)$, and multiply relation (2.14) by $\overline{y_0(x)}$.

$$\overline{y_0''(x)} y_1(x) + \left[\lambda_0^2 - 2\lambda_0 p(x) - q(x)\right] \overline{y_0(x)} y_1(x) = 0,$$

$$y_1''(x)\,\overline{y_0(x)} + \left[\lambda_0^2 - 2\lambda_0 p(x) - q(x)\right] \overline{y_0(x)} y_1(x) + \left[2\lambda_0 - 2p(x)\right] y_0(x)\overline{y_0(x)} = 0.$$

Subtract the second result from the first:

$$2\left[\lambda_0 - p(x)\right] y_0(x)\overline{y_0(x)} = \overline{y_0''(x)} y_1(x) - y_1''(x)\overline{y_0(x)}.$$

The last equality can be rewritten as

$$2\left[\lambda_0 - p(x)\right] |y_0(x)|^2 = \frac{d}{dx}\left[\overline{y_0'(x)} y_1(x) - y_1'(x)\overline{y_0(x)}\right].$$

After integrating this relation over $x$ from 0 to $\pi$, we get

$$2\int_0^{\pi} \left[\lambda_0 - p(x)\right] |y_0(x)|^2 dx = \left[\overline{y_0'(x)} y_1(x) - y_1'(x)\overline{y_0(x)}\right] \Big|_0^{\pi}$$

$$= \overline{y_0'(\pi)} y_1(\pi) - \overline{y_0'(0)} y_1(0) - y_1'(\pi)\overline{y_0(\pi)} + y_1'(0)\overline{y_0(0)}. \tag{2.15}$$

From the boundary conditions (1.2) and (2.2) for $y_0(x)$ and $y_1(x)$ we find $y_0'(0)$, $y_0'(\pi)$, $y_1'(0)$, $y_1'(\pi)$ and substitute in (2.15):

$$2\int_0^{\pi} \left[\lambda_0 - p(x)\right] |y_0(x)|^2 dx = \left(\omega\overline{y_0(0)} - \gamma\overline{y_0(\pi)}\right) y_1(\pi) + \left[\overline{(m{\lambda_0}^2 + \alpha\lambda_0 + \beta) y_0(0)}\right.$$

$$\left. + \overline{\bar\omega y_0(\pi)}\right] y_1(0) - \left[\bar\omega y_1(0) - \gamma y_1(\pi)\right] \overline{y_0(\pi)} - \left[(m{\lambda_0}^2 + \alpha\lambda_0 + \beta) y_1(0) + \omega y_1(\pi)\right.$$

$$\left. + (2m\lambda_0 + \alpha) y_0(0)\right] \overline{y_0(0)} = \omega\overline{y_0(0)} y_1(\pi) - \gamma\overline{y_0(\pi)} y_1(\pi) + m\lambda_o^2\overline{y_0(0)} y_1(0)$$

$$+ \alpha\overline{y_0(0)} y_1(0) + \overline{\omega}\overline{y_0(\pi)} y_1(0) + \beta y_1(0)\overline{y_0(0)} - \overline{\omega} y_1(0)\overline{y_0(\pi)} + \gamma y_1(\pi)\overline{y_0(\pi)}$$

$$- m\lambda_0^2 y_1(0)\overline{y_0(0)} - \alpha\lambda_0 y_1(0)\overline{y_0(0)} - \beta y_1(0)\overline{y_0(0)} - \omega y_1(\pi)\overline{y_0(0)}$$

$$- 2m\lambda_0 y_0(0)\overline{y_0(0)} - \alpha y_0(0)\overline{y_0(0)} = -(2m\lambda_0 + \alpha)|y_0(0)|^2.$$

From this we get

$$2\int_0^{\pi} \left[\lambda_0 - p(x)\right] |y_0(x)|^2 dx + (2m\lambda_0 + \alpha)|y_0(0)|^2 = 0$$

or $2V\lambda_0 - R = 0$ (see [11]), which contradicts the inequality (2.10).

The theorem is proved.

## 3 Asymptotics of the eigenvalues

Let $c(x,\lambda)$, $s(x,\lambda)$ be the fundamental system of solutions of equation (1.1), determined by the initial conditions

$$c(0,\lambda)=s'(0,\lambda)=1, c'(0,\lambda)=s(0,\lambda)=0. \tag{3.1}$$

For any $x$ the functions $c(x,\lambda)$, $s(x,\lambda)$, $c'(x,\lambda)$, $s'(x,\lambda)$ are entire functions (of exponential type) of the variable $\lambda$. The general solution of equation (1.1) is written as

$$y\left(x,\,\lambda\right)=A_1c\left(x,\,\lambda\right)+A_2s\left(x,\,\lambda\right), \tag{3.2}$$

where $A_1,\ A_2-$ are arbitrary constants. Taking into account the initial conditions (3.1), we obtain

$$y\left(0,\,\lambda\right)=A_1c\left(0,\,\lambda\right)+A_2s\left(0,\,\lambda\right)=A_1,$$
$$y'\left(0,\,\lambda\right)=A_1c'\left(0,\,\lambda\right)+A_2s'\left(0,\,\lambda\right)=A_2.$$

Substituting function (3.2) into boundary conditions (1.2) and using the last relations, we obtain the following system for $A_1$ and $A_2$ :

$$\begin{cases} A_1\left[m\lambda^2+\alpha\lambda+\beta+\omega c\left(\pi,\,\lambda\right)\right]+A_2\left[1+\omega s\left(\pi,\,\lambda\right)\right]=0,\\ A_1\left[-\bar{\omega}+\gamma c\left(\pi,\,\lambda\right)+c'\left(\pi,\,\lambda\right)\right]+A_2\left[\gamma s\left(\pi,\,\lambda\right)+s'\left(\pi,\,\lambda\right)\right]=0.\end{cases}$$

For the number $\lambda$ to be an eigenvalue of the boundary value problem $P$, it is necessary and sufficient that the last system has a nonzero solution. But this system has a nonzero solution if and only if its determinant is equal to zero. Therefore, the eigenvalues of the boundary value problem $P$ coincide with the zeros of the function

$$\varDelta\left(\lambda\right)=\begin{vmatrix} m\lambda^2+\alpha\lambda+\beta+\omega c\left(\pi,\,\lambda\right) & 1+\omega s\left(\pi,\,\lambda\right)\\ -\bar{\omega}+\gamma c\left(\pi,\,\lambda\right)+c'\left(\pi,\,\lambda\right) & \gamma s\left(\pi,\,\lambda\right)+s'\left(\pi,\,\lambda\right)\end{vmatrix}.$$

This function is called the characteristic function of the problem $P$. Let us expand this determinant and take into account the identity $c\left(x,\,\lambda\right)s'\left(x,\,\lambda\right)-c'\left(x,\,\lambda\right)s\left(x,\,\lambda\right)=1$ :

$$\begin{aligned}\varDelta\left(\lambda\right)&=\left[m\lambda^2+\alpha\lambda+\beta+\omega c\left(\pi,\,\lambda\right)\right]\cdot\left[\gamma s\left(\pi,\,\lambda\right)+s'\left(\pi,\,\lambda\right)\right]\\
&-\left[1+\omega s\left(\pi,\,\lambda\right)\right]\cdot\left[-\bar{\omega}+\gamma c\left(\pi,\,\lambda\right)+c'\left(\pi,\,\lambda\right)\right]=m\lambda^2\gamma s\left(\pi,\,\lambda\right)\\
&+m\lambda^2 s'\left(\pi,\,\lambda\right)+\alpha\lambda\gamma s\left(\pi,\,\lambda\right)+\alpha\lambda s'\left(\pi,\,\lambda\right)+\omega\gamma c\left(\pi,\,\lambda\right)s\left(\pi,\,\lambda\right)\\
&+\omega c\left(\pi,\,\lambda\right)s'\left(\pi,\,\lambda\right)+\beta\gamma s\left(\pi,\,\lambda\right)+\beta s'\left(\pi,\,\lambda\right)+\overline{\omega}-\gamma c\left(\pi,\,\lambda\right)\\
&-c'\left(\pi,\,\lambda\right)+\omega\overline{\omega}s(\pi,\lambda)-\omega\gamma s(\pi,\lambda)c(\pi,\lambda)-\omega s(\pi,\lambda)c'(\pi,\lambda)\\
&=\left[m\lambda^2+\alpha\lambda+\beta\right]s'(\pi,\lambda)+\left[m\lambda^2+\alpha\lambda+\beta\right]\gamma s(\pi,\lambda)\\
&+\omega\left[c(\pi,\lambda)\,s'(\pi,\lambda)-c'(\pi,\lambda)s(\pi,\lambda)\right]-\gamma c(\pi,\lambda)-c'(\pi,\lambda)\\
&+\left|\omega\right|^2\cdot s(\pi,\lambda)+\overline{\omega}=2Re\omega+\left|\omega\right|^2\cdot s(\pi,\lambda)\\
&+\left[m\lambda^2+\alpha\lambda+\beta\right]\left(s'\left(\pi,\,\lambda\right)+\gamma\,s\left(\pi,\,\lambda\right)\right)-\gamma c(\pi,\lambda)-c'(\pi,\lambda).\end{aligned}$$

Denote

$$\eta\left(\lambda\right)=c'\left(\pi,\,\lambda\right)+\gamma\,c\left(\pi,\,\lambda\right)\ ,\ \ \sigma\left(\lambda\right)=s'\left(\pi,\,\lambda\right)+\gamma\,s\left(\pi,\,\lambda\right)\ .$$

Then

$$\varDelta\left(\lambda\right)=2Re\omega-\eta\left(\lambda\right)+\left|\omega\right|^2s\left(\pi,\,\lambda\right)+\ \left(m\lambda^2+\alpha\lambda+\beta\right)\sigma\left(\lambda\right). \tag{3.3}$$

**Theorem 3.1** *For the eigenvalues $\mu_k$ $(k = \pm 0, \pm 1, \pm 2, ...)$ of the boundary value problem $P$ for $|k| \to \infty$ the following asymptotic formula holds :*

$$\mu_k = k - \frac{1}{2}\text{sign}k + a + \frac{A}{m\pi k} + \frac{m_k}{k}, \tag{3.4}$$

*where* $a = \frac{1}{\pi}\int_0^\pi p(t)dt$, $A = 1 + m\pi a_1 + m\gamma$, $a_1 = \frac{1}{2\pi}\int_0^\pi [q(t) + p^2(t)]dt$, $m_k \in l_2$.

**Proof.** It is known [10] that the following representations are valid for the functions $c(\pi, \lambda)$, $c'(\pi, \lambda)$, $s(\pi, \lambda)$ and $s'(\pi, \lambda)$:

$$c(\pi, \lambda) = \cos\pi(\lambda - a) - c_1\frac{\cos\pi(\lambda - a)}{\lambda} + \pi a_1\frac{\sin\pi(\lambda - a)}{\lambda} + \frac{1}{\lambda}\int_{-\pi}^{\pi}\psi_1(t)\, e^{i\lambda t}dt,$$

$$c'(\pi, \lambda) = -\lambda\sin\pi(\lambda - a) + c_0\sin\pi(\lambda - a) + \pi a_1\cos\pi(\lambda - a) + \frac{1}{\lambda}\int_{-\pi}^{\pi}\psi_2(t)\, e^{i\lambda t}dt,$$

$$s(\pi, \lambda) = \frac{\sin\pi(\lambda - a)}{\lambda} + c_0\frac{\sin\pi(\lambda - a)}{\lambda^2} - \pi a_1\frac{\cos\pi(\lambda - a)}{\lambda^2} + \frac{1}{\lambda^2}\int_{-\pi}^{\pi}\psi_3(t)\, e^{i\lambda t}dt,$$

$$s'(\pi, \lambda) = \cos\pi(\lambda - a) + c_1\frac{\cos\pi(\lambda - a)}{\lambda} + \pi a_1\frac{\sin\pi(\lambda - a)}{\lambda} + \frac{1}{\lambda}\int_{-\pi}^{\pi}\psi_4(t)\, e^{i\lambda t}dt,$$

where $c_0 = \frac{1}{2}[p(0) + p(\pi)]$, $c_1 = \frac{1}{2}[p(0) - p(\pi)]$, $\psi_m(t) \in L_2[-\pi, \pi]$, $m = 1, 2, 3, 4$.
From these representations and (3.3) according to the Paley–Wiener theorem [14, p. 69] we obtain that the characteristic function $\Delta(\lambda)$ of the boundary value problem $P$ has the form

$$\begin{aligned}\Delta(\lambda) &= 2Re\omega + m\lambda^2\cos\pi(\lambda - a) + \lambda[\sin\pi(\lambda - a) \\ &+ mc_1\cos\pi(\lambda - a) + m\pi a_1\sin\pi(\lambda - a) + \alpha\cos\pi(\lambda - a) \\ &+ m\gamma\sin\pi(\lambda - a)] - c_0\sin\pi(\lambda - a) - \pi a_1\cos\pi(\lambda - a) \\ &- \gamma\cos\pi(\lambda - a) + \alpha c_1\cos\pi(\lambda - a) + \alpha\pi a_1\sin\pi(\lambda - a) \\ &+ \beta\cos\pi(\lambda - a) + mc_0\gamma\sin\pi(\lambda - a) \\ &- m\gamma\pi a_1\cos\pi(\lambda - a) + \alpha\gamma\sin\pi(\lambda - a) + \lambda g_1(\lambda) + g_2(\lambda) \\ &= m\lambda^2\cos\pi(\lambda - a) + \lambda[(1 + m\pi a_1 + m\gamma)\sin\pi(\lambda - a) \\ &+ (mc_1 + \alpha)\cos\pi(\lambda - a) + g_1(\lambda)] \\ &+ (\alpha\pi a_1 - c_0 + mc_0\gamma + \alpha\gamma)\sin\pi(\lambda - a) \\ &+ (\alpha c_1 - \pi a_1 - \gamma + \beta - m\gamma\pi a_1)\cos\pi(\lambda - a) + g_2(\lambda) + 2Re\omega,\end{aligned} \tag{3.5}$$

where

$$g_j(\lambda) = \int_{-\pi}^{\pi}\tilde{g}_j(t)e^{i\lambda t}dt, \tilde{g}_j(t) \in L_2[-\pi; \pi],\ j = 1,\ 2.$$

We denote by $\Gamma_n$ the contour bounding the square

$$K_n = \{\lambda :\ |Re\lambda - a| \le n,\ \ |Im\lambda| \le n\}.$$

By virtue of relation (3.5) we have

$$\Delta(\lambda) = f(\lambda) + g(\lambda),$$

where $f(\lambda) = m\lambda^2 \cos \pi(\lambda - a)$,

$$g(\lambda) = \lambda[(1 + m\pi a_1 + m\gamma) \sin \pi(\lambda - a) + (mc_1 + \alpha) \cos \pi(\lambda - a) \\ +g_1(\lambda)] + (\alpha\pi a_1 - c_0 + mc_0\gamma + \alpha\gamma) \sin \pi(\lambda - a) + (\alpha c_1 - \pi a_1 - \gamma \\ +\beta - m\gamma\pi a_1) \cos \pi(\lambda - a) + g_2(\lambda) + 2Re\omega.$$

It is easy to prove that the inequality $|f(\lambda)| > |g(\lambda)|$ holds on $\Gamma_n$ for sufficiently large $n$. Then, by Rouché's theorem the square $K_n$ contains the same number of zeros $\Delta(\lambda)$ and $f(\lambda)$, i.e. $2n + 2$ zeros. Using representation (2.13) and Rouche's theorem, it is easy to establish that the roots $\mu_k$ $(k = \pm 0, \pm 1, \pm 2, ...)$ of the equation $\Delta(\lambda) = 0$ for $|k| \to \infty$ obey the asymptotics

$$\mu_k = k - \frac{1}{2}\text{sign}k + a + \varepsilon_k, \tag{3.6}$$

where $\varepsilon_k = O\left(k^{-1}\right)$. Taking into account the asymptotics (3.6) and the expansions $\cos x = 1 + O\left(x^2\right)$, $\sin x = x + O\left(x^3\right)$, $\frac{1}{1-x} = 1 + x + O\left(x^2\right)$ $(x \to 0)$, we have

$$\sin \pi(\mu_k - a) = (-1)^k \sin \pi\left(-\frac{1}{2}\text{sign}k + \varepsilon_k\right) = (-1)^{k+1} \text{sign}k \cos \pi\varepsilon_k$$

$$= (-1)^{k+1} \text{sign}k + O\left(\frac{1}{k^2}\right), \tag{3.7}$$

$$\cos \pi(\mu_k - a) = (-1)^k \cos \pi\left(-\frac{1}{2}\text{sign}k + \varepsilon_k\right) = (-1)^k \text{sign}k \sin \pi\varepsilon_k$$

$$= (-1)^k \pi\varepsilon_k \text{sign}k + O\left(\frac{1}{k^3}\right), \tag{3.8}$$

$$\frac{1}{\mu_k} = \frac{1}{k\left(1 - \frac{1}{2k}\text{sign}k + \frac{a}{k} + \frac{\varepsilon_k}{k}\right)} = \frac{1}{k}\left[1 + \frac{1}{2k}\text{sign}k - \frac{a}{k} + O\left(\frac{1}{k^2}\right)\right]$$

$$= \frac{1}{k} + O\left(\frac{1}{k^2}\right). \tag{3.9}$$

Moreover, using Lemma 1.4.3 in [16], we obtain the asymptotics

$$g_j(\mu_k) = \theta_{jk} + \frac{\rho_{jk}}{k}, \tag{3.10}$$

where $\{\theta_{jk}\}, \{\rho_{jk}\} \in l_2,\ j = 1,\ 2$. Substituting (3.6) into $\Delta(\mu_k) = 0$ and taking into account relations (3.7) - (3.10), we obtain the asymptotics

$$\varepsilon_k = \frac{A}{m\pi k} + \frac{m_k}{k}, m_k \in l_2. \tag{3.11}$$

Then from (3.6) by virtue of (3.11) the asymptotic formula (3.4) follows.

The theorem is proved.

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